Essential Relativistic Celestial Mechanics

V. A. Brumberg
Institute of Applied Astronomy of RAN, St.Petersburg, Russia

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V.A.Brumberg

Lecture 1. Elements of Riemannian geometry and tensor analysis

Main mathematical objects of GRT (general relativity theory) are Riemannian four–dimensional space and tensors. The purpose of this lecture is to remind and to recollect the basic formulas of Riemannian geometry and tensor analysis. It might seem that for people interested only in GRT applications there is no need in these mathematical subjects. But the correct understanding of GRT techniques and results can be hardly achieved without competence in these mathematical tools.

The simplest space studied in geometry is the affine space representing the set of points and vectors governed by the laws of ordinary vector algebra. This space is homogeneous and isotropic. In the affine n–dimensional space A_n a set of n linearly independent vectors $\mathbf{e}_i$ $(i = 1, 2, \ldots, n)$ emanating from an arbitrary point O constitutes an affine reference basis. An arbitrary vector $\mathbf{x}$ emanating from point O may be decomposed on the basis vectors

$$\mathbf{x} = x^i \mathbf{e}_i \,, \tag{1.1}$$

x^i $(i = 1, 2, \ldots, n)$ being affine coordinates of the vector. The Einstein summation rule (summation from 1 to n over every index occurring twice in any expression) is applied here and everywhere in what follows. The affine (linear) transformation from one basis to another one reads

$$\mathbf{e}_{i'} = A^i_{i'} \mathbf{e}_i \tag{1.2}$$

involving the corresponding transformation of the affine coordinates of a vector

$$x^{i'} = A^{i'}_i x^i \,, \qquad A^{k'}_i A^j_{k'} = \delta^j_i \tag{1.3}$$

with the Kronecker symbol

$$\delta^j_i = \begin{cases} 0\,, & i \neq j\,, \\ 1\,, & i = j\,. \end{cases} \tag{1.4}$$

The accented indices in (1.2), (1.3) refer to the new reference basis. It is seen that the the affine coordinates of a vector are transformed with the aid of the transposed inverse matrix of the generating basis transformation (1.2).

Generalization of the vectorial form may be carried out with the aid of tensors. Tensors enable one to define operations remaining invariant under the transformation of coordinates. All relations expressed in the tensorial form retain their form in any reference system (although the components of a tensor are distinct in the different reference systems). A tensor covariant of rank k and contravariant of rank m is defined as a collection of n^{k+m} quantities $a^{j_1 \ldots j_m}_{i_1 \ldots i_k}$ given in any reference system and transforming under the basis change (1.2) in accordance with

$$a^{j'_1 \ldots j'_m}_{i'_1 \ldots i'_k} = A^{j'_1}_{j_1} \ldots A^{j'_m}_{j_m} \, A^{i_1}_{i'_1} \ldots A^{i_k}_{i'_k} \, a^{j_1 \ldots j_m}_{i_1 \ldots i_k} \,. \tag{1.5}$$

The basic tensor operations are addition, multiplication, contraction (choosing all components for which a definite upper index is equal to a definite lower index), and change of indices (alteration of the order of arrangement of indices of a given tensor).

Euclidean n–dimensional space R_n represents the affine space A_n with the operation of scalar (inner) product of vectors. This product has properties of symmetry and indegeneracy. For the vectors of some affine reference basis it reads

$$g_{ij} = \mathbf{e}_i \mathbf{e}_j\,, \qquad g_{ij} = g_{ji}\,, \qquad g = \det \parallel g_{ij} \parallel \neq 0\,. \tag{1.6}$$

Then in virtue of (1.1) the scalar product of vectors $\mathbf{x}$ and $\mathbf{y}$, the scalar square and the length of vector $\mathbf{x}$ will be

$$\mathbf{x}\mathbf{y} = g_{ij} x^i y^j\,, \qquad \mathbf{x}^2 = g_{ij} x^i x^j\,, \qquad |\,\mathbf{x}\,| = (\mathbf{x}^2)^{1/2}\,. \tag{1.7}$$

Under transformation (1.2) the quantities g_{ij} act as the components of a covariant tensor of rank two. This metric tensor determines all structure of R_n at hand. For a new reference system the components of this tensor and the determinant of the corresponding matrix will be

$$g_{i'j'} = A^i_{i'} A^j_{j'}\, g_{ij}\,, \qquad g' = (\det \parallel A^i_{i'} \parallel)^2 g\,. \tag{1.8}$$

Elements g^{ij} of the inverse matrix of $\parallel g_{ij} \parallel$ are the components of the contravariant metric tensor. With the aid of g_{ij} and g^{ij} one can perform in R_n the operations of index raising and lowering such as

$$x_i = g_{ij} x^j\,, \qquad x_i = \mathbf{x}\mathbf{e}_i\,, \qquad x^i = g^{ij} x_j\,. \tag{1.9}$$

Depending on real or complex values for the scalar products of vectors Euclidean space R_n may be real or complex. Then, the real Euclidean space can be classified as proper Euclidean ($\mathbf{x}^2 > 0$ for all non–zero vectors) and pseudo–Euclidean where there exist the non–zero vectors with zero, positive and negative scalar squares ($\mathbf{x}^2 = 0\,, > 0\,, < 0$). Hence, in the pseudo–Euclidean space the length of a non–zero vector may be zero (isotropic vector), positive or purely imaginary.

A curve in A_n is represented analytically as a one–parametric set $x^i = x^i(t)$. The position vector of any point of the curve, its tangent vector and the arc length from M_1 to M_2 are

$$\mathbf{x}(t) = x^i(t)\mathbf{e}_i\,, \qquad \frac{d\mathbf{x}}{dt} = \frac{dx^i}{dt}\,\mathbf{e}_i\,, \qquad s = \int_{M_1}^{M_2} |d\mathbf{x}| = \int_{t_1}^{t_2} \left|\frac{d\mathbf{x}}{dt}\right| dt \tag{1.10}$$

with the differential of the arc length

$$ds = |d\mathbf{x}| = \left|\frac{d\mathbf{x}}{dt}\right| dt\,. \tag{1.11}$$

Depending on the sign of ds^2 the curves in R_n may be divided into three types as follows:

$$ds^2 = \begin{cases} > 0\,, & \text{real length curve,} \\ < 0\,, & \text{purely imaginary length curve,} \\ = 0\,, & \text{isotropic curve.} \end{cases}$$

In R_n one can construct an orthonormal basis characterized by $n(n+1)/2$ independent parameters. Such a basis for the pseudo–Euclidean space consists of k unit and $n-k$ imaginary unit vectors $(0 < k < n)$, the space index k being the same for any reference basis in given R_n. The spaces with the indices k and $n-k$ are not different since all the lengths of these spaces are distinguished by the common factor i.

For application to the theory of relativity the pseudo–Euclidean space R_4 of index 1 is of primary importance. It is used mostly as R_4 of index 3 with numeration $0\,,1\,,2\,,3$ (instead of $1\,,2\,,3\,,4$). In the orthonormal basis

$$\mathbf{e}_0^2 = 1\,, \qquad \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1 \tag{1.12}$$

of this space the scalar square of any vector is

$$\mathbf{x}^2 = {x^0}^2 - {x^1}^2 - {x^2}^2 - {x^3}^2\,. \tag{1.13}$$

Affine transformation of the reference basis consists of rotation of the pseudo–Euclidean plane $(\mathbf{e}_0\,, \mathbf{e}_1)$ and trivial rotation of the proper Euclidean space $R_3\,(\mathbf{e}_1\,, \mathbf{e}_2\,, \mathbf{e}_3))$. By omitting here this trivial rotation one has

$$\mathbf{e}_{0'} = \frac{\mathbf{e}_0 + \beta\mathbf{e}_1}{(1-\beta^2)^{1/2}}\,, \qquad \mathbf{e}_{1'} = \frac{\beta\mathbf{e}_0 + \mathbf{e}_1}{(1-\beta^2)^{1/2}}\,, \qquad \mathbf{e}_{2'} = \mathbf{e}_2\,, \qquad \mathbf{e}_{3'} = \mathbf{e}_3 \tag{1.14}$$

involving

$$x^{0'} = \frac{x^0 - \beta x^1}{(1-\beta^2)^{1/2}}\,, \qquad x^{1'} = \frac{-\beta x^0 + x^1}{(1-\beta^2)^{1/2}}\,, \qquad x^{2'} = x^2\,, \qquad x^{3'} = x^3\,, \tag{1.15}$$

β being a real constant such that $-1 < \beta < 1$. Affine transformation (1.15) underlies the famous Lorentz transformation of the theory of relativity.

All preceding formulas involve only affine coordinates permitting to express all relations in A_n and R_n in the most simple form. But it is possible to use also the curvilinear coordinates related with the affine ones by means of a non–singular, non–linear transformation. If x^i are such curvilinear coordinates it means that at any point M of A_n one can construct n coordinate lines x^i (along any such line the position vector $\mathbf{x}$ is a function of this coordinate alone) and the local reference basis $\mathbf{x}_i$ formed by the tangent vectors to the coordinate lines at the given point, i.e.

$$\mathbf{x} = \mathbf{x}(x^i)\,, \qquad \mathbf{x}_i(M) = \frac{d\mathbf{x}}{dx^i}(M). \tag{1.16}$$

Under the non–singular, non–linear transformation

$$x^{i'} = x^{i'}(x^1, ..., x^n), \qquad x^i = x^i(x^{1'}, ..., x^{n'}) \tag{1.17}$$

there results a new local reference basis

$$\mathbf{x}_{i'} = \frac{\partial x^i}{\partial x^{i'}}\mathbf{x}_i\,. \tag{1.18}$$

As an extension of (1.5) the components of a tensor with respect to the local basis are transformed under (1.17) as follows:

$$a^{j'_1\ldots j'_m}_{i'_1\ldots i'_k}(M) = \frac{\partial x^{j'_1}}{\partial x_{j_1}}(M)\ldots\frac{\partial x^{j'_m}}{\partial x_{j_m}}(M)\frac{\partial x^{i_1}}{\partial x_{i'_1}}(M)\ldots\frac{\partial x^{i_k}}{\partial x_{i'_k}}(M)\, a^{j_1\ldots j_m}_{i_1\ldots i_k}(M). \tag{1.19}$$

Algebraic operations are valid in curvilinear coordinates as well. But the absence of a unified affine reference basis manifests itself in parallel transporting the vectors. Consider at some point $M(x^i)$ of the curve $x^i = x^i(t)$ a constant vector $\boldsymbol{\xi} = \xi^k(t)\mathbf{x}_k$. Parallel transport of this vector to the neighbouring point

$$M(x^i) \to \tilde{M}(x^i + dx^i)$$

results due to the change of the local basis in new coordinates $\xi^k + d\xi^k$ of this vector. From the condition of parallel transport

$$d\boldsymbol{\xi} \equiv \mathbf{x}_k d\xi^k + \xi^j d\mathbf{x}_j = 0 \tag{1.20}$$

with

$$d\mathbf{x}_j = \mathbf{x}_{ij}dx^i\,, \qquad \mathbf{x}_{ij} = \frac{\partial \mathbf{x}_j}{\partial x^i} = \frac{\partial^2 \mathbf{x}}{\partial x^i \partial x^j} = \Gamma^k_{ij}\mathbf{x}_k \tag{1.21}$$

it follows

$$d\xi^k = -\Gamma^k_{ij}\xi^j dx^i\,, \tag{1.22}$$

quantities Γ^k_{ij} symmetrical in their lower indices are called the connection coefficients (Christoffel symbols) of the second kind. For affine coordinates $\mathbf{x}_i = \mathbf{e}_i$, $\mathbf{x}_{ij} = 0$, and $\Gamma^k_{ij} = 0$. Vanishing of Γ^k_{ij} is necessary and sufficient for curvilinear coordinates x^i in A_n to be affine. Transformation (1.17) leads to the transformation of the Christoffel symbols as follows:

$$\Gamma^{k'}_{i'j'} = \frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k}\Gamma^k_{ij}\,. \tag{1.23}$$

In using curvilinear coordinates the metric tensor of R_n is determined as the scalar product of the local basis vectors

$$g_{ij}(M) = \mathbf{x}_i(M)\mathbf{x}_j(M) \tag{1.24}$$

being the function of a point. If $x^i = x^i(t)$ and $\mathbf{x} = \mathbf{x}(x^1, \ldots, x^n)$ are some curve in R_n and a position vector of any point of this curve then

$$d\mathbf{x} = \mathbf{x}_i dx^i\,, \tag{1.25}$$

and the square of the differential of arc $ds^2 = d\mathbf{x}^2$ is expressed by the metric quadratic form

$$ds^2 = g_{ij}dx^i dx^j\,. \tag{1.26}$$

Given metric tensor g_{ij} or metric form ds^2, all the geometry of R_n may be determined. In particular, it is easy to find the connection coefficients. The relations

$$\mathbf{x}_{ij}\mathbf{x}_m = \Gamma_{mij}\,, \qquad \Gamma_{mij} = g_{km}\Gamma^k_{ij}\,, \qquad \Gamma^k_{ij} = g^{km}\Gamma_{mij}\,, \qquad \frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik} \tag{1.27}$$

result in the connection coefficients (Christoffel symbols) of the first kind

$$\Gamma_{mij} = \frac{1}{2}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right). \tag{1.28}$$

The mathematical tools considered until now and related to Euclidean (flat) n–dimensional space ($n = 4$) are adequate for SRT (special relativity theory). For GRT (general relativity theory) one has to deal with more sophisticated tools related to Riemannian (curved) n–dimensional space ($n = 4$). A Riemannian space V_n represents a manifold satisfying two conditions: 1) there exists one–to–one map into a domain M_n of n coordinates defined up to the transformation (1.17), and 2) for each point M of this manifold there exists a covariant symmetrical and non–singular metric tensor $g_{ij}(M) = g_{ij}(x^1, \ldots, x^n)$. A tensor at point M of V_n is defined as a set of quantities given in any reference system x^i and transformed to new coordinates with (1.19). In contrast to R_n there is no reference basis (global or local) in V_n. But the components of tensors in V_n in coordinates x^i may be interpreted as the tensor coordinates in the tangent Euclidean space R_n relative to the local reference basis appropriate to the coordinates x^i. V_n is the proper or pseudo–Riemannian space in the same sense as the tangent space R_n is the proper or pseudo–Euclidean. In the pseudo–Riemannian space there exist curves of real ($ds^2 > 0$), imaginary ($ds^2 < 0$) and zero ($ds^2 = 0$) length. Euclidean space R_n may be regarded as the particular case of V_n admitting such coordinates (affine) for which components g_{ij} are constant for the whole space (reducible to $0, \pm 1$).

Under the transformation (1.17) the determinant $g = ||g_{ij}||$ will be

$$g' = \left(\det \left|\left|\frac{\partial x^i}{\partial x^{i'}}\right|\right|\right)^2 g\,, \tag{1.29}$$

so that the n– dimensional integral taken over some domain Ω

$$W = \int_\Omega \sqrt{|\ g\ |}dx^1 ... dx^n \tag{1.30}$$

is invariant under the transformation (1.17) determining the volume of domain Ω in curvilinear coordinates in R_n and in V_n.

In R_n it is possible to introduce such coordinates (affine) that the connection coefficients Γ^k_{ij} vanish in the whole domain at hand. For V_n it is possible for Γ^k_{ij} to vanish at any given point M. Coordinates satisfying this condition are called geodesic. Equations determining the geodesic coordinates $x^{i'}$ in terms of x^i follow just from (1.23)

$$\frac{\partial^2 x^{m'}}{\partial x^i \partial x^j}(M) = \Gamma^k_{ij}(M)\frac{\partial x^{m'}}{\partial x^k}(M). \tag{1.31}$$

These equations may be satisfied by putting

$$x^{m'} = a_i^{m'}(x^i - x^i_M) + \tfrac{1}{2}a_k^{m'}\Gamma^k_{ij}(M)(x^i - x^i_M)(x^j - x^j_M) \tag{1.32}$$

with non–singular constant matrix $||a_i^{m'}||$, x^i_M being the coordinates of point M.

Parallel transport of vectors in V_n formally coincides with the corresponding operation (1.22) in curvilinear coordinates in R_n. But in contrast to R_n the parallel transport in V_n depends generally on the path.

Parallel transport is closely related to absolute differentiation, the most important operation of tensor analysis. Consider a tensor $a^{j_1\dots j_m}_{i_1\dots i_k}(t)$ at some point of the curve $x^i = x^i(t)$. At an infinitesimally close point this tensor has value $a^{j_1\dots j_m}_{i_1\dots i_k}(t+dt)$. However, these tensors are related to different local reference bases and cannot be compared directly, their difference being approximately equal to the non–tensorial differential $da^{j_1\dots j_m}_{i_1\dots i_k}(t)$. Denoting by $\tilde{a}^{j_1\dots j_m}_{i_1\dots i_k}(t)$ the tensor resulting from the parallel transport of $a^{j_1\dots j_m}_{i_1\dots i_k}(t+dt)$ to point t one has approximately

$$a^{j_1\dots j_m}_{i_1\dots i_k}(t+dt) - a^{j_1\dots j_m}_{i_1\dots i_k}(t) \approx da^{j_1\dots j_m}_{i_1\dots i_k}(t), \quad \tilde{a}^{j_1\dots j_m}_{i_1\dots i_k}(t) - a^{j_1\dots j_m}_{i_1\dots i_k}(t) \approx Da^{j_1\dots j_m}_{i_1\dots i_k}(t), \quad (1.33)$$

D standing for the absolute differential leading to a tensor of the same structure as the initial one. For vectors there results

$$Da^k = da^k + \Gamma^k_{ij} a^j dx^i\,, \qquad Da_j = da_j - \Gamma^k_{ij} a_k dx^i\,. \tag{1.34}$$

Generalizing these relations for a tensor of arbitrary structure one has

$$\begin{aligned} Da^{j_1\dots j_m}_{i_1\dots i_k} =& da^{j_1\dots j_m}_{i_1\dots i_k} + \Big(\Gamma^{j_1}_{rs} a^{s j_2\dots j_m}_{i_1\dots i_k} + \dots + \Gamma^{j_m}_{rs} a^{j_1\dots j_{m-1} s}_{i_1\dots i_k} - \\ & - \Gamma^s_{r i_1} a^{j_1\dots j_m}_{s i_2\dots i_k} - \dots - \Gamma^s_{r i_k} a^{j_1\dots j_m}_{i_1\dots i_{k-1} s}\Big) dx^r\,. \end{aligned} \tag{1.35}$$

Hence, under parallel transport of a tensor its absolute differential vanishes. Formula (1.35) may be rewritten in the form

$$Da^{j_1\dots j_m}_{i_1\dots i_k} = \nabla_r a^{j_1\dots j_m}_{i_1\dots i_k}\, dx^r\,. \tag{1.36}$$

The quantity $\nabla_r a^{j_1\dots j_m}_{i_1\dots i_k}(t)$, denoted often as $a^{j_1\dots j_m}_{i_1\dots i_k;r}(t)$ represents the tensor with an extra lower index and is called the absolute or covariant derivative. For scalars and vectors one has respectively

$$\nabla_r a = \frac{\partial a}{\partial x^r}\,, \qquad \nabla_r a^k = \frac{\partial a^k}{\partial x^r} + \Gamma^k_{rs} a^s\,, \qquad \nabla_r a_k = \frac{\partial a_k}{\partial x^r} - \Gamma^s_{rk} a_s\,. \tag{1.37}$$

By virtue of the definition of the Christoffel symbols one gets

$$\nabla_k g_{ij} = \nabla_k g^{ij} = \nabla_k \delta^i_j = 0\,. \tag{1.38}$$

A curve $x^i = x^i(t)$ in V_n is called geodesic if any vector $\xi^k = \xi^k(t)$ tangent to it at some point remains tangent in parallel transport along it. By virtue of the collinearity of tangent vectors one has

$$\frac{dx^k}{dt} = \alpha(t)\xi^k(t), \qquad d\lambda = \alpha(t)dt \tag{1.39}$$

with the scalar factor $\alpha(t)$ dependent on a point of the curve. By means of (1.22) the differential equations of the geodesic line referred to the canonical parameter λ such that $d\lambda = \alpha(t)dt$ will be

$$d\frac{dx^k}{d\lambda} = -\Gamma^k_{ij}\frac{dx^j}{d\lambda}dx^i\,, \qquad D\frac{dx^k}{d\lambda} \equiv \frac{d^2x^k}{d\lambda^2} + \Gamma^k_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0\,. \tag{1.40}$$

Since a parallel transported vector retains its length one has along the geodesic

$$g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = C \tag{1.41}$$

with a constant

$$C = \begin{cases} 1, & \lambda = s \text{ (geodesic of real length)}, \\ -1, & \lambda = s/\mathrm{i} \text{ (geodesic of imaginary length)}, \\ 0, & \text{(isotropic geodesic)}. \end{cases}$$

The isotropic geodesic referred to an arbitrary parameter $t = t(\lambda)$ is determined in accordance with (1.40) and (1.41) by the equations

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = -\frac{d^2t}{d\lambda^2}\left(\frac{dt}{d\lambda}\right)^{-2}\frac{dx^k}{dt}\,, \qquad g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0\,. \tag{1.42}$$

The equations for the non–isotropic geodesic may be obtained from the variational principle

$$\delta\int ds = 0\,. \tag{1.43}$$

Indeed, from (1.43) one gets

$$ds = \sqrt{f}d\lambda\,, \qquad f = g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\,, \qquad \frac{d}{d\lambda}\frac{\partial\sqrt{f}}{\partial(dx^i/d\lambda)} - \frac{\partial\sqrt{f}}{\partial x^i} = 0 \tag{1.44}$$

and finally

$$\frac{d^2x^k}{d\lambda^2} + \Gamma^k_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = \frac{1}{2}\frac{d\ln f}{d\lambda}\frac{dx^k}{d\lambda} \tag{1.45}$$

generalizing (1.40) for an arbitrary (non–canonical) parameter λ, e.g. in choosing λ among coordinates x^i.

Principle (1.43) gives Lagrangian $\sqrt{f}$ in irrational form. There exists also the variational principle in rational form (including isotropic geodesics)

$$\delta\int f\,d\lambda = 0\,, \qquad f = \left(\frac{ds}{d\lambda}\right)^2\,. \tag{1.46}$$

If a curve is not geodesic then a vector tangent to it at some point does not remain tangent in parallel transport along this curve. Transport which enables a vector ξ^k to remain tangent for any real curve $x^i = x^i(s)$ is called Fermi–Walker transport. This transport is determined by the equations

$$\frac{D\xi^i}{ds} = W^{ij}\xi_j\,, \qquad W^{ij} = A^i u^j - A^j u^i \tag{1.47}$$

with

$$u^i = \frac{dx^i}{ds}\,, \qquad A^i = \frac{Du^i}{ds}\,, \qquad u_i u^i = 1\,, \qquad u_i A^i = 0\,. \tag{1.48}$$

If $\xi^i\,,\eta^i$ are two vectors satisfying the Fermi–Walker transport then

$$\frac{D}{ds}(\xi_i \eta^i) = 0 \tag{1.49}$$

demonstrating that the Fermi–Walker transport preserves the scalar products of vectors and hence the angles between vectors and their lengths.

All operations with absolute differentials and derivatives of the first order are performed in the same manner as with ordinary differentials and derivatives. But even for the absolute differentials and derivatives of the second order this is not true. If $D\,,d$ are operations of absolute and ordinary differentiation in displacement from a given point x^i along some definite direction and derivatives $\tilde{D}\,,\tilde{d}$ are analogous operators in displacement along some other direction, then from (1.34) there results

$$\tilde{D}Da^j - D\tilde{D}a^j = -R_{kmi.}^{\cdot\cdot\cdot j}\,a^i\tilde{d}x^k dx^m\,, \qquad \tilde{D}Da_i - D\tilde{D}a_i = R_{kmi.}^{\cdot\cdot\cdot j}\,a_j\tilde{d}x^k dx^m \tag{1.50}$$

with the curvature (Riemann–Christoffel) tensor

$$R_{kmi.}^{\cdot\cdot\cdot j} = \frac{\partial\Gamma^j_{ki}}{\partial x^m} + \Gamma^j_{mn}\Gamma^n_{ki} - \frac{\partial\Gamma^j_{mi}}{\partial x^k} - \Gamma^j_{kn}\Gamma^n_{mi}\,, \qquad R_{mki.}^{\cdot\cdot\cdot j} = -R_{kmi.}^{\cdot\cdot\cdot j}\,. \tag{1.51}$$

In terms of absolute derivatives the relations (1.50) may be rewritten

$$(\nabla_k\nabla_m - \nabla_m\nabla_k)a^j = -R_{kmi.}^{\cdot\cdot\cdot j}\,a^i\,, \qquad (\nabla_k\nabla_m - \nabla_m\nabla_k)a_i = R_{kmi.}^{\cdot\cdot\cdot j}\,a_j\,. \tag{1.52}$$

The curvature tensor satisfies the Ricci identities

$$R_{kmi.}^{\cdot\cdot\cdot j} + R_{mik.}^{\cdot\cdot\cdot j} + R_{ikm.}^{\cdot\cdot\cdot j} = 0 \tag{1.53}$$

and the Bianchi identities

$$\nabla_k R_{mni.}^{\cdot\cdot\cdot j} + \nabla_m R_{nki.}^{\cdot\cdot\cdot j} + \nabla_n R_{kmi.}^{\cdot\cdot\cdot j} = 0\,. \tag{1.54}$$

By lowering an upper index one obtains the covariant curvature tensor

$$R_{kmij} = g_{jn}R_{kmi.}^{\cdot\cdot\cdot n} = \tfrac{1}{2}(g_{kj,mi} + g_{mi,kj} - g_{mj,ki} - g_{ki,mj}) + g_{pq}(\Gamma^p_{kj}\Gamma^q_{mi} - \Gamma^p_{mj}\Gamma^q_{ki})\,. \tag{1.55}$$

The covariant curvature tensor satisfies the relations

$$R_{ijkm} = R_{kmij}\,, \qquad R_{mkij} = -R_{kmij}\,, \qquad R_{kmji} = -R_{kmij} \tag{1.56}$$

and the Ricci identities take the form

$$R_{kmij} + R_{mikj} + R_{ikmj} = 0 \tag{1.57}$$

admitting cyclic permutation of any three indices. The number of the significantly different components of the curvature tensor in V_n reduces from n^4 to $n^2(n^2-1)/12$. Along with tensor (1.55) one obtains on the basis of the curvature tensor the symmetrical Ricci tensor

$$R_{ij} = R^{\cdot\cdot\cdot k}_{kij.} = g^{km} R_{kjim} \tag{1.58}$$

and the scalar curvature

$$R = g^{ij} R_{ij}\,. \tag{1.59}$$

In particular, the curvature tensor determines the behaviour of close geodesics in V_n. For a one–parametric set of curves $x^i = x^i(t\,,\,p)$ with parameter p one has from (1.34)

$$\frac{Du^i}{dp} = \frac{Dq^i}{dt}\,, \qquad \frac{D^2 q^i}{dt^2} = \frac{D}{dp}\frac{Du^i}{dt} + R^{\cdot\cdot\cdot i}_{kmj.} u^m u^j q^k\,, \tag{1.60}$$

where $u^i = \partial x^i/\partial t$ represents a vector tangent to the reference curve of the set and $q^i = \partial x^i/\partial p$ represents a vector characterizing the deviation from the reference curve. Deviation of curves is determined by the infinitesimal vector $\eta^i = q^i dp$. For the canonical parameter $t = \lambda$ the final equations of the deviation of the geodesics take the form

$$\frac{D^2\eta^i}{d\lambda^2} = R^{\cdot\cdot\cdot i}_{kmj.} u^m u^j \eta^k\,. \tag{1.61}$$

Definitions and generally all exposition techniques employed here are based on the textbook by Rashevsky (1953).

Lecture 2. Elements of the special relativity theory

One of the greatest scientific achievements to open the 20th century was the creation of special relativity theory (SRT) by Albert Einstein in 1905. Nowadays, it is even difficult to imagine the astonishment and admiration of the scientific community caused by SRT. In its further development the 20th century involved so much novelty into human life (both positively and negatively), that people seemed to have lost the capability to be surprised by anything. But, in the beginning of the 20th century, SRT and the resulting revolutionary change of the physical description of the world was met by mankind in a quite adequate manner. Indeed, for two preceding centuries Newtonian mechanics and Newtonian gravitation theory have successfully advanced in the description of the observed world phenomena and prediction of observable effects (it is sufficient to remind the triumph of celestial mechanics when Neptune got discovered based on analysis of the relevant perturbations in motion of Uranus). Therefore, the concepts of Newtonian physics seemed to be quite true. These concepts were:

1) the absolute (homogeneous) time – one and the same regardless of the reference system (RS) of its actual measuring. In particular, the time interval between two events is the same in any RS. One may intuitively consider RS as a laboratory equipped by clock and some devices to measure linear quantities (a local physical RS) or angular distances in the background of fixed celestial reference objects (a global astronomical RS);

2) the absolute space described by three–dimensional Euclidean geometry. This space is maxi–homogeneous (no privileged points) and maxi–isotropic (no privileged directions). In particular, the distance between two points is the same regardless of RS of its actual measuring;

3) the laws of Newtonian mechanics. The first one of them is the inertia law. A RS where this law is valid is called inertial RS. Any RS moving uniformly and rectilinearly with respect to a given inertial RS also represents an inertial RS. The laws of Newtonian mechanics are valid in any inertial RS resulting in the Galileo principle of relativity. Mathematically this principle is provided by the invariance of the equations of Newtonian mechanics under the Galileo transformations describing the transfer in the three dimensional Euclidean space between two inertial reference systems. The above-mentioned properties of absolute time (homogeneity) and absolute space (homogeneity and isotropy) are related just to inertial systems;

4) the Newton law of universal gravitation. In developing Newtonian gravitation theory (NGT) this law has become into use in the alternative form as well, i.e. as the Newtonian gravitational field of material bodies.

Newton law of universal gravitation and Newtonian mechanics, within the concepts of absolute time and absolute space, were completely sufficient to satisfy scientific and technical demands of human society during two centuries. The first difficulties were met in the middle of 19th century while attempting to represent experimental data of electrodynamics and optics of moving bodies (Maxwell theory of electromagnetism and wave theory of light). Retaining the concepts of 1) homogeneity of space and time and 2) isotropy of space, these experimental data caused the crisis of Newtonian mechanics in the beginning of the 20th century and resulted in two new statements: 3) the principle of special relativity (invariance of the laws of nature in all inertial reference systems) and 4) the postulate of the constancy of light velocity (the speed of light c in vacuum is constant having the same magnitude in all inertial reference systems). The first two statements are common for Newtonian mechanics and SRT. The last two statements, specific of SRT of Albert Einstein and formulated in his famous paper 'On the electrodynamics of moving bodies' were published in September 1905, in the journal 'Annalen der Physik'.

Let $S(t,\, x,\, y,\, z)$ and $S'(t',\, x',\, y',\, z')$ be two inertial systems, S' being in motion relative to S with constant velocity $\mathbf{v}$. The Galileo transformation $S \to S'$ reads

$$t' = t\,, \qquad \mathbf{r}' = \mathbf{r} - \mathbf{v}t\,. \tag{2.1}$$

Adoption of the special principle of relativity and the postulate of the constancy of the velocity of light has radically changed Newtonian conceptions on space and time. The set of four statements indicated above results in the invariability for any two events $M_1(ct_1, x_1, y_1, z_1)$ and $M_2(ct_2, x_2, y_2, z_2)$ the square of length of the 4–vector M_1M_2 (the square of the space–time interval between two events)

$$(M_1M_2)^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2. \tag{2.2}$$

For infinitesimal events this means the invariability of

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2\,. \tag{2.3}$$

From this it follows that the SRT space–time is described by geometry of the four–dimensional pseudo–Euclidean space of index 1. Transformation between different inertial systems thus reduces to the transformation of the orthonormal reference bases. In the general case for $n = 4$ this transformation depends on $n(n+1)/2 = 10$ parameters. Four of them are due to the possibility of arbitrary choice of the initial point (homogeneity of space and time). Three parameters enable one to perform arbitrary rotation of the spatial axes (isotropy of space). The remaining three parameters are the components of the translatory velocity of one inertial system with respect to another one. Within the trivial transformation of translation and rotation, for the particular case of coinciding axes of S and S' and motion along the x–axis with constant velocity $v = \beta c$ ($v^2 < c^2$) this change reduces by (1.15) to the famous Lorentz transformation

$$t' = \frac{t - c^{-2}vx}{(1 - v^2/c^2)^{1/2}}\,, \qquad x' = \frac{-vt + x}{(1 - v^2/c^2)^{1/2}}\,, \qquad y' = y\,, \qquad z' = z\,. \tag{2.4}$$

A more general six–parameter Lorentz transformation valid for arbitrary direction of the translatory velocity $\mathbf{v}$ and including the rotation of the spatial axes has the form

$$t' = \frac{t - c^{-2}(\mathbf{v}\mathbf{r})}{(1 - v^2/c^2)^{1/2}}\,, \tag{2.5}$$

$$T\mathbf{r}' = \mathbf{r} + \left[\left(\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\right)\frac{\mathbf{v}\mathbf{r}}{v^2} - \frac{t}{(1 - v^2/c^2)^{1/2}}\right]\mathbf{v}\,, \tag{2.6}$$

T being the orthogonal 3×3 rotation matrix. In the absence of rotation, T is replaced by the unit matrix E. In the limit $c \to \infty$ the Lorentz transformation reduces to the Galileo transformation. The inverse transformation from S' to S is determined by the same formulas by replacing $T \to T^{-1}$, $t \leftrightarrow t'$, $\mathbf{r} \leftrightarrow \mathbf{r}'$, $\mathbf{v} \to \mathbf{v}'$, $\mathbf{v}'$ being the velocity of S relative to S' satisfying the relations

$$T\mathbf{v}' = -\mathbf{v}\,, \qquad v' = |\mathbf{v}'| = v = |\mathbf{v}|\,. \tag{2.7}$$

The Lorentz transformations represent the affine transformations of R_4 and assume a specific class of reference systems, i.e. inertial systems (reflected in the name 'special relativity theory'). They involve a set of kinematical consequences to demonstrate the relativity of the space–time observational data in dependence of RS of actual measurement. The most important consequences are as follows:

1) Contraction of the linear sizes of the moving bodies.

The proper length of a body, i.e. the result of the length measurement of the body in the RS where the body is at rest is greater than the result of its length measurement in the RS where the body is in motion. For a particular case considered by (2.4) the length d of a material rod at rest in S' (its proper length) and its coordinate length d' in S are related by

$$d = \frac{d'}{(1 - v^2/c^2)^{1/2}}\,, \qquad d = x_2' - x_1'\,, \qquad d' = x_2 - x_1\,. \tag{2.8}$$

2) Relativity of simultaneity.

Two events simultaneous with respect to the clock readings of some RS and occurring in different spatial points of this system are not simultaneous with respect to another RS moving relative the first RS. Moreover, if in some RS the time interval between two events is less than the time needed for light to pass the distance between them (in accordance with SRT no interaction can propagate with a velocity greater than the light velocity) then their consecutive order can be different in different systems admitting the existence of a system where such events are simultaneous. On the other hand, if two events are such that the time interval between them is greater than the time needed for light to pass the distance between them then their consecutive order remains the same in any RS and there exists a RS where these events occur in the same spatial point. Indeed, for two events M_1 and M_2 in accordance with (2.4) one has

$$t_2' - t_1' = \frac{t_2 - t_1 - c^{-2}v(x_2 - x_1)}{(1 - v^2/c^2)^{1/2}}\,. \tag{2.9}$$

Dependent on the sign of the four–dimensional interval (2.2) one has two cases of quasi-simultaneous events or consequtive events described above

$$(M_1M_2)^2 = \begin{cases} < 0\,, & \text{spacelike interval, possibility of } t_1' = t_2'\,, \\ > 0\,, & \text{timelike interval, possibility of } x_1' = x_2'\,. \end{cases}$$

3) Retardation of a moving clock.
The time interval τ measured by a clock at rest in the given RS (proper time of RS) is less compared with its coordinate time T for a clock moving in the given RS. For a clock at rest in system S' one has be means of the inverse Lorentz transformation

$$\tau = (1 - v^2/c^2)^{1/2} T \,, \qquad \tau = t'_2 - t'_1 \,, \qquad T = t_2 - t_1 \,. \tag{2.10}$$

4) Relativistic addition of velocities.
The law of velocity addition in SRT differs significantly from the simple algebraic addition of velocities of Newtonian mechanics. This law involves new effects as compared with Newtonian mechanics. For example, if there are three inertial systems such that the second system is in translatory rectilinear motion with respect to the first one (no rotation) while the third system moves in the same way with respect to the second one (no rotation) then the spatial axes of the third system turn out to rotate with respect to the spatial axes of the first system (Thomas precession).

Consider a moving point having velocities $\mathbf{u} = d\mathbf{r}/dt$ and $\mathbf{u}' = d\mathbf{r}'/dt'$ in systems S and S', respectively. Let systems S and S' be related by the Lorentz transformation (2.5), (2.6) without rotation, i.e. $T = E$. By applying the inverse Lorentz transformation in terms of differentials of $t, t', \mathbf{r}, \mathbf{r}'$ one has the addition formula

$$\mathbf{u} = \frac{(1 - v^2/c^2)^{1/2}}{1 + c^{-2}(\mathbf{v}\mathbf{u}')} \left\{ \mathbf{u}' + \left[\left(\frac{1}{(1 - v^2/c^2)^{1/2}} - 1 \right) \frac{\mathbf{v}\mathbf{u}'}{v^2} + \frac{1}{(1 - v^2/c^2)^{1/2}} \right] \mathbf{v} \right\} . \tag{2.11}$$

For the particular case of collinear velocities this involves the Einstein formula

$$\mathbf{u} = \frac{\mathbf{u}' + \mathbf{v}}{1 + c^{-2}\mathbf{v}\mathbf{u}'} \,, \qquad (\mathbf{u}' \parallel \mathbf{v}). \tag{2.12}$$

Due to the non–commutability of the Lorentz transformation the velocities $\mathbf{v}$ and $\mathbf{u}'$ enter into (2.11) non–symmetrically (unless they are parallel directed). But the absolute magnitude of $\mathbf{u}$ is commutative since as resulted from (2.11)

$$\mathbf{u}^2 = \frac{(\mathbf{u}' + \mathbf{v})^2 - c^{-2}|\mathbf{v} \times \mathbf{u}'|^2}{(1 + c^{-2}\mathbf{v}\mathbf{u}')^2} \,. \tag{2.13}$$

Formula (2.11) enables one to solve a set of problems of motion in SRT. For example, let us find the relative velocity of two particles having in some system S' velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Evidently, the velocity of the second particle relative to the first one represents the velocity of the second particle in a system S in which the first particle is at rest. Hence, S moves relative to S' with velocity $\mathbf{v}_1$. Putting in (2.11) $\mathbf{v} = -\mathbf{v}_1$, $\mathbf{u}' = \mathbf{v}_2$ one finds the needed velocity $\mathbf{u}$. Assuming that the particles move with close velocities, $\mathbf{v}_2 = \mathbf{v}_1 + d\mathbf{v}$, one finds from (2.13) the square of the relative velocity

$$ds^2 \equiv \mathbf{u}^2 = \frac{(1 - v^2/c^2) d\mathbf{v}^2 + c^{-2}(\mathbf{v}d\mathbf{v})^2}{(1 - v^2/c^2)^2} \,, \qquad (\mathbf{v} \equiv \mathbf{v}_1) \tag{2.14}$$

where, for the sake of convenience, $\mathbf{v}_1$ is designated by $\mathbf{v}$. The space with such metric is called Lobachevsky space.

Another important application of (2.11) is to derive the above mentioned Thomas precession. One may regard a point at hand (moving with velocities $\mathbf{u}$ and $\mathbf{u}'$ in S and S', respectively) as being at rest in some inertial system S'' defined by coordinates t'', $\mathbf{r}''$. Denoting the Lorentz transformation without rotation by a double arrow ($\Rightarrow$) we have

$$S \Rightarrow S'(\mathbf{v}), \qquad S' \Rightarrow S''(\mathbf{u}'), \qquad S \rightarrow S''(\mathbf{u})$$

with the inverse transformations

$$S' \Rightarrow S(-\mathbf{v}), \qquad S'' \Rightarrow S'(-\mathbf{u}'), \qquad S'' \rightarrow S(\mathbf{u}''),$$

$\mathbf{u}''$ being the velocity of S relative to S''. In accordance with (2.7)

$$T\mathbf{u}'' = -\mathbf{u} \tag{2.15}$$

with the spatial rotation matrix T of the general–case Lorentz transformation $S \rightarrow S''$. The velocity $\mathbf{u}$ is expressed in terms of $\mathbf{v}$ and $\mathbf{u}'$ by (2.11). The velocity $\mathbf{u}''$ is expressed by the same formula replacing $\mathbf{v}$ and $\mathbf{u}'$ by $-\mathbf{u}'$ and $-\mathbf{v}$, respectively. If $\mathbf{u}'$ is infinitesimal then within the terms of the first order in $\mid \mathbf{u}'|$ by applying (2.11) to $(\mathbf{u}, \mathbf{v}, \mathbf{u}')$ and to $(\mathbf{u}'', -\mathbf{u}', -\mathbf{v})$ one gets

$$\mathbf{u} = \mathbf{v} + d\mathbf{v}\,, \qquad d\mathbf{v} = (1 - v^2/c^2)^{1/2}\left[\mathbf{u}' + \left((1 - v^2/c^2)^{1/2} - 1\right)\frac{(\mathbf{v}\mathbf{u}')}{v^2}\mathbf{v}\right], \tag{2.16}$$

$$\mathbf{u}'' = -[\mathbf{v} + \mathbf{u}' - c^{-2}(\mathbf{v}\mathbf{u}')\mathbf{v}]. \tag{2.17}$$

Using the relation

$$\mathbf{v}d\mathbf{v} = (1 - v^2/c^2)(\mathbf{v}\mathbf{u}') \tag{2.18}$$

resulting from (2.16) one can present (2.17) in the form

$$\mathbf{u}'' = -\mathbf{v} - d\mathbf{v} + \left(\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\right)\frac{1}{v^2}[\mathbf{v} \times (\mathbf{v} \times d\mathbf{v})]. \tag{2.19}$$

Comparison with (2.15) shows that rotation T of the resulting transformation from S to S'' represents the infinitesimal rotation around vector Ω by angle $\mid \Omega|$

$$T\mathbf{r} = \mathbf{r} + (\Omega \times \mathbf{r}), \tag{2.20}$$

$$\Omega = -\frac{1}{v^2}[(1 - v^2/c^2)^{-1/2} - 1](\mathbf{v} \times d\mathbf{v}). \tag{2.21}$$

Let a particle moving in system S be a gyroscope. If the velocity of the particle at moment t is $\mathbf{v}$ then for moment $t + dt$ this velocity will be $\mathbf{u} = \mathbf{v} + d\mathbf{v}$, $d\mathbf{v} = \dot{\mathbf{v}}dt$. S' and S'' are in this case the inertial rest systems of the particle at moments t and $t + dt$, respectively. The direction of rotation of the gyroscope at moment t in S' coincides with its direction at moment $t + dt$ in S'' provided that there is no external angular momentum. According to (2.20) the axis of gyroscope in S undergoes precession with angular velocity

$$\boldsymbol{\omega} = -\frac{1}{v^2}[(1 - v^2/c^2)^{-1/2} - 1](\mathbf{v} \times \dot{\mathbf{v}}). \tag{2.22}$$

This effect is called the Thomas precession.

5) Aberration of light.

Aberration of light and Doppler effect are kinematical effects treated in SRT more accurately as compared with Newtonian physics.

Directions towards one and the same star as measured in two different reference systems differ from one another by aberration. Returning to (2.11) let $\mathbf{u}$ and $\mathbf{u}'$ be velocities of a light photon in S and S'. Since the speed of light is the same in any system we have $\mathbf{u} = c\mathbf{p}$, $\mathbf{u}' = c\mathbf{p}'$, p, p' being the unit vectors characterizing the direction of light. The relation between $\mathbf{p}$ and $\mathbf{p}'$ follows from (2.11)

$$\mathbf{p} = \frac{(1 - v^2/c^2)^{1/2}}{1 + c^{-1}(\mathbf{v}\mathbf{p}')}\left\{\mathbf{p}' + \left[\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\right]\frac{(\mathbf{v}\mathbf{p}')}{v^2}\mathbf{v} + \frac{1}{c(1 - v^2/c^2)^{1/2}}\mathbf{v}\right\}. \tag{2.23}$$

Inverting this formula one gets

$$\mathbf{p}' = \frac{(1 - v^2/c^2)^{1/2}}{1 - c^{-1}(\mathbf{v}\mathbf{p})}\left\{\mathbf{p} + \left[\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\right]\frac{(\mathbf{v}\mathbf{p})}{v^2}\mathbf{v} - \frac{1}{c(1 - v^2/c^2)^{1/2}}\mathbf{v}\right\} \tag{2.24}$$

with a checking relation

$$\frac{1 + c^{-1}(\mathbf{v}\mathbf{p}')}{(1 - v^2/c^2)^{1/2}} = \frac{(1 - v^2/c^2)^{1/2}}{1 - c^{-1}(\mathbf{v}\mathbf{p})}\,. \tag{2.25}$$

Different particular cases of SRT aberration formula may be obtained from (2.23) or (2.24). The terms of the first order with respect to v/c describe the Newtonian aberration of light.

6) Doppler effect

Doppler effect reveals as a difference between the proper frequency $\Delta t'$ of a light emission at the spatial origin of S' ($\mathbf{r}' = 0$) and its registered frequency Δt^* in S. If the time of the light emission is t then the time of the light registration will be

$$t^* = t + \frac{r(t)}{c}\,, \tag{2.26}$$

$r(t)$ being the distance between the emitter and the receiver. It involves the relation between the frequencies

$$\Delta t^* = \left(1 + \frac{v_r}{c}\right)\Delta t\,, \qquad v_r = \frac{dr}{dt} \tag{2.27}$$

with radial velocity v_r. The inverse Lorentz transformation

$$t = \frac{t'}{(1 - v^2/c^2)^{1/2}} \tag{2.28}$$

enables one to relate the frequency Δt^* in S to the proper frequency $\Delta t'$ in S' resulting in the Doppler effect formula

$$\Delta t^* = \frac{1 + c^{-1} v_r}{(1 - v^2/c^2)^{1/2}} \Delta t' \,. \tag{2.29}$$

Again the first–order terms in v/c describe the Doppler effect in Newtonian theory.

These are basic kinematical effects associated with the Lorentz transformations. Their more detailed physical meaning is exposed in numerous textbooks on SRT, in particular, in the monograph by Møller (1972) underlying our exposition.

Dynamics of a particle in SRT is characterized by the following basic equations: For the world line of any material point one has

$$ds^2 = {dx^0}^2 - \delta_{ij} dx^i dx^j \; => 0 \,, \qquad x^0 = ct \,. \tag{2.30}$$

Taking s as the parameter of the world line of a material particle one gets

$$x^\alpha = x^\alpha(s) \,, \qquad u^\alpha = \frac{dx^\alpha}{ds} \,, \qquad v^i = \frac{dx^i}{dt} \,, \qquad \mathbf{u}^2 = 1 \,. \tag{2.31}$$

Here and everywhere below the greek indices run from 0 to 3 and the latin indices run from 1 to 3. Therefore, the unit tangent vector u^α defines the 4–velocity of the particle, v^i being its 3–velocity. The proper time τ of the moving particle is determined by

$$ds = c d\tau = c(1 - v^2/c^2)^{1/2} dt \,. \tag{2.32}$$

The relationship between 4–velocity and 3–velocity reads

$$u^0 = \frac{1}{(1 - v^2/c^2)^{1/2}} \,, \qquad u^i = \frac{v^i}{c(1 - v^2/c^2)^{1/2}} \,. \tag{2.33}$$

A 4–vector $\mathbf{A}$ with components

$$A^\alpha = \frac{du^\alpha}{ds} = \frac{1}{c(1 - v^2/c^2)^{1/2}} \frac{du^\alpha}{dt} \,, \qquad \mathbf{uA} = 0 \tag{2.34}$$

represents the 4–acceleration of the moving particle.

By the second law of mechanics the equations of motion of the material particle in some inertial system are written in the form

$$\frac{d}{ds}(m_0 u^\alpha) \equiv m_0 A^\alpha + u^\alpha \frac{dm_0}{ds} = c^{-2} \chi(v) F^\alpha \,, \tag{2.35}$$

the rest mass $m_0 =$ constant is a quantity characterizing the inertial mass of the particle. F^α are components of the external force. The 3–vector $\mathbf{F} = (F^i)$ represents an ordinary vector of force. The constancy of m_0 implies $F^0 = c^{-1}\mathbf{F}\mathbf{v}$ and

$$\chi(v) = (1 - v^2/c^2)^{-1/2} \,. \tag{2.36}$$

Finally, the SRT dynamics equations (2.35) take the form

$$\frac{dE}{dt} = \mathbf{F}\mathbf{v}\,, \qquad \frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{2.37}$$

with kinetic energy $E = mc^2$, impulse $\mathbf{p} = m\mathbf{v}$ and mass m of the moving point

$$m = \frac{m_0}{(1 - v^2/c^2)^{1/2}} \,. \tag{2.38}$$

The dynamical characteristic of the moving point is the energy–momentum vector $E_0\mathbf{u}$ with $E_0 = m_0c^2$ being the rest energy. The time component of this vector is the energy of the point $E_0u^0 = mc^2$. The space components are components of the impulse multiplied by c, i.e. $E_0u^i = mv^ic$.

For a more complicated case of matter, if $d\omega_0$ and dm_0 are elementary rest volume and rest mass in a co–moving system S_0 then in S

$$d\omega = (1 - v^2/c^2)^{1/2}\, d\omega_0\,, \qquad dm = \frac{dm_0}{(1 - v^2/c^2)^{1/2}} \,. \tag{2.39}$$

Mass density may be characterized by any one of the quantities

$$\rho^* = \frac{dm_0}{d\omega_0}\,, \qquad \rho = \frac{dm_0}{d\omega}\,, \qquad \tilde{\rho} = \frac{dm}{d\omega}\,, \tag{2.40}$$

ρ^* being the rest mass density in a co–moving system S_0 (invariant density), ρ representing the rest mass density in S and $\tilde{\rho}$ being the mass density in S (this is the total density including the mass corresponding to the kinetic energy of particles). The tensor of mass of a stream of dust–like non–interacting particles is determined by

$$T^{\alpha\beta} = \rho^* u^\alpha u^\beta \tag{2.41}$$

with u^α being the 4–vector of velocity. The components of the energy–momentum tensor $c^2T^{\alpha\beta}$ are

$$c^2T^{00} = c^2\tilde{\rho}\,, \qquad c^2T^{0i} = c\tilde{\rho}v^i\,, \qquad c^2T^{ij} = \tilde{\rho}v^iv^j\,. \tag{2.42}$$

The most important property of the total energy–momentum tensor is the conservation law. Analytically this law may be formulated in the form

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{2.43}$$

valid in any curvilinear coordinates. For the dust matter described by (2.41) this may be easily verified

$$\nabla_\beta T^{\alpha\beta} = u^\alpha \frac{\partial(\rho^* u^\beta)}{\partial x^\beta} + \rho^* u^\beta \frac{\partial u^\alpha}{\partial x^\beta} \,. \tag{2.44}$$

The first term vanishes in virtue of the equation of continuity (conservation of the rest mass)

$$\frac{\partial(\rho^* u^\beta)}{\partial x^\beta} = c^{-1} \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i} \right) = 0 \,. \tag{2.45}$$

Besides,

$$u^\beta \frac{\partial u^\alpha}{\partial x^\beta} = \frac{1}{c(1 - v^2/c^2)^{1/2}} \left(\frac{\partial u^\alpha}{\partial t} + v^i \frac{\partial u^\alpha}{\partial x^i} \right) = \frac{1}{c(1 - v^2/c^2)^{1/2}} \frac{du^\alpha}{dt} = A^\alpha = 0 \,, \tag{2.46}$$

4–acceleration A^α being zero due to the absence of external forces.

Inside macroscopic bodies one may assume the validity of the equation of continuity and the equations of motion of a continuous medium (internal equations)

$$\rho \frac{dv^i}{dt} = \rho F^i + \frac{\partial p^{ij}}{\partial x^j} \,, \tag{2.47}$$

F^i being the external force and p^{ij} being the stress tensor. For an ideal fluid $p^{ij} = -p\delta^{ij}$. An expression for the energy–momentum tensor of macroscopic bodies is obtained by generalizing (2.30) taking into account pressure p (Fock, 1955)

$$c^2 T^{\alpha\beta} = (c^2 \rho^* + \rho^* \Pi + p) u^\alpha u^\beta - p \eta^{\alpha\beta} \tag{2.48}$$

with potential compressional energy

$$d\Pi = \frac{p}{\rho^{*2}} d\rho^* \,, \qquad \Pi = -\frac{p}{\rho^*} + \int_0^p \frac{dp}{\rho^*} \,. \tag{2.49}$$

Tensor $\eta^{\alpha\beta}$ is called Minkowski tensor

$$\eta^{00} = 1 \,, \qquad \eta^{0i} = 0 \,, \qquad \eta^{ij} = -\delta^{ij} \,, \tag{2.50}$$

$$\eta_{00} = 1 \,, \qquad \eta_{0i} = 0 \,, \qquad \eta_{ij} = -\delta_{ij} \tag{2.51}$$

permitting to write the SRT metric in the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.52}$$

Presently SRT represents a theory experimentally checked in all of its aspects. Moreover, it is a working theory used now in many domains of applied science from astronavigation (by means of navigation satellites) to physics of elementary particles. In the distant future described now in scientific fiction SRT might play the major role as scientific base for interstellar flights with the use of photon rockets.

Historical remarks.

Nowadays, there are no explicit opponents of SRT (although at the present time of broad activity of pseudo-science one can listen from time to time about sensational ‘discoveries’ claiming to argue against the postulate of the light constancy). But there are still attempts to doubt about the priority of Einstein in the foundation of this theory. One can even meet the implicit accusations of Einstein in plagiarism. But the responsibility of all statements of such kind should be addressed to the conscience of their authors. The history of the creation and development of SRT is well known. This history started with the works by Lorentz and Poincaré. The paper by Poincaré ‘On dynamics of electron’ appeared in draft form in June 1905 and in complete form in January 1906. When comparing with SRT many statements of this paper may be considered equivalent to the SRT statements. But such comparison might be done only having the formulations of SRT. As mentioned above the basic paper by Einstein formulating in complete form all physical statements of SRT came out of press in September 1905. Later on Minkowski contributed into mathematical outlook of this theory. Replying the question about the priority in the foundation of SRT Einstein wrote to his biographer Carl Seelig in February 1955: ‘Remembering the history of the development of SRT we can firmly say that its discovery had been prepared by 1905. Lorentz knew already that the transformation called later after him was of significant importance for the analysis of Maxwell equations. Poincaré advanced this thought. As for me, I knew only the fundamental work by Lorentz written in 1895 but was not aware of his later paper and research by Poincaré related with it. In this aspect my work was independent. Its novelty was to formulate that the Lorentz transformation is of importance beyond the scope of the Maxwell equations and concerns the structure of the space and time. Another new point was the conclusion that the ‘Lorentz invariance’ is the general demand for any physical theory.’

SRT and celestial mechanics.

For Einstein SRT was of importance not only as a theory of space and time in the absence of gravitation but also as a starting point to elaborate a theory of space, time and gravitation. This theory, completed in 1915 and called GRT, forms now a physical foundation for contemporary celestial mechanics. From purely operational point of view GRT extends SRT demonstrating that all space-time characteristics at the point of observation in some RS depend not only on the velocity of this point but on the value of the gravitational potential (and its higher moments) at this point. Neglecting gravitation and returning to SRT one may indicate some problems of celestial mechanics under the beneficial effect of SRT:

1) Hierarchy of reference systems and time scales. Since 1991 International Astronomical Union (IAU) recommends to use the GRT-based hierarchy of four-dimensional reference systems mutually interrelated by the generalized (non-linear) Lorentz transformations. The fourth coordinate of any RS serves as the corresponding time scale. In the framework of SRT such four-dimensional systems generalize the spatial inertial systems of Newtonian astronomy. In case of SRT these astronomical reference systems are realized by inertial systems of SRT interrelated by the ordinary Lorentz transformations. All kinematical effects of Newtonian astrometry such as velocity–dependent rates of clocks, aberration of light, Doppler effect, etc., can be easily calculated just from the sequence of

the Lorentz transformations from some global RS to a local RS of an observer. One meets therewith a specific SRT effect, i.e. Thomas precession, mentioned above. Treatment in the GRT framework generalizes all these effects taking into account gravitation but the first distinction from Newtonian astrometry reveals already at the SRT level.

2) Photon rocket motion. It was already mentioned that the study of the photon rocket motion demands mainly to apply SRT dynamics combining the kinematical effects due to the four-dimensional space-time with the dynamical effects due to the variable mass. Investigations of motion of a particle with variable mass have long been studied in celestial mechanics. SRT extends and stimulates such investigations.

3) KS (Kustaanheimo–Stiefel) regularization. SRT was of benefit for celestial mechanics (and generally for theoretical mechanics) even outside the scope of relativity. SRT introduces the fourth dimension quite naturally. It turns out that many problems of Newtonian dynamics can be treated more effectively in extending the dimension of the space of events from three to four. Although the famous KS regularization has been introduced without explicit reference to SRT there is no doubt that the use of the mathematical techniques of SRT facilitated greatly in developing KS transformation and its generalizations.

4) SRT as a prelude of GRT. If celestial mechanics relativistic effects of kinematical origin can be well understood in the SRT framework the dynamical relativistic effects and hence relativistic celestial mechanics as a whole are based on GRT underlying the primary role of SRT as a stemming theory for GRT.

To conclude this lecture let us remind once again that SRT is one of the greatest scientific achievements of human civilization. It is of interest that there exist alternative theories of gravitation aiming (unsuccessfully so far) to replace GRT. In this respect the position of SRT is more solid because there is no serious alternative theory contesting the fundamental role of SRT in describing the physical processes (ignoring gravitation) in the flat four-dimensional spacetime.

Lecture 3. Essential general relativity theory

The basic idea of GRT is that the properties of space and time, i.e. the space–time metric, are determined by motion and distribution of masses and, conversely, motion and distribution of masses are governed by the field metric:

[space-time properties] $\leftrightarrow$ [motion and distribution of masses].

Some people believe that, currently, it might be possible to come to the main idea of GRT from experimental results. Yet Einstein derived the basic statements of GRT by logical considerations proceeding from SRT and the fundamental law of equality of gravitational and inertial mass.

Having completed SRT Einstein successively put forward the principle of equivalence and the principle of general covariance. According to the principle of equivalence all physical processes follow the same pattern both in an inertial system under the action of the homogeneous gravitational field and in a non–inertial uniformly accelerated system in the absence of gravitation. The principle of equivalence is strictly local in contrast to the law of identity of gravitational and inertial mass underlying it. The principle of general covariance implies that equations of physics should have the same form in all reference systems. Combination of these two principles enabled Einstein to formulate the principle of general relativity.

The space–time of GRT is the four–dimensional pseudo–Riemannian space with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3.1}$$

(let us remember again that greek indices can have values 0, 1, 2, 3, while latin indices take the values 1, 2, 3).

In SRT it is possible, if one wishes, to put all equations in the covariant form and to use any curvilinear coordinates. But the space of events in SRT is flat (affine), i.e. its tensor of curvature is zero. In this space there are preferable coordinate systems (affine) defined up to affine transformations (Lorentz transformations). In such affine systems the SRT metric takes the form (2.52) with values (2.51). The coordinates providing values (2.51) are called Galilean. In GRT one cannot introduce the global Galilean coordinates (valid for the whole space–time). Instead, one may use quasi–Galilean coordinates such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \eta_{00} = 1\,,\ \eta_{0i} = 0\,,\ \eta_{ij} = -\delta_{ij}\,, \tag{3.2}$$

$h_{\mu\nu}$ being functions of x^0, x^1, x^2, x^3 with $|\, h_{\mu\nu}| \ll 1$. More specifically, h_{00} and h_{ij} are of the second order in v/c and h_{0i} are of the third order (v being the characteristic velocity of bodies with $v^2 \sim U$, U being the characteristic Newtonian potential of bodies). We will write simply

$$h_{00}\,,\ h_{ij}\ \sim c^{-2}\,,\ h_{0i}\ \sim c^{-3}$$

taking in mind the dimensionless small quantities indicated above. Representation (3.2) makes it evident that the GRT pseudo–Riemannian metric differs little from the pseudo–Euclidean metric of SRT. Quasi–Galilean coordinates x^μ admit not only linear transformation but any non–linear transformation of the type

$$\tilde{x}^\alpha = x^\alpha + \xi^\alpha(x^\beta), \qquad \left|\frac{\partial \xi^\alpha}{\partial x^\beta}\right| \ll 1 \tag{3.3}$$

since metric (3.1) expressed in new coordinates $\tilde{x}^\alpha$ is again little distinct from the Galilean metric. But it should be underlined that this distinction looks different for each reference system.

At every point M of the GRT space–time one may introduce in accordance with (1.32) the locally geodesic coordinates ensuring $g_{\mu\nu} = \eta_{\mu\nu}$ and $g_{\mu\nu,\alpha} = 0$ at M (here and everywhere below the partial derivative with respect to some coordinate is denoted by a comma accompanied by the appropriate index). Moreover, the locally geodesic coordinates may be introduced in some vicinity of a given world line. Thus, in these coordinates in displacing from point M to an infinitesimally close point functions $h_{\mu\nu}$ are at least of second order with respect to differences $x^\mu - x^\mu(M)$. Neglecting these infinitesimal terms of the second order one has in the infinitesimal vicinity of point M the space–time of SRT. Hence, the locally geodesic coordinates determine a local inertial reference system where there is no gravitational field and the GRT coordinates x^μ may be interpreted as the SRT inertial coordinates ct, x, y, z. Such a possibility of introducing the locally geodesic coordinates is due to the principle of equivalence valid only locally. The relation $x^0 = ct$ is used in GRT just as transformation to a time–type mathematical coordinate (coordinate time t).

The interrelation between the properties of space and time from the one hand and motion and distribution of masses from the other hand is revealed in the field equations for determining the tensor $g_{\mu\nu}$. Einstein set up the form of the GRT field equations using the considerations as follows:

(a) the right–hand members of the field equations may contain only the mass tensor,

(b) the field equations should have the covariant form,

(c) in analogy with the Poisson equation for the Newtonian potential the field equations might be of the second order,

(d) in virtue of $\nabla_\beta T^{\alpha\beta} = 0$ of SRT the left–hand members should satisfy the same conditions.

These considerations lead to the following form of the field equations:

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = -\kappa T^{\alpha\beta} \tag{3.4}$$

with the Einstein tensor

$$G^{\alpha\beta} = R^{\alpha\beta} - \tfrac{1}{2} R g^{\alpha\beta} \tag{3.5}$$

expressed in terms of the Ricci tensor $R^{\alpha\beta}$, scalar curvature R and metric tensor $g^{\alpha\beta}$. The Bianchi identities (1.54) imply

$$\nabla_\beta R = 2 g_{\beta\mu} \nabla_\nu R^{\mu\nu} \,, \qquad \nabla_\beta G^{\alpha\beta} = 0 \,. \tag{3.6}$$

The field equations (3.4) contain a constant κ determined below in passing to the limiting case of the Newtonian field and the cosmological constant Λ which is of no importance in most problems of relativistic celestial mechanics and astrometry. Assuming $\Lambda = 0$ one has

$$G^{\alpha\beta} = -\kappa T^{\alpha\beta} \,. \tag{3.7}$$

In virtue of

$$R = \kappa T \,, \qquad T = g_{\alpha\beta} T^{\alpha\beta} \tag{3.8}$$

the field equations may be presented also in the form

$$R^{\alpha\beta} = -\kappa(T^{\alpha\beta} - \tfrac{1}{2}Tg^{\alpha\beta})\,. \tag{3.9}$$

In the domain outside the gravitating masses the tensor $T^{\alpha\beta}$ vanishes and the field equations in vacuum may be reduced simply to the vanishing Ricci tensor

$$R_{\alpha\beta} = 0\,. \tag{3.10}$$

The field equations represent ten non–linear second–order partial equations of the hyperbolic type. They determine ten unknown functions: six components of $g_{\alpha\beta}$ (four components remain arbitrary due to general covariance of the field equations), three velocity components v^i of the matter, and mass density ρ (an equation of state relating the density and pressure should be given separately).

To derive the solution of the field equations in some definite coordinate system it is necessary to add (explicitly or implicitly) four non–tensorial equations called coordinate conditions. For any problem at hand there may exist coordinate conditions which are preferable mathematically (but no way physically) for solving this problem. In problems of relativistic celestial mechanics one often uses harmonic conditions determined by the equations

$$\frac{\partial(\sqrt{-g}g^{\alpha\beta})}{\partial x^\beta} = 0\,. \tag{3.11}$$

The problem of coordinate conditions has for a long while been the subject of rather emotional discussions, particularly, between Fock (1955) and Infeld (Infeld and Plebanski, 1960). At present such discussion seems meaningless. Each specific celestial mechanics problem may be solved in any coordinates (taking into account the possibility of mathematical simplification under some definite coordinate conditions) but in the final comparison with observations one should obtain coordinate–independent results. The problem of comparison of calculated and measurable quantities will be considered below.

For application in celestial mechanics the most important problem of GRT is the problem of motion of material bodies. The early history of this problem is exposed in (Chazy, 1928, 1930; Kottler, 1922). In 1927 it was first demonstrated by Einstein and Grommer that in contrast to NGT (Newton gravitation theory) the equations of motion of material bodies in GRT are closely related to the field equations. Starting in 1938, the group of Einstein from one side and the group of Fock from another side succeeded in elaborating practical methods to derive the GRT equations of motion of material bodies. These two different techniques, i.e. EIH (Einstein–Infeld–Hoffman) technique for vacuum equations (3.10) and Fock technique for equations (3.7) with a mass tensor, are treated in (Brumberg, 1972). In more detail the equations of motion will be considered further.

For a test particle, i.e. for a particle with infinitesimal rest mass producing no influence on the surrounding field, the equations of motion are determined by the geodesic principle implying that the motion of such a particle is performed on the geodesic line in a given field. This law results from the field equations (3.7) or (3.10). The most simple derivation is related to (3.7) with the mass tensor (2.41) for a stream of dust–like non–interacting particles. Considering

$$u^\alpha = \frac{dx^\alpha}{ds}\,, \qquad g_{\alpha\beta}u^\alpha u^\beta = 1 \tag{3.12}$$

and expressing the law of the conservation of rest mass in covariant form

$$\nabla_\beta(\rho^* u^\beta) = 0 \tag{3.13}$$

one gets from (2.43)

$$u^\beta \nabla_\beta u^\alpha = 0 \tag{3.14}$$

or

$$\frac{Du^\alpha}{ds} \equiv \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,. \tag{3.15}$$

Hence, the trajectory $x^\mu = x^\mu(s)$ of a test material particle is a non–isotropic geodesic. Due to the principle of equivalence the propagation of light is performed on isotropic geodesics.

Thus, there is no gravitational force in the Newtonian sense in GRT. The motion of test particles is presented as the free inertial motion along the geodesic lines of the pseudo–Riemannian space with metric determined by the gravitating masses. Components of the metric tensor act as gravitation potentials. In this respect the field equations (3.7) and (3.10) may be regarded as generalizing Poisson and Laplace equations for the Newtonian potential. Irrespective of the smallness of the relativity effects in specific celestial mechanics problems the explanation of gravitation first achieved only in GRT is of paramount scientific importance. Celestial mechanics primarily devoted to the motion of celestial bodies under gravitation and astrometry devoted to the orientation of celestial bodies in space and time should be inevitable relativistic, i.e. based on GRT.

The field equations (3.7) may be derived from the condition of stationarity of some scalar invariant. This variational principle is of importance both from the theoretical point of view and for applications.

Let us find the variation of the integral

$$S_g = \int \sqrt{-g}\, R d\Omega \tag{3.16}$$

with changing $g_{\mu\nu}$ provided that the variations of $g_{\mu\nu}$ and their first derivatives vanish on the boundary of the 4–domain of integration. The variation of the Ricci tensor

$$R_{\mu\nu} = \frac{\partial\Gamma^\alpha_{\alpha\nu}}{\partial x^\mu} - \frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu} - \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} \tag{3.17}$$

reads

$$\delta R_{\mu\nu} = \nabla_\mu \delta\Gamma^\alpha_{\alpha\nu} - \nabla_\alpha \delta\Gamma^\alpha_{\nu\nu}\,. \tag{3.18}$$

The differential of the determinant g is

$$dg = g g^{\alpha\beta} dg_{\alpha\beta}\,. \tag{3.19}$$

From the definition of the Christoffel symbols it follows

$$\Gamma^\alpha_{\alpha\mu} = \tfrac{1}{2} g^{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^\mu} = \frac{1}{2g}\frac{\partial g}{\partial x^\mu} = \frac{\partial \ln\sqrt{-g}}{\partial x^\mu}\,. \tag{3.20}$$

Therefore for any tensor a^μ

$$\nabla_\mu a^\mu = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}a^\mu)}{\partial x^\mu} \,. \qquad (3.21)$$

There results

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} [\sqrt{-g}(g^{\mu\nu}\delta\Gamma^\alpha_{\alpha\nu} - g^{\alpha\nu}\delta\Gamma^\mu_{\alpha\nu})] \,. \qquad (3.22)$$

In the variation of the integral (3.16)

$$\delta S_g = \int \sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu} d\Omega + \int R_{\mu\nu}\delta(\sqrt{-g}\, g^{\mu\nu}) d\Omega \qquad (3.23)$$

the first term makes no contribution since $\delta g_{\mu\nu}$ together with their first derivatives vanish on the domain boundary. Using the relations

$$g_{\mu\nu}g^{\beta\nu} = \delta^\beta_\mu \,, \qquad g_{\mu\nu}dg^{\beta\nu} = -g^{\beta\nu}dg_{\mu\nu} \,, \qquad dg = -g g_{\alpha\beta}dg^{\alpha\beta} \,, \qquad (3.24)$$

$$dg^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}dg_{\mu\nu} \,, \qquad a_{\alpha\beta}dg^{\alpha\beta} = -a^{\alpha\beta}dg_{\alpha\beta} \,, \qquad (3.25)$$

$$\delta(\sqrt{-g}\, g^{\mu\nu}) = \sqrt{-g}(\delta g^{\mu\nu} - \tfrac{1}{2} g_{\alpha\beta}dg^{\alpha\beta}) \,, \qquad (3.26)$$

$$R_{\mu\nu}\delta(\sqrt{-g}\, g^{\mu\nu}) = \sqrt{-g}(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu})\delta g^{\mu\nu} \qquad (3.27)$$

the variation of (3.16) may be presented in the form

$$\delta S_g = -\int (R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu})\sqrt{-g}\, \delta g_{\mu\nu} d\Omega \qquad (3.28)$$

resulting to the left–hand member of the field equations (3.7). In addition, the integrand of S_g may be transformed to the form not containing the second–order derivatives of the metric tensor components. In virtue of the relation (Fock, 1955; Brumberg, 1972)

$$\sqrt{-g}R = \frac{\partial}{\partial x^\mu} [\sqrt{-g}(g^{\mu\nu}\Gamma^\alpha_{\alpha\nu} - g^{\alpha\nu}\Gamma^\mu_{\alpha\nu})] - \sqrt{-g}J \,, \qquad (3.29)$$

$$J = g^{\mu\nu}(\Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\alpha\mu} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}) \qquad (3.30)$$

it is evident that S_g may be rewritten as

$$S_g = -\int \sqrt{-g}\, J d\Omega \,. \qquad (3.31)$$

It may be noted that J as a function of $g^{\mu\nu}$ and derivatives $\partial g^{\mu\nu}/\partial x^\alpha$ is Lagrangian for the left–side of the field equations (3.7).

The right–hand side of (3.7), i.e. the mass tensor, also may be obtained as a variation of the appropriate action integral

$$S_m = \int \rho^*(c^2 + \Pi)\sqrt{-g}\, d\Omega \qquad (3.32)$$

with ρ^* being an invariant density satisfying the equation of continuity (3.13) and Π being the potential compressional energy. With the rest mass density ρ the equation of continuity takes the standard form

$$\frac{\partial}{\partial x^\alpha}(\sqrt{-g}\rho^* u^\alpha) = 0\,, \qquad \sqrt{-g}\rho^* = \rho\frac{ds}{dx^0}\,. \tag{3.33}$$

The variation of ρ^* with respect to $g_{\mu\nu}$ yields

$$\delta\rho^* = \tfrac{1}{2}\rho^*(u^\alpha u^\beta - g^{\alpha\beta})dg_{\alpha\beta} \tag{3.34}$$

resulting to

$$\delta S_m = \int \left[\left(c^2 + \Pi + \frac{p}{\rho^*}\right)\delta\rho^* + \frac{1}{2g}\,\rho^*(c^2+\Pi)\delta g\right]\sqrt{-g}\,d\Omega \tag{3.35}$$

or else

$$\delta S_m = \tfrac{1}{2}c^2\int T^{\mu\nu}\sqrt{-g}\,\delta g_{\mu\nu} d\Omega \tag{3.36}$$

with

$$c^2 T^{\mu\nu} = (c^2\rho^* + \rho^*\Pi + p)u^\mu u^\nu - p g^{\mu\nu} \tag{3.37}$$

generalizing (2.48). Combination of (3.28) and (3.36) enables one to conclude that the field equations (3.7) follow from the variational principle

$$\delta(2c^{-2}\kappa S_m - S_g) = 0\,. \tag{3.38}$$

The simplest approximate solution of the field equations is obtained in the linearized theory with respect to $h_{\mu\nu}$. Separating linear and non–linear parts in the Ricci tensor one has

$$R_{\mu\nu} = \tfrac{1}{2}\eta^{\alpha\beta}(h_{\alpha\beta,\mu\nu} + h_{\mu\nu,\alpha\beta} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) + L'_{\mu\nu}\,, \tag{3.39}$$

$$\begin{aligned} L'_{\mu\nu} =& \tfrac{1}{2}h^{\alpha\beta}(h_{\alpha\beta,\mu\nu} + h_{\mu\nu,\alpha\beta} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) + \\ &+ g^{\alpha\beta}g^{\lambda\delta}(\Gamma_{\lambda\mu\nu}\Gamma_{\delta\alpha\beta} - \Gamma_{\lambda\alpha\mu}\Gamma_{\delta\beta\nu}). \end{aligned} \tag{3.40}$$

Based on (3.9) the field equations may be rewritten in the form

$$R_{\alpha\beta} = -\kappa T^*_{\alpha\beta}\,, \qquad T^*_{\alpha\beta} = T_{\alpha\beta} - \tfrac{1}{2}T g_{\alpha\beta}\,. \tag{3.41}$$

Substituting the Galilean values of the metric tensor into (3.37), ignoring the terms dependent on the internal structure ($p = \Pi = 0$) and retaining only the first–order terms in v/c one gets

$$T^{00} = \rho\,, \qquad T^{0i} = c^{-1}\rho v^i\,, \qquad T^{ij} = 0\,, \qquad T = \rho\,, \tag{3.42}$$

$$T_{00} = \rho\,, \qquad T_{0i} = -c^{-1}\rho v^i\,, \qquad T_{ij} = 0 \tag{3.43}$$

resulting in the right–hand members of (3.41)

$$T^*_{00} = \tfrac{1}{2}\rho\,, \qquad T^*_{0i} = -c^{-1}\rho v^i\,, \qquad T^*_{ij} = \tfrac{1}{2}\rho\delta_{ij}\,. \tag{3.44}$$

Denoting

$$L_{\mu\nu} = L'_{\mu\nu} + \kappa T^*_{\mu\nu} \tag{3.45}$$

the field equations may be rewritten as follows:

$$h_{00,ss} - 2h_{0s,0s} + h_{ss,00} = 2L_{00}\,, \tag{3.46}$$

$$h_{0m,ss} - h_{0s,ms} + h_{ss,0m} - h_{ms,0s} = 2L_{0m}\,, \tag{3.47}$$

$$\begin{aligned} h_{mn,ss} - h_{mn,00} + h_{ss,mn} - h_{ms,ns} - h_{ns,ms} - h_{00,mn} + \\ + h_{0m,0n} + h_{0n,0m} = 2L_{mn}\,. \end{aligned} \tag{3.48}$$

In the linearized theory $L'_{\mu\nu}$ are ignored and the right–hand members of (3.46)–(3.48) become the known functions determined by (3.43). Equations (3.46)–(3.48) represent in this case the second–order linear partial differential equations. They are significantly simplified under harmonic conditions (3.11). Defining $h^{\mu\nu}$ by

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \tag{3.49}$$

one gets

$$\begin{aligned} h^{00} &= -h_{00} + (h_{00})^2 + \ldots\,, \\ h^{0m} &= h_{0m} - h_{00}h_{0m} + h_{0s}h_{ms} + \ldots\,, \\ h^{mn} &= -h^{mn} - h^{ms}h^{ns} + \ldots\,. \end{aligned} \tag{3.50}$$

It will be clear below that in arbitrary non–rotating quasi–Galilean coordinates the components h_{00} and h_{mn} are of the second order in v/c and h_{0m} are of the third order. Expressions (3.50) represent the relations between $h_{\mu\nu}$ and $h^{\mu\nu}$ up to the terms of fifth order inclusively. With the same accuracy

$$-g = 1 + h_{00} - h_{ss} - h_{00}h_{ss} + \tfrac{1}{2}(h_{ss})^2 - \tfrac{1}{2}h_{rs}h_{rs} + \ldots\,, \tag{3.51}$$

$$\begin{aligned} \sqrt{-g} =& 1 + \tfrac{1}{2}(h_{00} - h_{ss}) - \tfrac{1}{8}(h_{00})^2 - \tfrac{1}{4}h_{00}h_{ss} + \\ &+ \tfrac{1}{8}(h_{ss})^2 - \tfrac{1}{4}h_{rs}h_{rs} + \ldots\,, \end{aligned} \tag{3.52}$$

$$\begin{aligned} \sqrt{-g}g^{00} =& 1 - \tfrac{1}{2}(h_{00} + h_{ss}) + \tfrac{3}{8}(h_{00})^2 + \tfrac{1}{4}h_{00}h_{ss} + \\ &+ \tfrac{1}{8}(h_{ss})^2 - \tfrac{1}{4}h_{rs}h_{rs} + \ldots\,, \end{aligned} \tag{3.53}$$

$$\sqrt{-g}g^{0m} = h_{0m} - \tfrac{1}{2}(h_{00} + h_{ss})h_{0m} + h_{0s}h_{ms} + \ldots\,, \tag{3.54}$$

$$\sqrt{-g}g^{mn} = -\,\delta_{mn} - h_{mn} - \tfrac{1}{2}(h_{00} - h_{ss})\delta_{mn} + \\ + [\tfrac{1}{8}(h_{00})^2 + \tfrac{1}{4}h_{00}h_{ss} - \tfrac{1}{8}(h_{ss})^2 + \tfrac{1}{4}h_{rs}h_{rs}]\delta_{mn} - \\ - \tfrac{1}{2}(h_{00} - h_{ss})h_{mn} - h_{ms}h_{ns} + \ldots\,. \tag{3.55}$$

By differentiating these expressions one easily deduces the harmonic conditions up to the terms of fifth order inclusively. The harmonic conditions for the linearized theory are

$$h_{00,0} + h_{ss,0} - 2h_{0s,s} = 0\,, \tag{3.56}$$

$$h_{00,m} - h_{ss,m} + 2h_{ms,s} - 2h_{0m,0} = 0\,. \tag{3.57}$$

Under these conditions the linearized field equations take the form of the wave equations

$$h_{\mu\nu,ss} - h_{\mu\nu,00} = 2L_{\mu\nu}\,. \tag{3.58}$$

As is known the solution of the wave equation

$$\psi_{,ss} - \psi_{,00} = -4\pi\sigma(t, \mathbf{r}) \tag{3.59}$$

is expressed by the volume integral

$$\psi = \int \frac{\sigma(t', \mathbf{r}')}{\mid \mathbf{r} - \mathbf{r}'\mid}\, d^3x'\,, \qquad t' = t - c^{-1} \mid \mathbf{r} - \mathbf{r}'\mid \tag{3.60}$$

with d^3x' being an elementary 3–volume and the density being evaluated at the current point $\mathbf{r}'$ at the retarded moment t'. The exact solution (3.60) is applied now in some high–accuracy applications as, e.g., the light propagation in the field of moving bodies. But in most applications under assumption ($\mid \mathbf{r} - \mathbf{r}'\mid \ll ct$) this exact solution may be expanded in series :

$$\psi = \int \frac{\sigma(t, \mathbf{r}')}{\mid \mathbf{r} - \mathbf{r}'\mid}\, d^3x' - c^{-1}\frac{d}{dt}\int \sigma(t, \mathbf{r}')d^3x' + \tfrac{1}{2}c^{-2}\frac{d^2}{dt^2}\int \sigma(t, \mathbf{r}') \mid \mathbf{r} - \mathbf{r}'\mid d^3x' + \ldots\,. \tag{3.61}$$

In the linearized theory it is sufficient to take only the first term resulting to

$$h_{\mu\nu} = -\frac{\kappa}{2\pi}\int \frac{(T^{*}_{\mu\nu})'}{\mid \mathbf{r} - \mathbf{r}'\mid}\, d^3x' \tag{3.62}$$

with primes indicating that the appropriate value is to be taken for the moment t at the current point $\mathbf{r}'$. The comparison with the Newtonian equations of motion of a test particle results in a determining constant κ

$$\kappa = 8\pi G c^{-2}\,. \tag{3.63}$$

With the aid of (3.44) the solution (3.62) of the linearized field equations in harmonic coordinates is presented in the form

$$h_{00} = -c^{-2}2U\,, \qquad h_{0i} = c^{-3}4U^i\,, \qquad h_{ij} = -c^{-2}2U\delta_{ij} \tag{3.64}$$

with Newtonian potential U

$$U = G \int \frac{\rho'}{\mid \mathbf{r} - \mathbf{r}' \mid} \, d^3x' \tag{3.65}$$

and vector–potential U^i

$$U^i = G \int \frac{\rho' v'^i}{\mid \mathbf{r} - \mathbf{r}' \mid} \, d^3x' \, . \tag{3.66}$$

One may add to it the relation

$$cU_{,0} + U^i_{,i} = 0 \, . \tag{3.67}$$

In all celestial mechanics applications of GRT v^2/c^2 and U/c^2 are small quantities ($\sim 10^{-8}$ for the Earth orbital motion). From (3.64) it follows that h_{00} and h_{ij} are of second order and h_{0i} are of third order of smallness. This is true in harmonic coordinates and in all practically employed quasi–Galilean coordinates. Returning to the linearized equations (3.46)–(3.48) in arbitrary quasi–Galilean coordinates and denoting the solution in harmonic coordinates by an asterisk one obtains the solution in arbitrary coordinates in the form

$$h_{00} = h^*_{00} \, , \qquad h_{0i} = h^*_{0i} + a_{0,i} + a_{i,0} \, , \qquad h_{ij} = h^*_{ij} + a_{i,j} + a_{j,i} \, , \tag{3.68}$$

a_μ being four arbitrary functions of x^0, x^1, x^2, x^3. In dealing with (3.68) a_0 is to be regarded as a third–order function and a_i as second–order functions. In substituting (3.68) into (3.46)–(3.48) all terms with first– and second–order derivatives with respect to x^0 should be rejected as having higher orders than required (differentiating with respect to x^0 increases the order of smallness of the corresponding quantity). Starting with the solution in harmonic coordinates $x^{*\mu}$ expressions (3.68) may be obtained by the coordinate transformation

$$x^{*0} = x^0 + a_0 \, , \qquad x^{*i} = x^i - a_i \, , \qquad a_0 = \mathrm{O}(c^{-3}), \qquad a_i = \mathrm{O}(c^{-2}) \, . \tag{3.69}$$

Indeed, the usual tensor formula

$$g_{\mu\nu}(x^\delta) = \frac{\partial x^{*\alpha}}{\partial x^\mu} \frac{\partial x^{*\beta}}{\partial x^\nu} g^*_{\alpha\beta}(x^{*\delta}) \tag{3.70}$$

gives (3.68) within the adopted accuracy. Finally, the linearized metric in arbitrary quasi-Galilean coordinates reads

$$\begin{aligned} ds^2 =& (1 - c^{-2}2U)c^2dt^2 + 2(c^{-3}4U^i + a_{0,i} + a_{i,0})cdtdx^i + \\ &+ [-(1 + c^{-2}2U)\delta_{ij} + a_{i,j} + a_{j,i}]dx^idx^j \, . \end{aligned} \tag{3.71}$$

With $h_{\mu\nu} = 0$ this metric becomes Galilean (Minkowski) metric of SRT yielding the Newtonian equations of light propagation (motion in a straight line). Retaining in (3.71) only the term $h_{00} = -c^{-2}2U$ and rejecting all $h_{0i} \, , h_{ij}$ one gets the metric giving the Newtonian equations of motion and incorrect approximation for the light propagation equations. The correct post–Newtonian equations of light propagation are obtained with the full metric (3.71). This metric is called weak–field (linearized) GRT metric and it is quite sufficient for present relativistic astrometry based on the post–Newtonian equations of light propagation and post–Newtonian (generalized Lorentz) transformations between different four–dimensional reference systems (including the post–Newtonian theory of the time scales). It is widely believed that to derive the post–Newtonian equations of material bodies underlying present relativistic celestial mechanics it is necessary to include in h_{00} also the non–linear fourth–order term $\underset{4}{h}_{00}$.

Indeed, the post–Newtonian equations of motion of a test particle in a given gravitational field resulting from the geodesic principle (motion on a geodesic line) demand $\underset{4}{h}_{00}$ (e.g., Section 2.2.3 of Brumberg, 1991). The derivation of the equations of motion in the post–Newtonian N–body problem by the PNA (post–Newtonian approximations) techniques also demands this term. This term is explicitly indicated in the IAU (2000) Resolution B1 (IAU, 2001) on reference systems and time scales. Moreover, it is combined with $\underset{2}{h}_{00}$ term by introducing generalized (relativistic) potential instead of 'out-of-date' Newtonian potential.

Presently, relativistic celestial mechanics is far beyond the linearized metric (3.71). Thanks to the PNA techniques one has now in practical disposition the expansions of the metric coefficients for the for the gravitational N–body field as follows:

$$h_{00} = c^{-2}\,\underset{2}{h}_{00} + c^{-4}\,\underset{4}{h}_{00} + c^{-5}\,\underset{5}{A}_{00} + c^{-6}\,\underset{6}{h}_{00} + c^{-7}\,\underset{7}{A}_{00} + \mathrm{O}(c^{-8})\,, \tag{3.72}$$

$$h_{0i} = c^{-3}\,\underset{3}{h}_{0i} + c^{-5}\,\underset{5}{h}_{0i} + c^{-6}\,\underset{6}{A}_{0i} + \mathrm{O}(c^{-7})\,, \tag{3.73}$$

$$h_{ij} = c^{-2}\,\underset{2}{h}_{ij} + c^{-4}\,\underset{4}{h}_{ij} + c^{-5}\,\underset{5}{A}_{ij} + \mathrm{O}(c^{-6})\,. \tag{3.74}$$

The $A_{\mu\nu}$ terms are due to gravitational radiation of the N–body system presenting a qualitative difference from the Newtonian N–body problem (see references in Brumberg, 1991). Such advanced expansion is needed in studying the motion in a strong gravitational field (e.g., the binary pulsar problem). Most present applications related to the solar system deal with the post–Newtonian metric resulting from (3.71) by adding the fourth–order correction term. In harmonic coordinates this term reads (Fock, 1955)

$$\underset{4}{h}_{00} = 2(U^2 - \tilde{U} - c^2\chi_{,00}) \tag{3.75}$$

with

$$\chi_{,ss} = U\,, \tag{3.76}$$

$$\tilde{U}_{,ss} = -4\pi G(\tfrac{3}{2}\rho v^2 - \rho U + \rho\Pi + 3p)\,. \tag{3.77}$$

In converting by (3.69) to arbitrary quasi–Galilean coordinates it is necessary also to transform in (3.70) $g^*_{\mu\nu}(x^{*\alpha})$ to $g^*_{\mu\nu}(x^\alpha)$. For the general case of the gravitational field created by N bodies with coordinates x^s_A and velocity components v^s_A $(A = 1, 2, \ldots, N)$ this transformation implies

$$g^*_{\mu\nu}(x^{*\delta},\, {x^*_A}^s,\, {v^*_A}^s) = g^*_{\mu\nu}(x^\delta,\, {x_A}^s,\, {v_A}^s) + g^*_{\mu\nu,\alpha}(x^{*\alpha} - x^\alpha) + \\ + \sum_A \left(\frac{\partial g^*_{\mu\nu}}{\partial x^s_A}({x^*_A}^s - x^s_A) + \frac{\partial g^*_{\mu\nu}}{\partial v^s_A}({v^*_A}^s - v^s_A) \right) + \ldots\,. \tag{3.78}$$

Denoting (3.75) by $\underset{4}{h}{}^{*}_{00}$ one finds the correction term $\underset{4}{h}{}_{00}$ in arbitrary coordinates

$$c^{-4}\,\underset{4}{h}{}_{00} = c^{-4}\,\underset{4}{h}{}^{*}_{00} + 2a_{0,0} + c^{-2}\,2U_{,s}a_s + c^{-2}\,2\sum_A \frac{\partial U}{\partial x^s_A}(\tilde{a}_s)_A\,, \tag{3.79}$$

$\tilde{a}_s$ being the regular part of a_s after substituting $x^k = x^k_A$. The coordinate conditions with non–zero a_i are used now very seldom. But together with harmonic choice $a_0 = 0$ one may often meet the so called SPN (standard post-Newtonian) gauge with

$$a_0 = c^{-2}\nu\chi_{,0}\,, \qquad \nu = \begin{cases} 0\,, & \text{harmonic gauge} \\ 1\,, & \text{SPN gauge} \end{cases}\,. \tag{3.80}$$

resulting to

$$h_{00} = -c^{-2}\,2U + c^{-4}\,2\Big(U^2 - \tilde{U} + (\nu-1)c^2\chi_{,00}\Big), \tag{3.81}$$

$$h_{0i} = c^{-3}(4U^i + \nu c\chi_{,0i})\,. \tag{3.82}$$

Instead of Newtonian potential U and vector–potential U^i satisfying the equations

$$U_{,ss} = -4\pi G\rho\,, \qquad U^i_{,ss} = -4\pi G\rho v^i \tag{3.83}$$

one often uses now generalized (relativistic) potential w and vector–potential w^i with compact representation

$$h_{00} = -c^{-2}\,2w + c^{-4}\,2w^2\,, \qquad w = U + c^{-2}\Big(\tilde{U} + (1-\nu)c^2\chi_{,00}\Big), \tag{3.84}$$

$$h_{0i} = c^{-3}w^i\,, \qquad w^i = U^i + \tfrac{1}{4}\nu c\chi_{,0i} \tag{3.85}$$

and defining equations

$$w_{,ss} - (1-\nu)w_{,00} = -4\pi G\sigma\,, \qquad \sigma = \rho\left[1 + c^{-2}\left(\tfrac{3}{2}v^2 - U + \Pi + \frac{3p}{\rho}\right)\right], \tag{3.86}$$

$$w^i_{,ss} = -4\pi G\sigma^i\,, \qquad \sigma^i = \rho v^i - \frac{1}{16\pi G}\nu c U_{,0i}. \tag{3.87}$$

In case of harmonic coordinates σ and σ^i may be expressed only in terms of the mass–tensor components. Indeed, from (3.33) and (3.37) it follows

$$\rho^* = \rho[1 - c^{-2}(\tfrac{1}{2}v^2 + 3U) + \ldots], \tag{3.88}$$

and

$$\begin{aligned} T^{00} &= \rho[1 + c^{-2}(\tfrac{1}{2}v^2 - U + \Pi) + \ldots], \\ cT^{0i} &= \rho v^i\left[1 + c^{-2}\left(\tfrac{1}{2}v^2 - U + \Pi + \frac{p}{\rho}\right) + \ldots\right], \\ c^2T^{ij} &= \rho v^i v^j + p\delta_{ij} + \ldots \end{aligned} \tag{3.89}$$

resulting in

$$\sigma = T^{\mu\mu}\,, \qquad \sigma^i = cT^{0i}\,. \tag{3.90}$$

But is should be reminded that the $T^{\mu\nu}$ components depend generally on the metric tensors components $g_{\mu\nu}$ and only by substituting some approximate values of $g_{\mu\nu}$ in the process of iterations they become functions of the matter variables alone. Therefore, the expressions of σ and σ^i in terms of the matter variables may correspond to some step of approximation.

In spite of all these results it may be reminded that as far back as 1957 Infeld (1957) suggested to derive the post–Newtonian equations of the N–body problem from the variational principle for the GRT field equations. This technique turned out to be the most economical one not demanding the use of $\underset{4}{h}_{00}$. In doing so Infeld used the mass tensor with δ–functions. In (Brumberg, 1972) the same technique was applied in combination with the Fock mass tensor for ideal liquid. The possibility to use just the GRT linearized metric for relativistic celestial mechanics simplifies greatly the use of GRT in practical astronomy. Very regretfully, this idea was forgotten (rather frequent case in any science in all times).

Now it is possible to return to (3.38) to find the Lagrangian L of the post–Newtonian N–body problem in harmonic coordinates (or SPN coordinates considering that function a_0 does not affect the post–Newtonian equations of motion). The variational principle (3.38) after multiplication by $c^2/2\kappa$ may be presented in the form

$$\delta \int \Big[\frac{c^4}{16\pi G}\sqrt{-g}J + c^2(1 + c^{-2}\Pi)\rho\frac{ds}{dx^0}\Big]d\Omega = 0\,. \tag{3.91}$$

Since the elementary 4–volume is $d^\Omega = cdt\,d^3x$ it seems reasonable to rewrite this principle

$$\delta \int Ldt = 0 \tag{3.92}$$

to treat L as the Lagrangian of the N–body problem equations (see Infeld and Plebansky, 1960 for mathematical reasoning). Needless to say, this may be done in the post–Newtonian and post–post–Newtonian approximations in (3.72)–(3.74) but not for the radiation approximation when the equations of motion cannot be presented in the Lagrange form due to the presence of the radiative (dissipative) terms. Changing the sign one gets from (3.91) and (3.92)

$$L = -\int \Big[\frac{c^4}{16\pi G}\sqrt{-g}J + c^2(1 + c^{-2}\Pi)\rho\frac{ds}{dx^0}\Big]d^3x\,, \tag{3.93}$$

the integral is to be considered as the sum of integrals taken over the volumes of the bodies.

By using expansions (3.72)–(3.74) in the post–Newtonian approximation and taking into account the values (3.64) one finds (Brumberg, 1972)

$$c^2\,\frac{ds}{dx^0} = c^2 - (\tfrac{1}{2}v^2 + U) + c^{-2}(-\tfrac{1}{8}v^4 - \tfrac{3}{2}v^2U - \tfrac{1}{2}U^2 + 4U^kv^k + \tfrac{1}{2}\underset{4}{h}_{00}) + \dots\,, \tag{3.94}$$

$$\begin{aligned} c^4\sqrt{-g}J =& 2(U_{,i})^2 + c^{-2}[8U(U_{,i})^2 + 6(cU_{,0})^2 + 16\,U_{,i}\,cU^i_{,0} - \\ &- 4(U^i_{,k} - U^k_{,i})^2 - 2U_{,i}\,\underset{4}{h}_{00,i}] + \dots\,. \end{aligned} \tag{3.100}$$

The function $c^2(1 - ds/dx^0)$ represents the Lagrangian of the geodesic equations of a test particle in a given field. It is seen that both expressions (3.94) and (3.95) entering into Lagrangian (3.93) contain explicitly $\underset{4}{h}_{00}$. The last expression may be rewritten as follows:

$$
\begin{aligned}
c^4\sqrt{-g}J = &-2UU_{,ss} + c^{-2}[-4U^2U_{,ss} - 2(U^i_{,i})^2 + 8U^iU^i_{,ss} + 2\,\underset{4}{h}_{00}U_{,ss}]+\\
&+[(2U + 4c^{-2}U^2 - 2c^{-2}\,\underset{4}{h}_{00})U_{,i} + 8c^{-2}(U^iU^k)_{,k} - 8c^{-2}U^kU^k_{,i}]_{,i}+\\
&+16c^{-1}(U^iU_{,i})_{,0} + \dots\,. \qquad (3.96)
\end{aligned}
$$

The terms under the sign of the derivative $\partial/\partial x^i$ vanish upon integrating (3.93). The term representing the time derivative does not affect the equations of motion and may be omitted. Considering the equations for U , U^i and χ one may transform the integrand $(U^i_{,i})^2$ as follows:

$$
\begin{aligned}
&\int (U^i_{,i})^2\,d^3x = -c\int U^i_{,i}\,U_{,0}\,d^3x = c\int U^i\,U_{,0i}\,d^3x = c\int U^i\,\chi_{,0iss}\,d^3x =\\
&= -c\int U^i_{,s}\,\chi_{,0is}\,d^3x = c\int U^i_{,ss}\,\chi_{,oi}\,d^3x = -4\pi Gc\int \rho v^i\chi_{,0i}\,d^3x\,. \qquad (3.97)
\end{aligned}
$$

Therefore, within the post–Newtonian accuracy there results

$$
\begin{aligned}
L = \int \rho\Big\{&[\tfrac{1}{2}v^2 + U + c^{-2}(\tfrac{1}{8}v^4 + \tfrac{3}{2}v^2U + \tfrac{1}{2}U^2 - 4U^iv^i - \tfrac{1}{2}\,\underset{4}{h}_{00})]+\\
&+[-\tfrac{1}{2}U + c^{-2}(-U^2 + 2v^iU^i - \tfrac{1}{2}cv^i\chi_{,0i} + \tfrac{1}{2}\,\underset{4}{h}_{00})]+\\
&+\Pi[-1 + c^{-2}(\tfrac{1}{2}v^2 + U)]\Big\}d^3x\,. \qquad (3.98)
\end{aligned}
$$

As mentioned above the expression in the first square brackets resulting from (3.94) represents the Lagrangian of the geodesic motion equations of a test particle in a given field. The expression in the second square brackets results from (3.96). Adding of these two expressions results in remarkable simplification of the Lagrangian of the post–Newtonian N–body problem, i.e. cancelling $\underset{4}{h}_{00}$ terms. The terms dependent on the internal structure of the bodies may be treated as in (Brumberg, 1972) by using

$$
\rho\Pi = \rho\Pi^* - c^{-2}p\,(\tfrac{1}{2}v^2 + 3U) + \dots\,, \qquad \Pi^* = -\frac{p}{\rho} + \int_0^p \frac{dp}{\rho}\,, \qquad (3.99)
$$

Π^* being the Newtonian value of Π. Taking into account that such internal structure terms result only in re–definition of parameters (e.g., introducing 'effective' masses, etc.) and omitting for the sake of simplicity such terms we have finally

$$
L = \sum_A \int_A \rho\,[\tfrac{1}{2}v^2 + \tfrac{1}{2}U + c^{-2}(\tfrac{1}{8}v^4 + \tfrac{3}{2}v^2U - \tfrac{1}{2}U^2 - 2U^iv^i - \tfrac{1}{2}cv^i\chi_{,0i})]\,d^3x \qquad (3.100)
$$

with summation over all gravitating bodies labelled $A, B, C, \ldots$.

Integration in (3.100) may be performed by standard technique by Fock separating U, U^i, χ into internal and external parts with respect to body A and expanding the external parts in the vicinity of body A (Fock, 1955; Brumberg, 1972). In particular, for the model of point masses there results the well–known Lagrangian of the EIH (Einstein–Infeld–Hoffman) equations

$$L = \sum_A \Bigg\{ \tfrac{1}{2} m_A \mathbf{v}_A^2 + \tfrac{1}{2} G \sum_{B \neq A} \frac{m_A m_B}{r_{AB}} + c^{-2} \Bigg[\tfrac{1}{8} m_A (\mathbf{v}_A^2)^2 + \tfrac{1}{4} G \sum_{B \neq A} \frac{m_A m_B}{r_{AB}} \Big(3\mathbf{v}_A^2 + \\ + 3\mathbf{v}_B^2 - 7\mathbf{v}_a \mathbf{v}_b - (\mathbf{v}_A \mathbf{r}_{AB})(\mathbf{v}_B \mathbf{r}_{AB}) \frac{1}{r_{AB}^2} \Big) - \tfrac{1}{4} G^2 \sum_{B \neq A} \frac{m_A m_B (m_A + m_B)}{r_{AB}^2} - \\ - \tfrac{1}{6} G^2 \sum_{B \neq A} \sum_{C \neq A,B} m_A m_B m_C \Big(\frac{1}{r_{AB} r_{AC}} + \frac{1}{r_{BA} r_{BC}} + \frac{1}{r_{CA} r_{CB}} \Big) \Bigg] \Bigg\} \qquad (3.101)$$

with $\mathbf{r}_{AB} = \mathbf{x}_A - \mathbf{x}_B$, $\mathbf{v}_A = \dot{\mathbf{x}}_A$, $\mathbf{x}_A$ denoting the position vector of body A. Assuming in the EIH equations one of the N masses to be zero one gets the equations of the restricted $N-1$ problem (motion of a test particle in the field of $N-1$ gravitating bodies).

To take into account the non–point structure of the bodies using more complicated expressions for U, U^i and χ it is to be reminded that such body characteristics (rotation velocity, multipole moments, etc.) should be considered in a body reference system. In so doing it is sufficient to use the post–Newtonian theory of reference system transformations in the equations obtained in a global reference system (e.g., Bretagnon and Brumberg, 2003) not demanding $\underset{4}{h_{00}}$. It involves relativistic contributions to the parameters occurring in U, U^i and χ. The post–Newtonian equations of rotation of bodies also may be derived from the variational principle. Now it is possible to improve the first results obtained in this way in (Michalska, 1960a,b).

It is to be noted once again that for high accuracy research (strong gravitational field, high accurate relationship between proper time and coordinate time, etc.) one should know $\underset{4}{h_{00}}$ (and even more advanced terms) in the expansions of the metric coefficients. But the results obtained half a century ago (Infeld, 1957) and reminded here clearly show that for post–Newtonian celestial mechanics and astrometry there is no need in it.

It might seem that this conclusion is in contradiction to the IAU (2000) Resolution B1 (IAU, 2001) on reference systems and time scales claiming the necessity of $\underset{4}{h_{00}}$ term. As stated above this term is relevant for the geodesic principle and PNA techniques implicitly envisaged by this resolution. In this respect it is of interest that the original EIH technique (unsurpassed by its elegance) demands not only $\underset{4}{h_{00}}$ but also $\underset{4}{h_{ij}}$ and $\underset{5}{h_{0i}}$. No one resolution can restrict the search for different techniques to solve problems.

Let us underline that the word 'relativistic' in these lectures means always the synonym of the word 'Einsteinian' involving the use of the general relativity theory by Einstein. However, there exist many alternative theories of gravitation pretending to be called 'relativistic' since they admit pseudo–Riemannian space–time metrics but different from GRT metrics. In the 70th of the last century much attention was given to PPN (Parametrized Post–Newtonian) formalism involving a set of numerical coefficients in the space–time metric to be determined from observations. Until now all PPN coefficients determined from observations turned out in full agreement with the values corresponding to GRT. Two main PPN parameters γ and β (extending formally the Eddington–Robertson metric for the one–body problem considered in Lecture 6) enter into $\underset{2}{h}_{ij}$ and $\underset{4}{h}_{00}$, respectively. But as follows from above discussion the term $\underset{4}{h}_{00}$ is not needed at all in deriving the post–Newtonian equations of motion. If the basic equations of relativistic celestial mechanics were derived from the very beginning without this term then parameter β might not been introduced into equations of motion at all. The GRT variational principle underlines once again the compactness of GRT as compared with alternative theories of gravitation.

In dealing with PPN parameters one should not mix two different approaches. If one works, for example, within the framework of the scalar-tensor theory then parameter β is present in the equations of motion regardless of the way of their derivation, e.g. from the geodesic principle or from the variational principle. The key point is that there is no cancellation of $\underset{4}{h}_{00}$ term in the variational principle for the scalar-tensor theory (this cancellation of $\underset{4}{h}_{00}$ is characteristic for GRT). On the other hand, if the PPN parameter β is introduced just empirically in $\underset{4}{h}_{00}$ as it takes place in most applications of the PPN formalism) the corresponding metric cannot be considered anymore as the GRT metric. No wonder that the equations of motion resulted from the geodesic principle and from the GRT variational principle are different (coinciding only for the GRT value $\beta = 1$). PPN formalism played some positive role in its time stimulating the interest to GRT (in vain hope to reveal some observational discrepancies with it). Now the mixture of GRT and PPN formalism might lead to ambiguities.

Needless to say, all practical GRT tools such as EIH planetary equations, equations of Earth's satellite and lunar motion, Earth's rotation equations, algorithms of reference systems transformations, etc., are well known now but it is of importance that all these tools might be obtained based on a very simple metric. Moreover, as indicated by Infeld, if one is interested in generalizing the EIH equations for the post–post–Newtonian order it may be done with the available values $\underset{4}{h}_{00}$ $\underset{5}{h}_{0i}$ $\underset{4}{h}_{ij}$ for (3.72)–(3.74) (given, for example, in Brumberg, 1972) using the field variational principle. The standard PNA techniques or the geodesic principle (for a test particle) would demand also $\underset{6}{h}_{00}$.

To conclude this lecture let's say several words about GRT and astronomy. Generating of GRT resulted in developing three new branches of astronomy, i.e.

1) relativistic cosmology,

2) relativistic astrophysics, and

3) relativistic celestial mechanics.

There is no doubt that the most significant scientific prediction of GRT was given by the theory of the expanding Universe developed by A.A.Friedman based on the Einstein field equations. The phenomenon of the expanding Universe was discovered from observations in 1929. Nowadays, relativistic cosmology presents an intensively developing branch of astronomy based on GRT, on the one hand, and on vast observational data, on the other hand.

In application to astrophysics GRT enables one to analyze phenomena not compatible within the framework of Newtonian concepts. It is sufficient to give two examples. GRT predicts the existence of qualitatively new objects like black holes. GRT made it possible to calculate the binary pulsar motion (as a problem of relativistic celestial mechanics). Binary pulsars observations confirmed the conclusion of GRT about the energy loss due to gravitational radiation. Although gravitational waves predicted by GRT have not been registered so far the agreement of theoretical and observational results in the binary pulsar problem demonstrates indirectly their existence.

The place of GRT in celestial mechanics is quite peculiar. In contrast to cosmology and astrophysics relativistic celestial mechanics does not deal with so impressive and unusual events. But celestial mechanics has an indisputable merit to have an extraordinary precision of observations unattainable in cosmology and astrophysics. That's why relativistic celestial mechanics and related to it relativistic astrometry are so important in checking the consequences of GRT. Moreover, GRT acts here as a working theory for high accurate dynamical theories of motion and high precision analysis of observations.

Lecture 4. Celestial Mechanics

Aspects of celestial mechanics

In most brief terms celestial mechanics is a science to study the motion of celestial bodies. This laconic and nevertheless very broad definition involves a lot of ambiguities. What is to be meant by celestial bodies? Does this term include both the actually existing physical bodies and model mathematical objects? In case of artificial celestial bodies (satellites, space probes, etc.) do the problems of the guidance motion lie into the scope of celestial mechanics? Celestial mechanics is doubtless one of the most ancient sciences but from the antique times until Newton epoch it managed to describe only the kinematical aspects of motion of celestial bodies (Ptolemeus theory of motion of planets, the Sun and the Moon, Kepler laws). Only since the Newton epoch the dynamical aspects of motion began prevail in celestial mechanics. Actually, celestial mechanics became a science about the motion of the solar system bodies under the Newton gravitation law. In 18th-19th centuries celestial mechanics was advancing with permanent success in developing high-accuracy theories of motion of planets and the Moon. This advance resulted in the triumphal discovery of Neptune based on the analysis of perturbations caused by Neptune in the motion of Uranus. In the end of 19th century Poincar who contributed so much into the development of celestial mechanics formulated its aim as the solution of the question if the Newton gravitation law alone is sufficient to explain all observed motions of celestial bodies.

Starting with the middle of the 20th century celestial mechanics became much more versatile than it was before. It lost the title of theoretical astronomy (historical title when astronomy was restricted only by astrometry and celestial mechanics representing its observational and theoretical parts, respectively) but became related much closer to physics and mathematics. In fact, modern celestial mechanics deals with four interrelated groups of topics, as follows:

1) physics of motion, i.e. investigation of the physical nature of forces affecting the motion of celestial bodies and formulation of a physical model for a specific celestial mechanics problem. The final aim in this domain is to derive the differential equations of motion of celestial bodies and of light propagation. The global physical model underlying contemporary celestial mechanics is the Einstein general relativity theory (GRT). Within present physics Newtonian celestial mechanics is regarded as a completed science since the equations of motion for any Newtonian problem are known and the problem is reduced to the mathematical investigation of these equations;

2) mathematics of motion, i.e. investigation of the mathematical characteristics of the solutions of the differential equations of motion of celestial bodies (various forms of solution representation, asymptotic behaviour, stability, convergence, etc.). Within this domain a problem of celestial mechanics is considered as solved if the general solution form and qualitative picture of motion are known;

3) computation of motion, i.e. the actual determination of the quantitative characteristics of motion. In many natural sciences this problem domain presents no difficulty and is not treated as a separate section. This is not so in celestial mechanics. For instance, if it is known that some problem may be solved in form of a power/trigonometric series then the actual determination of the necessary amount of the terms of such series and their

summation is not a trivial problem when the amount of the terms ranges of many hundred and thousand. Numerical integration of the equations of motion of celestial bodies over a long interval of time is not also a non-trivial problem. Analytical and numerical techniques of celestial mechanics have been permanently improved over all history of celestial mechanics. In its turn it was stimulatory for many branches of mathematics (the theory of special functions, linear algebra, differential equations, theory of approximation including series compression techniques, etc.). The computation tools employed now in celestial mechanics range from *Numerical Recipes* to computer algebra systems like *Mathematica*;

4) astronomy of motion, i.e. application of mathematical solution of a problem to a specific celestial body, comparison with the results of observations, determination of initial values and parameters of motion, and pre-computation of motion for future. By comparing the theoretical (computed) and observational results one may make conclusion about adequacy of physical and mathematical model to the observed picture of motion. If this adequacy is not satisfactory the investigation of the problem is returning to one of the previous steps (improvement of a physical model and mathematical solution).

In the first three items celestial mechanics acts as a fundamental science. The fourth section characterizes celestial mechanics as an applied science although eventually just the results of the fourth section investigations (agreement or disagreement with observations) are crucial for the development of celestial mechanics as a whole. Needless to say, this classification of the philosophy of celestial mechanics is rather conventional.

Keplerian two–body problem

The simplest and at the same time the most practically important problem of celestial mechanics is the problem of motion of two material points mutually attracted in accordance with the Newtonian law of gravitation. Kepler was the first to derive (empirically) the resulted laws of motion. That is why this problem is often called the Keplerian two–body problem.

If m_i and $\mathbf{r}_i$ $(i = 1, 2)$ are masses and position vectors of two bodies (regarded as material points) then according to Newtonian mechanics the equations of motion of these bodies in an arbitrary inertial coordinate system have the form

$$\ddot{\mathbf{r}}_1 = -Gm_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{r^3}, \qquad \ddot{\mathbf{r}}_2 = -Gm_1 \frac{\mathbf{r}_2 - \mathbf{r}_1}{r^3} \tag{4.1}$$

where G is the gravitational constant and $r = |\mathbf{r}_2 - \mathbf{r}_1|$ is the mutual distance between the bodies. Introducing instead of $\mathbf{r}_1$ and $\mathbf{r}_2$ the position vector $\mathbf{r}_0$ of the centre of mass of the bodies

$$\mathbf{r}_0 = \frac{1}{m}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2), \qquad m = m_1 + m_2, \tag{4.2}$$

and the relative position vector of the second body with respect to the first one

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \tag{4.3}$$

one can replace (4.1) by two independent systems

$$\ddot{\mathbf{r}}_0 = 0 \tag{4.4}$$

and

$$\ddot{\mathbf{r}} = -Gm\frac{\mathbf{r}}{r^3}\,. \tag{4.5}$$

Equations (4.4) show that the centre of mass of the two bodies is in the uniform rectilinear motion. A coordinate system in which

$$\mathbf{r}_0 = \dot{\mathbf{r}}_0 = 0 \tag{4.6}$$

is called the barycentric coordinate system. Equations (4.5) describe the relative motion of the two bodies. Just these equations are characteristic for the Keplerian two–body problem.

The trajectories of the two–body problem are the conic sections. In dependence on the type of the conic section one speaks about elliptic, parabolic or hyperbolic type of motion. In the degenerate case one meets also the rectilinear type of motion.

From mathematical point of view the general solution of equations depends on time and six arbitrary constants. One may adopt for these constants the values of the coordinates and velocity components at the initial moment of time but to visualize the general picture of motion it is reasonable to choose for these constants some quantities characterizing the orbit. Such quantities are called orbital elements. The most widespread form of the general solution of the two–body problem is given by the set of relations

$$\mathbf{r} = \mathbf{r}(t;\, a, e, i, \Omega, \omega, M_0), \tag{4.7}$$

and

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}(t;\, a, e, i, \Omega, \omega, M_0), \tag{4.8}$$

expressing the position and velocity vectors in terms of time t and six Keplerian elements a, e, i, Ω, ω and M_0. The first two elements, the semi–major axis a and eccentricity e, characterize the size and the form of the orbit, respectively. Just these two elements determine the type of the motion as exposed below:

type	elliptical	parabolic	hyperbolic	rectilinear
a	> 0	∞	< 0	finite
e	< 1	1	> 1	1

Elements a, e and inclination i are the action elements of the two–body problem. The remaining three quantities are the angular elements:

Ω, the longitude of node
ω, the argument of pericentre
M_0, the mean anomaly at epoch

In addition, other angular quantities of the two–body problem are as follows:

$\pi = \Omega + \omega$, the longitude of pericentre
$\varepsilon = M_0 + \pi$, the mean longitude at epoch
v, true anomaly
$u = v + \omega$, the argument of latitude
g, eccentric anomaly

$M = M_0 + n(t - t_0) = n(t - \tau)$, mean anomaly
τ, the instant of pericentre passage
$\lambda = M + \pi = \varepsilon + n(t - t_0)$, mean longitude
The mean motion n related to a by the third Kepler's law

$$n = \frac{(Gm)^{1/2}}{a^{3/2}}$$

involves the period of motion

$$T = \frac{2\pi}{n}\,.$$

Equations (4.5) admit the first integrals, i.e. area vector integral

$$\mathbf{c} = \mathbf{r} \times \dot{\mathbf{r}}\,, \tag{4.9}$$

Laplace vector integral

$$\mathbf{f} = \left(\dot{\mathbf{r}}^2 - \frac{Gm}{r}\right)\mathbf{r} - (\mathbf{r}\dot{\mathbf{r}})\dot{\mathbf{r}}\,, \tag{4.10}$$

and energy integral

$$\dot{\mathbf{r}}^2 = Gm\left(\frac{2}{r} - \frac{1}{a}\right)\,. \tag{4.11}$$

In virtue of (4.9) one has

$$\mathbf{c}^2 = \mathbf{r}^2\dot{\mathbf{r}}^2 - (\mathbf{r}\dot{\mathbf{r}})^2\,. \tag{4.12}$$

Seven scalar constants of (4.9)–(4.11) satisfy two relationships

$$\mathbf{cf} = 0\,, \qquad Gm\mathbf{c}^2 + a\mathbf{f}^2 = (Gm)^2 a\,. \tag{4.13}$$

One often makes use of the unit vectors related to the two–body problem

$$\mathbf{l} = \begin{pmatrix} \cos\Omega \\ \sin\Omega \\ 0 \end{pmatrix}, \qquad \mathbf{m} = \begin{pmatrix} -\cos i \sin\Omega \\ \cos i \cos\Omega \\ \sin i \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} \sin i \sin\Omega \\ -\sin i \cos\Omega \\ \cos i \end{pmatrix}, \tag{4.14}$$

and

$$\mathbf{P} = \mathbf{l}\cos\omega + \mathbf{m}\sin\omega\,, \qquad \mathbf{Q} = -\mathbf{l}\sin\omega + \mathbf{m}\cos\omega\,. \tag{4.15}$$

In particular,

$$\mathbf{c} = na^2\eta\,\mathbf{k}\,, \qquad \mathbf{f} = n^2a^3e\,\mathbf{P}\,, \qquad \eta = \sqrt{1 - e^2}\,. \tag{4.16}$$

In terms of these quantities the general solution of the two–body problem may be presented in the closed form as follows:

$$\mathbf{r} = r(\mathbf{l}\cos u + \mathbf{m}\sin u) = r(\mathbf{P}\cos v + \mathbf{Q}\sin v) = a[\mathbf{P}(\cos g - e) + \mathbf{Q}\eta\sin g], \tag{4.17}$$

$$\dot{\mathbf{r}} = \frac{na}{\eta}[-\mathbf{l}(\sin u + e\sin\omega) + \mathbf{m}(\cos u + e\cos\omega)] = \frac{na}{\eta}[-\mathbf{P}\sin v + \mathbf{Q}(\cos v + e)] \tag{4.18}$$

with radius–vector r and orbital parameter p

$$r = \frac{p}{1 + e\cos v} = a(1 - e\cos g)\,, \qquad p = a\eta^2, \qquad \tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{g}{2}\,. \tag{4.19}$$

One often meets also the orbital coordinates

$$X = r\cos v = a(cos g - e), \qquad Y = r\sin v = a\eta\sin g\,. \tag{4.20}$$

Relation to time t is given by the differential equations

$$r^2\dot{u} = na^2\eta\,, \qquad \dot{r} = \frac{na}{\eta}e\,\sin v \tag{4.21}$$

resulting in the Kepler's equation

$$g - e\sin g = M\,. \tag{4.22}$$

The coordinates of the two–body problem may be expressed as explicit functions of time by means of the Hansen series (trigonometric series in the mean anomaly or mean longitude with Hansen coefficients dependent only on e):

$$\Phi_{nm} \equiv \left(\frac{r}{a}\right)^n \exp\mathrm{i}\,mv = \sum_{s=-\infty}^{\infty} X_s^{n,m}(e)\exp\mathrm{i}\,sM\,, \tag{4.23}$$

$$\begin{aligned} x + \mathrm{i}\,y =& a\sum_{s=-\infty}^{\infty} X_{s+1}^{1,1}(e)\big\{\cos^2\frac{i}{2}\exp\mathrm{i}[(s+1)\lambda - s\pi] + \\ &+ \sin^2\frac{i}{2}\exp\mathrm{i}[(-s-1)\lambda + s\pi + 2\Omega]\big\}, \end{aligned} \tag{4.24}$$

$$z = a\sin i\sum_{s=-\infty}^{\infty} X_{s+1}^{1,1}(e)\sin[(s+1)\lambda - s\pi - \Omega], \tag{4.25}$$

$$r = a\sum_{s=-\infty}^{\infty} X_s^{1,0}(e)\cos(s\lambda - s\pi). \tag{4.26}$$

In virtue of

$$X_s^{n,m}(e) = X_{-s}^{n,-m}(e) = \mathrm{O}(e^{|m-s|})$$

the terms of the Hansen series satisfy D'Alembert characteristics being of the structure

$$\mathrm{O}\big(e^{|s_2|}(\sin i)^{|s_3|}\big)\exp\mathrm{i}(s_1\lambda + s_2\pi + s_3\Omega)$$

The coordinates of the two–body problem are holomorphic in terms of $e\cos\pi$, $e\sin\pi$, $\sin i\cos\Omega$, $\sin i\sin\Omega$.

For large values of eccentricity the Hansen series are not too effective. Alternative series in true or eccentric anomaly might be more compact:

$$\Phi_{nm} = \sum_{s=-\infty}^{\infty} Y_s^{n,m}(e)\exp\mathrm{i}\,sv\,, \tag{4.27}$$

$$\Phi_{nm} = \sum_{s=-\infty}^{\infty} Z_s^{n,m}(e)\exp\mathrm{i}\,sg\,. \tag{4.28}$$

But the most compressed series for celestial mechanics problems are given by using the fast converging elliptic functions expansions (Brumberg and Brumberg, 1999). Application of these expansions is possible in cases as follows:

If $f(x,y)$ represents some function admitting a Fourier expansion

$$f(x,\, y) = \sum_{m=-\infty}^{\infty} f_m(x) \exp \mathrm{i}\, my\,, \qquad 0 \le x \le 1\,, \quad 0 \le y \le 2\pi \tag{4.29}$$

converging rather slowly for large values of x and if there exists a transformation of variables

$$(x,\, y) \to (k,\, u)\,, \tag{4.30}$$

reducing function $f(x,y)$ to Jacobi elliptic function $g(k,\, u)$ with modulus k and argument u then the corresponding Fourier expansion

$$g(k,\, u) = \sum_{m=-\infty}^{\infty} g_m(q) \exp \mathrm{i}\, mw\,, \qquad w = \frac{\pi u}{2K(k)} + \mathrm{const}\,, \tag{4.31}$$

$$q = \exp\left(-\frac{\pi K'}{K}\right), \qquad K' = K(k'), \qquad k' = \sqrt{1-k^2}\,, \tag{4.32}$$

is generally much more compact than (4.29). $K = K(k)$ stands here for the complete elliptic integral of the first kind. Jacobi nome q remains comparatively small even for large values of k.

Just this case takes place in the two–body problem. It is possible to introduce here elliptic anomaly w by putting

$$k = e\,, \qquad \sin g = -\operatorname{cn} u\,, \quad \cos g = \operatorname{sn} u\,.$$

Then function (4.23) admits the expansion

$$\Phi_{nm} = \sum_{s=-\infty}^{\infty} B_s^{n,m}(q) \exp \mathrm{i}\, sw\,, \qquad w = \frac{\pi u}{2K} - \frac{\pi}{2}\,. \tag{4.33}$$

Expansion (4.33) is much more compact that (4.23), (4.27), or (4.28) (moreover, coefficients of (4.33) may be computed by the closed formulas). A similar anomaly related with elliptic functions may be introduced also in planetary problems with the big ratio a/a' of the semi–major axes (Brumberg, 1995).

Problems with the dominant role of the two–body motion may be presented in the form of the perturbed two–body problem

$$\ddot{\mathbf{r}} = -\frac{Gm}{r^3}\mathbf{r} + \mathbf{F}\,, \tag{4.34}$$

$$F^i = \frac{\partial R}{\partial x^i} - \frac{d}{dt}\frac{\partial R}{\partial \dot{x}^i}\,, \qquad R = R(\mathbf{r},\,\dot{\mathbf{r}},\,t). \tag{4.35}$$

These equations may be rewritten in the Lagrange form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0\,, \qquad L = \tfrac{1}{2}\dot{x}^i\dot{x}^i + U + R\,, \qquad U = \frac{Gm}{r}\,, \qquad r^2 = x^i x^i\,, \tag{4.36}$$

or in the canonical (Hamiltonian) form

$$\dot{x}^i = \frac{\partial H}{\partial p^i}\,, \qquad \dot{p}^i = -\frac{\partial H}{\partial x^i}\,, \tag{4.37}$$

$$H = \tfrac{1}{2}p^i p^i - U - V\,, \qquad V = R + \frac{1}{2}\frac{\partial R}{\partial \dot{x}^i}\frac{\partial R}{\partial \dot{x}^i}\,, \qquad p^i \equiv \frac{\partial L}{\partial \dot{x}^i} = \dot{x}^i + \frac{\partial R}{\partial \dot{x}^i}\,. \tag{4.38}$$

The relationship between the Hamiltonian H and Lagrangian L is given by

$$H \equiv p^i\dot{x}^i - L = \tfrac{1}{2}\dot{x}^i\dot{x}^i + \dot{x}^i\frac{\partial R}{\partial \dot{x}^i} - U - R\,. \tag{4.39}$$

Needless to say, in these formulas we use again the Einstein summation rule.

The perturbed two–body problem may be investigated by means of the vectorial or Keplerian elements. In contrast to the two–body problem these elements are some definite functions of time satisfying differential equations resulted from (4.34). Variations of the vectorial elements read

$$\dot{\mathbf{c}} = \mathbf{r}\times\mathbf{F}\,, \tag{4.40}$$

$$\dot{\mathbf{f}} = 2(\dot{\mathbf{r}}\mathbf{F})\mathbf{r} - (\mathbf{r}\mathbf{F})\dot{\mathbf{r}} - (\mathbf{r}\dot{\mathbf{r}})\mathbf{F} = [\mathbf{F}\times(\mathbf{r}\times\dot{\mathbf{r}})] + [\dot{\mathbf{r}}\times(\mathbf{r}\times\mathbf{F})]. \tag{4.41}$$

In the variations for the Keplerian elements one meets the components of the perturbing forces

$$S = \frac{1}{r}(\mathbf{r}\mathbf{F}), \qquad T = \frac{1}{r}(\mathbf{k}\times\mathbf{r})\mathbf{F}\,, \qquad W = \mathbf{k}\mathbf{F} \tag{4.42}$$

so that

$$T = \mathbf{Q}\mathbf{F}\cos v - \mathbf{P}\mathbf{F}\sin v\,, \qquad S = \mathbf{Q}\mathbf{F}\sin v + \mathbf{P}\mathbf{F}\cos v\,, \tag{4.43}$$

$$\mathbf{r}\mathbf{P} = r\cos v\,, \qquad \mathbf{r}\mathbf{Q} = r\sin v\,, \qquad \dot{\mathbf{r}}\mathbf{P} = -\frac{na}{\eta}\sin v\,, \qquad \dot{\mathbf{r}}\mathbf{Q} = \frac{na}{\eta}(\cos v + e)\,, \tag{4.44}$$

and

$$\dot{\mathbf{f}}\mathbf{P} = na^2\eta[\sin v\,S + (\cos v + \cos g)T], \quad \dot{\mathbf{f}}\mathbf{Q} = na^2\eta\Big[-\cos v\,S + \Big(1 + \frac{r}{p}\Big)\sin v\,T\Big]. \tag{4.45}$$

Equations for the variations of the Keplerian elements result from the relations

$$\dot{\mathbf{c}} = \frac{na}{2\eta}\frac{dp}{dt}\mathbf{k} + na^2\eta\Big(\mathbf{l}\sin i\frac{d\Omega}{dt} - \mathbf{m}\frac{di}{dt}\Big), \tag{4.46}$$

$$\dot{\mathbf{f}} = n^2a^3\frac{de}{dt}\mathbf{P} + n^2a^3e\Big[\Big(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\Big)\mathbf{Q} + \Big(\sin\omega\frac{di}{dt} - \cos\omega\sin i\frac{d\Omega}{dt}\Big)\mathbf{k}\Big], \tag{4.47}$$

by multiplicating them by $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, $\mathbf{P}$, $\mathbf{Q}$, respectively.

Each angular anomaly (true, eccentric, mean or elliptic) should be related to time. By differentiating (4.17) and combining with (4.18) one finds

$$\frac{dv}{dt} = \frac{na^2}{r^2}\eta - \left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right). \tag{4.48}$$

Then

$$\frac{dg}{dt} = \frac{na}{r} - \frac{r}{a\eta}\left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt} + \frac{\sin v}{\eta^2}\frac{de}{dt}\right), \tag{4.49}$$

$$M = M_0 + \int_{t_0}^{t} n dt\,, \qquad \frac{dM}{dt} = n + \frac{dM_0}{dt}\,, \tag{4.50}$$

$$\frac{dM_0}{dt} = -\left(1 + \frac{r}{a\eta^2}\right)\sin g\frac{de}{dt} - \frac{r^2}{a^2\eta}\left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right), \tag{4.51}$$

and finally

$$\begin{aligned}\frac{dw}{dt} = \frac{\pi}{2K\,\mathrm{dn}\,u}\Bigg[&\frac{na}{r} + \frac{1}{\eta^2}\left(\frac{1}{k}Z(u)\,\mathrm{dn}\,u + \frac{r}{a}\,\mathrm{cn}\,u\right)\frac{de}{dt}\\ &- \frac{r}{a\eta}\left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right)\Bigg],\end{aligned} \tag{4.52}$$

$Z(u)$ being Jacobi zeta function , i,e, a periodic part of the elliptic integral of the second kind

$$E(\mathrm{am}\,u, k) = \frac{E}{K}u + Z(u). \tag{4.53}$$

Sometimes it might be useful to have the variation of the radius–vector in terms of any of these anomalies

$$dr = \frac{r}{a}da - a\cos v\,de + \frac{ae}{\eta}\sin v\,dM\,, \tag{4.54}$$

$$dr = \frac{r}{a}da - a\cos g\,de + ae\sin g\,dg\,, \tag{4.55}$$

$$dr = \frac{r}{a}da - \frac{r}{\eta^2}\left[e + (\cos v + e)\frac{r}{a\eta^2}\right]de + \frac{r^2}{a\eta^2}e\sin v\,dv\,, \tag{4.56}$$

$$\begin{aligned}dr = &\frac{r}{a}da + \frac{a}{\eta^2}\,\mathrm{dn}\,u\big(Z(u)\,\mathrm{cn}\,u - \mathrm{sn}\,u\;\mathrm{dn}\,u\big)de\\ &- \frac{2K}{\pi}ak\,\mathrm{cn}\,u\;\mathrm{dn}\,u\,dw\,.\end{aligned} \tag{4.57}$$

Osculating elements permitting to express the coordinates and velocities of the perturbed two–body problem by means of the formulas of the two–body problem satisfy equations

$$\frac{da}{dt} = \frac{2}{n\eta}\left(Se\sin v + T\,\frac{p}{r}\right) = \frac{2}{na}\frac{\partial R}{\partial M_0}\,, \tag{4.58}$$

$$\frac{de}{dt} = \frac{\eta}{na}[S\sin v + T(\cos v + \cos g)] = \frac{\eta^2}{na^2 e}\frac{\partial R}{\partial M_0} - \frac{\eta}{na^2 e}\frac{\partial R}{\partial \omega}\,, \tag{4.59}$$

$$\frac{di}{dt} = \frac{r\cos u}{na^2\eta}W = \frac{\cot i}{na^2\eta}\frac{\partial R}{\partial \omega} - \frac{\csc i}{na^2\eta}\frac{\partial R}{\partial \Omega}\,, \tag{4.60}$$

$$\frac{d\Omega}{dt} = \frac{r\sin u}{na^2\eta\sin i}W = \frac{\csc i}{na^2\eta}\frac{\partial R}{\partial i}\,, \tag{4.61}$$

$$\begin{aligned}\frac{d\omega}{dt} &= -\cos i\frac{d\Omega}{dt} + \frac{\eta}{nae}\left[-S\cos v + T\left(1+\frac{r}{p}\right)\sin v\right] = \\ &= -\frac{\cot i}{na^2\eta}\frac{\partial R}{\partial i} + \frac{\eta}{na^2 e}\frac{\partial R}{\partial e}\,,\end{aligned} \tag{4.62}$$

$$\frac{dM_0}{dt} = -\eta\left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right) - S\frac{2r}{na^2} = -\frac{\eta^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}\,. \tag{4.63}$$

When the right–hand members are expressed in terms of the components S, T, W the corresponding equations are called Gauss equations. In case when the perturbing force $\mathbf{F}$ admits the perturbing function R dependent only on the coordinates and time

$$\mathbf{F} = \frac{\partial R}{\partial \mathbf{r}}\,, \qquad R = R(t,\mathbf{r}) \tag{4.64}$$

the corresponding equations with the right–hand members expressed in terms of the derivatives of R are called Lagrange equations.

One may note therewith useful differential relations

$$\sin u\frac{di}{dt} - \cos u\sin i\frac{d\Omega}{dt} = 0\,, \tag{4.65}$$

$$\frac{d\mathbf{l}}{dt} = (\mathbf{m}\cos i - \mathbf{k}\sin i)\frac{d\Omega}{dt}\,, \tag{4.66}$$

$$\frac{d\mathbf{m}}{dt} = -\mathbf{l}\cos i\frac{d\Omega}{dt} + \mathbf{k}\frac{di}{dt}\,, \tag{4.67}$$

$$\frac{d\mathbf{k}}{dt} = \mathbf{l}\sin i\frac{d\Omega}{dt} - \mathbf{m}\frac{di}{dt}\,, \tag{4.68}$$

$$\frac{d\mathbf{P}}{dt} = \mathbf{Q}\left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right) + \mathbf{k}\left(\sin\omega\frac{di}{dt} - \cos\omega\sin i\frac{d\Omega}{dt}\right)\,, \tag{4.69}$$

$$\frac{d\mathbf{Q}}{dt} = -\mathbf{P}\left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt}\right) + \mathbf{k}\left(\cos\omega \frac{di}{dt} + \sin\omega \sin i \frac{d\Omega}{dt}\right) . \tag{4.70}$$

There results also

$$\dot{\mathbf{P}} \cos v + \dot{\mathbf{Q}} \sin v = \frac{1}{n^2 a^3 e}(-\mathbf{P} \sin v + \mathbf{Q} \cos v)\dot{\mathbf{f}}\,\mathbf{Q} . \tag{4.71}$$

The most typical example of the celestial mechanics perturbations is provided by the three–body problem. The disturbing function of this problem has the form

$$R = Gm'\left(\frac{1}{\Delta} - \frac{\mathbf{r}\mathbf{r}'}{r'^3}\right) , \tag{4.72}$$

$\Delta = \mid \mathbf{r} - \mathbf{r}' \mid$ being the mutual distance between the perturbed and perturbing bodies. The most general expansion of the perturbing function reads

$$R = Gm' \sum A_{qq'ss'j} \cos\theta_{qq'ss'j} \tag{4.72}$$

with angular arguments

$$\begin{aligned} \theta_{qq'ss'j} &= qM + q'M' + s\omega + s'\omega' + j(\Omega - \Omega') + \\ &= q\lambda + q'\lambda' + (s-q)\pi + (s'-q')\pi' + (j-s)\Omega + (-j-s')\Omega' \end{aligned} \tag{4.73}$$

and coefficients

$$A_{qq'ss'j} = \mathrm{O}\left(e^{|q-s|} e'^{|q'-s'|} (\sin\tfrac{i}{2})^{|s-j|} (\sin\tfrac{i'}{2})^{|s'+j|}\right) . \tag{4.74}$$

In dependence on the values of the trigonometric indices the perturbations of the first order are divided in three main groups, i.e. secular terms $\delta\Omega_s$, $\delta\omega_s$, δM_s with

$$q = q' = s = s' = j = 0 ,$$

quasi–secular terms in δe , δi , $\delta\Omega$, $\delta\omega$, δM with

$$q = q' = 0 , \ \mid s \mid + \mid s' \mid + \mid j \mid \neq 0 ,$$

and short–period terms in the variations of all six elements with

$$\mid q \mid + \mid q' \mid \neq 0 .$$

Quasi–secular terms in planetary theories are secular terms only formally. In fact, they result from replacing the long–period terms by their power series expansions. In satellite theories they retain the form of the long–period terms.

In classical planetary theories only mean anomalies (or mean longitudes) are considered as trigonometric variables involving in the process of integration the divisors

$$\nabla_{qq'000} = qn + q'n' \qquad \text{(planetary case)}.$$

In satellite theories, particularly in lunar theory, all trigonometric arguments of (4.72), (4.73) are regarded as functions of time involving in the process of integration the divisors

$$\nabla_{qq'ss'j} = q(n + \delta\dot{M}_s) + q'(n' + \delta\dot{M}'_s) + s\delta\dot{\omega}_s + s'\delta\dot{\omega}'_s + j(\delta\dot{\Omega}_s - \delta\dot{\Omega}'_s) \qquad \text{(lunar case)}.$$

The main difficulties in classical theories are related with small divisors resulted from the quasi–commensurability of the angular frequencies.

As an example of more modern approach to represent the motion of the major planets let's mention GPT, general (long–term) planetary theory (Brumberg, 1995). This theory is based on combining the Von Zeipel's separation of fast and slow variables and Birkhoff's normalization, i.e. reduction to the secular autonomous system. It means that instead of representation of the general solution by series GPT deals with the trigonometric series involving only fast variables (mean longitudes of the planets) and resulting to the secular autonomous system to describe the behaviour of the slow variables (longitudes of perihelia and nodes). This system may be investigated by different techniques specifically adequate for such case. The formulas below are given just for the sake of illustration. If x_i, y_i, z_i, A_i and λ_i $(i = 1, \ldots, N,\ N = 8)$ are heliocentric rectangular coordinates, semi–major axes and mean longitudes of the planets then the transformation

$$x_i + \mathrm{i}\, y_i = A_i(1 - p_i) \exp \mathrm{i}\, \lambda_i\,, \qquad z_i = A_i w_i\,, \tag{4.75}$$

$$\lambda_i = n_i t + \varepsilon_i\,, \qquad n_i^2 A_i^3 = G(m_0 + m_i) \tag{4.76}$$

enables one to deal with the dimensionless variables p_i (complex) and w_i (real) of the order of planetary eccentricities and inclinations, respectively. These variables may be expanded in power series in new variables a_i, $\bar{a}_i$, b_i, $\bar{b}_i$ with coefficients dependent only on the mean longitudes (fast variables)

$$p_i = \sum_{m=0}^{\infty} \sum_{p+q+r+s=m} p^{(i)}_{pqrs}(t) \prod_{j=1}^{N} a_j^{p_j} \bar{a}_j^{q_j} b_j^{r_j} \bar{b}_j^{s_j}\,, \tag{4.77}$$

$$w_i = \sum_{m=1}^{\infty} \sum_{p+q+r+s=m} w^{(i)}_{pqrs}(t) \prod_{j=1}^{N} a_j^{p_j} \bar{a}_j^{q_j} b_j^{r_j} \bar{b}_j^{s_j}\,, \tag{4.78}$$

p, q, r, s being multi–indices of the corresponding scalar indices. Being simply transformed

$$a_i = \alpha_i \exp \mathrm{i}\, \lambda_i\,, \qquad b_i = \beta_i \exp \mathrm{i}\, \lambda_i \tag{4.79}$$

these new variables permit to get a secular autonomous system

$$\begin{aligned} \dot{\alpha} &= \mathrm{i}\mathcal{N}\big[\mathcal{A}\alpha + \Phi(\alpha,\, \bar{\alpha},\, \beta,\, \bar{\beta})\big], \\ \dot{\beta} &= \mathrm{i}\mathcal{N}\big[\mathcal{B}\beta + \Psi(\alpha,\, \bar{\alpha},\, \beta,\, \bar{\beta})\big]. \end{aligned} \tag{4.80}$$

Here $\mathcal{N}$ is the matrix of the mean motions n_i, $\mathcal{A}$ and $\mathcal{B}$ are constant matrices dependent only on semi–major axes and masses of the planets, Φ and Ψ are power series of their arguments. In neglecting Φ and Ψ, and retaining only linear terms with respect to the planetary masses in $\mathcal{A}$ and $\mathcal{B}$ one comes back to the equations of the linear trigonometric theory of secular perturbations by Lagrange and Laplace.

Presently, the most important practical problems of celestial mechanics are as follows:

1) Theories of motion of the major planets and the Moon. The most developed analytical theories of the major planets and the Moon were elaborated in IMCCE (Institut de mécanique celeste et calcul des éphémérides, Paris). These theories are known under the names VSOP87 (Variations Séculaires des Orbites Planétaires) for major planets and ELP2000 (Éphémérides Lunaires Parisiennes) for the Moon. These theories may be found in Internet at www.imcce.fr . They are used in different celestial mechanics problems demanding analytical expressions for planetary–lunar coordinates and velocities. General form of representation of the planetary coordinates in these theories is as follows:

$$x^i(t) = \sum_{\alpha} t^{\alpha} \left[\sum_{k} X^{\alpha}_{ik} \cos(\psi^{\alpha}_k + \nu^{\alpha}_k t) \right]. \tag{4.81}$$

Yet, the space programs related to the computation of the high–accuracy orbits of space probes and the discussion of the high–precision observations make use of the numerical planetary–lunar theories such as DE/LE JPL (Jet Propulsion Laboratory, Pasadena, USA) and EPM IAA (Institute of Applied Astronomy, Saint–Petersburg). These theories are exposed at www.jpl.nasa.gov and www.ipa.nw.ru , respectively. It is to be noted that these theories are not purely Newtonian because they are constructed taking into account the main GRT effects.

2) Earth's rotation theory. The most advanced analytical rigid-body Earth's rotation theory is SMART97 (Solution du Mouvement de l'Axe de Rotation de la Terre), also elaborated in IMCCE basing on VSOP87 theories. This theory has been complemented by corrections due to a more sophisticated Earth's body model and by main GRT terms.

3) Space dynamics. Due to specific character of space flights most space dynamics solutions are given by short–term numerical theories (to precise possible approximate analytical solutions of the designing stage).

4) Uncompleted problems of Newtonian celestial mechanics. In spite of its completeness in opinion of many physicists Newtonian celestial mechanics still involves a lot of uncompleted and interesting problems. First of all, it concerns the problem of evolution of motion in the n–body problem of comparable masses. Even for the case of one dominant mass (the case of the solar system) the representation of a solution valid for long intervals of time still presents an actual problem. In this respect it seems reasonable to investigate further the possibilities of GPT by using the compact elliptic function expansions (Brumberg and Brumberg, 1999, 2001) and to extend GPT to represent both the heliocentric motion of the planets and their axial rotation. The elliptic function expansions might be useful also for the short–term theories of motion. In general, going beyond the material point model the motion of the rigid or liquid bodies in relationship to their axial rotation presents a vast field of investigation. Needless to say, celestial mechanics at present has

lost its former actuality but this is a common destiny of any science having nothing to do with mathematical or astronomical completeness of celestial mechanics. Some celestial mechanics problems still awaiting their solutions (reflecting the point of view of the author) are outlined in (Brumberg, 2008).

Lecture 5. Relativity in Celestial Mechanics and Astrometry

As seen from the previous lecture Newtonian celestial mechanics is characterized by four mutually independent conceptions, i.e.
(1) absolute time (invariance of dt),
(2) absolute space (invariance of $ds^2 = dx^i dx^i$),
(3) linear field theory of potential (linear Laplace and Poisson equations),
(4) Newtonian dynamics (ordinary differential equations of motion).
Special relativity theory (SRT) combines first two items into unique object, i.e. the four–dimensional flat space–time characterized by metric (2.52) in inertial coordinates. However, SRT does not consider gravitation.

General theory of relativity (GRT) simplifies methodological foundations of celestial mechanics combining all four items, i.e. time–space properties from the one hand and motion and distribution of matter from the other hand into the field equations (3.4). These equations determine the metric form (3.1) of the pseudo–Riemannian space–time together with the location and motion of the gravitating masses.

From the purely operational point of view there are two distinctions between NGT (Newton gravitation theory) and GRT:
(1) mathematically: the structure of the field equations and equations of motion,
(2) physically: the way to compare the results of computation and observational data (O−C problem).

The first question is the subject of relativistic celestial mechanics. It involves the solution of the field equations and then the equations of motion of celestial bodies. In some respect it may be regarded as one more branch of the celestial mechanics perturbation theory. The second question is the subject of relativistic astrometry. Only the consistent simultaneous treatment of the both questions leads to the physically meaningful results. The specific feature of the second question is that in contrast to the inertial coordinates of Newtonian mechanics the GRT coordinates have no physical meaning and cannot be considered as the physically measurable quantities. Therefore, the results of the relativistic dynamical theories expressed in terms of the coordinates are not unique, depend on the type of the employed coordinates and cannot be directly confirmed or refuted by observations. Only in terms of the measurable quantities the conclusions of the dynamical theories become unique and may be compared with observations.

There exist three main possible ways to overcome difficulties caused by the intrusion of coordinate dependent quantities into ephemeris astronomy, i.e.
(1) constructing theories only in terms of measurable quantities,
(2) using arbitrary coordinates with indicating the metric employed enabling one to develop an unambiguous procedure to compare measurable and calculated quantities,
(3) using one and the same type of coordinates both in constructing dynamical solution and in discussion of observations.

Considering present applications of these ways one may call them conventionally as physical, mathematical and astronomical approach, respectively. For ephemeris astronomy, first of all, for problems related to reference frames, time scales, astronomical constants, etc., the third way seems to be quite appropriate. But in applying it under the formulations by the IAU (International Astronomical Union) resolutions one should never forget that it presents only one of the possible way of research and even within its own framework there might be different approaches to solve astronomical problems.

The concept of reference frame is often differently used in physics and astronomy leading sometimes to misunderstanding. In astronomy the reference (coordinate) system (RS) is the primary mathematical construction to be given in GRT by a metric form. The reference frame results from the matching of the reference system to some reference astronomical objects ('materialization' of a reference system). In GRT one may use any coordinates in constructing a reference system. But if a coordinate system is not dynamically adequate to the class of problems under consideration then both the solution of the dynamical problems (the subject of relativistic celestial mechanics) and the transformation to the observational data (the subject of relativistic astrometry) will contain a number of extra terms caused only by the inadequate choice of the reference system. These terms cancel out in the expressions for the measurable quantities (time intervals, angular distances, frequency ratios, etc.) and the resulting relativistic effects turn out to be much smaller than the relativistic perturbations in the coordinate solution of the dynamical problems. On the contrary, if the coordinate system is dynamically adequate, then the coordinate solution of the dynamical problems will not contain any large terms of the non–dynamical origin and will insignificantly change in converting to the measurable quantities. These considerations underlying the contemporary theory of relativistic reference systems for astronomy will be explicitly exposed in Lecture 7.

Post–Newtonian equations of motion of a test particle

As seen from Lecture 3 for relativistic celestial mechanics in the post–Newtonian approximation it is sufficient to know only the linearized GRT metric. Nevertheless, if the complete post–Newtonian metric is known (i.e. metric with $\underset{2}{h_{00}}$, $\underset{2}{h_{ij}}$, $\underset{3}{h_{0i}}$ and $\underset{4}{h_{00}}$ then one may apply the geodesic principle to write the post–Newtonian equations of motion of a test particle in a given gravitational field (3.1), (3.2). Moreover, we will start with the exact equations of the geodesic motion

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,. \tag{5.1}$$

The transformation of the independent argument from s to x^0 performed with the derivatives

$$\frac{dx^i}{dx^0} = \frac{dx^i}{ds}\frac{ds}{dx^0}\,, \qquad \frac{d^2x^i}{dx^{0\,2}} = \frac{d^2x^i}{ds^2}\left(\frac{ds}{dx^0}\right)^2 + \frac{dx^i}{ds}\frac{d^2s}{dx^{0\,2}}\,,$$

$$\frac{ds}{dx^0} = \left(\frac{dx^0}{ds}\right)^{-1}\,, \qquad \frac{d^2s}{dx^{0\,2}} = -\frac{d^2x^0}{ds^2}\left(\frac{dx^0}{ds}\right)^{-3}$$

results in the equations

$$\frac{d^2x^i}{dx^{0\,2}} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{dx^0}\frac{dx^\nu}{dx^0} + \Gamma^0_{\mu\nu}\frac{dx^\mu}{dx^0}\frac{dx^\nu}{dx^0}\frac{dx^i}{dx^0} \tag{5.2}$$

or else

$$\begin{aligned}\ddot{x}^i = &- c^2\Gamma^i_{00} - 2c\Gamma^i_{0k}\dot{x}^k + c\Gamma^0_{00}\dot{x}^i - \Gamma^i_{km}\dot{x}^k\dot{x}^m + \\ &+ 2\Gamma^0_{0k}\dot{x}^k\dot{x}^i + c^{-1}\Gamma^0_{km}\dot{x}^k\dot{x}^m\dot{x}^i\,.\end{aligned} \tag{5.3}$$

These equations are exact. If one wants to apply them in the gravitational radiation approximation, i.e. within c^{-5} accuracy, then $\Gamma^\alpha_{\mu\nu}$ should be found within the accuracy as follows:

$$\Gamma^i_{00} \sim c^{-7}\,, \qquad \Gamma^i_{0k}\,,\ \Gamma^0_{00} \sim c^{-6}\,, \qquad \Gamma^i_{km}\,,\ \Gamma^0_{0k} \sim c^{-5}\,, \qquad \Gamma^0_{km} \sim c^{-4}\,.$$

Using the general expression

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta})$$

with

$$g_{\mu\nu,0} = c^{-1}\frac{\partial g_{\mu\nu}}{\partial t}\,, \qquad g_{\mu\nu,i} = \frac{\partial g_{\mu\nu}}{\partial x^i}$$

and the expansions

$$g^{00} = 1 - h_{00} + (h_{00})^2 + \dots\,, \tag{5.4}$$

$$g^{0m} = h_{0m} - h_{00}h_{0m} + h_{0s}h_{ms} + \dots\,, \tag{5.5}$$

$$g^{mn} = -\delta_{mn} - h_{mn} - h_{ms}h_{ns} + \dots \tag{5.6}$$

one finds

$$\Gamma^0_{00} = \tfrac{1}{2}h_{00,0} - \tfrac{1}{2}h_{00}h_{00,0} - \tfrac{1}{2}h_{0s}h_{00,s} + \dots\,, \tag{5.7}$$

$$\Gamma^0_{0k} = \tfrac{1}{2}h_{00,k} - \tfrac{1}{2}h_{00}h_{00,k} + \dots\,, \tag{5.8}$$

$$\Gamma^0_{km} = \tfrac{1}{2}(h_{0k,m} + h_{0m,k} - h_{mk,0}) + \dots \tag{5.9}$$

and

$$\begin{aligned}\Gamma^i_{00} =& \tfrac{1}{2}h_{00,i} + \underline{\tfrac{1}{2}h_{is}h_{00,s}} - h_{0i,0} + \tfrac{1}{2}h_{0i}h_{00,0} - h_{is}h_{0s,0} + \\ &+ \tfrac{1}{2}h_{ir}h_{rs}h_{00,s} + \dots\,,\end{aligned} \tag{5.10}$$

$$\Gamma^i_{0k} = \tfrac{1}{2}(h_{0k,i} - h_{0i,k} - h_{ik,0}) + \tfrac{1}{2}h_{0i}h_{00,k} + \tfrac{1}{2}(h_{0k,s} - h_{0s,k} - h_{ks,0})h_{is} + \dots\,, \tag{5.11}$$

$$\Gamma^i_{km} = \tfrac{1}{2}(h_{km,i} - h_{ik,m} - h_{im,k}) + \tfrac{1}{2}(h_{km,s} - h_{ks,m} - h_{ms,k})h_{is} + \dots,. \tag{5.12}$$

Expressions (5.9)–(5.12) are sufficient for writing the geodesic equations in the gravitational radiation approximation. Here we are interested only in the post–Newtonian equations. Retaining in (5.9)–(5.12) only linear terms and one quadratic (underlined) term we get

$$\begin{aligned}\ddot{x}^i =& -\tfrac{1}{2}c^2 h_{00,i} - \tfrac{1}{2}c^2 h_{is}h_{00,s} + c^2 h_{0i,0} + \tfrac{1}{2}c h_{00,0}\dot{x}^i + \\ &+ c(h_{0i,s} - h_{0s,i} + h_{is,0})\dot{x}^s + h_{00,s}\dot{x}^s\dot{x}^i + \\ &+ (h_{ir,s} - \tfrac{1}{2}h_{rs,i})\dot{x}^r\dot{x}^s + \mathrm{O}(c^{-4})\,.\end{aligned} \tag{5.13}$$

h_{00} in the first term of the right–hand side should be given up to the fourth–order terms inclusive. For all other terms one may use the values from the linearized theory. With the aid of the geodesic variational principle these equations may be put in the Lagrange form with the Lagrangian $L = c^2(1 - ds/dx^0)$. Considering that

$$\left(\frac{ds}{dx^0}\right)^2 = 1 + h_{00} - c^{-2}\dot{x}^s\dot{x}^s + 2c^{-1}h_{0k}\dot{x}^k + c^{-2}h_{rs}\dot{x}^r\dot{x}^s \tag{5.14}$$

one has

$$\begin{aligned} L = \tfrac{1}{2}\dot{x}^s\dot{x}^s - \tfrac{1}{2}c^2h_{00} + \tfrac{1}{8}c^2(h_{00})^2 - ch_{0s}\dot{x}^s - \\ - \tfrac{1}{2}h_{rs}\dot{x}^r\dot{x}^s - \tfrac{1}{4}h_{00}\dot{x}^s\dot{x}^s + \tfrac{1}{8}c^{-2}(\dot{x}^s\dot{x}^s)^2 + \mathrm{O}(c^{-4})\,. \end{aligned} \tag{5.15}$$

In taking the time–derivative the relation

$$\dot{h}_{\mu\nu} = ch_{\mu\nu,0} + h_{\mu\nu,s}\dot{x}^s \tag{5.16}$$

is to be used. Sometimes one makes use of the equations of motion referred to s. From (5.14) it follows

$$\begin{aligned} \frac{dx^0}{ds} = 1 - \frac{1}{2}h_{00} + \frac{1}{2}\frac{dx^k}{ds}\frac{dx^k}{ds} + \frac{3}{8}(h_{00})^2 - \frac{1}{4}h_{00}\frac{dx^k}{ds}\frac{dx^k}{ds} - \\ - \frac{1}{2}h_{km}\frac{dx^k}{ds}\frac{dx^m}{ds} - \frac{1}{8}\left(\frac{dx^k}{ds}\frac{dx^k}{ds}\right)^2 - h_{0k}\frac{dx^k}{ds} + \ldots \end{aligned} \tag{5.17}$$

. There results the post–Newtonian equations of the test particle referred to s

$$\begin{aligned} \frac{d^2x^i}{ds^2} = -\tfrac{1}{2}h_{00,i} + \tfrac{1}{2}h_{00}h_{00,i} - \tfrac{1}{2}h_{ik}h_{00,k} + h_{0i,0} + (h_{0i,k} - h_{0k,i} + h_{ik,0})\frac{dx^k}{ds} - \\ - \tfrac{1}{2}h_{00,i}\frac{dx^k}{ds}\frac{dx^k}{ds} + (h_{ik,m} - \tfrac{1}{2}h_{km,i})\frac{dx^k}{ds}\frac{dx^m}{ds}\,. \end{aligned} \tag{5.18}$$

Again h_{00} is needed here within c^{-4} accuracy.

Solution of equations (5.13) determines the motion of a test particle in some specific coordinate system given by components $g_{\mu\nu}$. For comparison with observations it is necessary then to describe the procedure of observations in the same coordinate system. This aim is achieved with the use of the equations of light propagation. Only afterwards is it possible to exclude the physically meaningless coordinates x^i and to deal with the directly measurable quantities in the proper time of an observer.

Post–Newtonian equations of light propagation

The propagation of light referred to the coordinate time t is described by the equations of isotropic geodesics (1.42), i.e.

$$\ddot{x}^i + \Gamma^i_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\frac{d^2t}{d\lambda^2}\left(\frac{dt}{d\lambda}\right)^{-2}\dot{x}^i\,, \tag{5.19}$$

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \tag{5.20}$$

with λ being the canonical parameter. In more detail, relation (5.20) reads

$$c^2(1+h_{00}) + 2ch_{0k}\dot{x}^k + (-\delta_{km} + h_{km})\dot{x}^k\dot{x}^m = 0\,. \tag{5.21}$$

Differentiating with respect to time and considering that in light propagation $\dot{x}^i = \mathrm{O}(c)$, $\ddot{x}^i = \mathrm{O}(1)$ one obtains

$$2\dot{x}^k\ddot{x}^k = c^3h_{00,0} + c^2h_{00,k}\dot{x}^k + c(2h_{0k,m} + h_{km,0})\dot{x}^k\dot{x}^m + h_{ik,m}\dot{x}^i\dot{x}^k\dot{x}^m + \mathrm{O}(c^{-1})\,. \tag{5.22}$$

Substituting into (5.19) the Christoffel symbols one finds within the same accuracy

$$\begin{aligned}\ddot{x}^i + \tfrac{1}{2}c^2h_{00,i} + c(h_{0k,i} - h_{0i,k}h_{ik,0})\dot{x}^k + (\tfrac{1}{2}h_{km,i} - h_{ik,m})\dot{x}^k\dot{x}^m = \\ = -\frac{d^2t}{d\lambda^2}\left(\frac{dt}{d\lambda}\right)^{-2}\dot{x}^i\,.\end{aligned} \tag{5.23}$$

Considering that for the Newtonian approximation $\dot{x}^k\dot{x}^k = c^2$ and $d^2t/d\lambda^2 = 0$ one gets from (5,22), (5.23)

$$-\frac{d^2t}{d\lambda^2}\left(\frac{dt}{d\lambda}\right)^{-2} = \tfrac{1}{2}ch_{00,0} + h_{00,k}\dot{x}^k + c^{-1}(h_{0k,m} - \tfrac{1}{2}h_{km,0})\dot{x}^k\dot{x}^m \tag{5.24}$$

resulting in combination with (5.23) in the post–Newtonian equations of light propagation

$$\begin{aligned}\ddot{x}^i = -\tfrac{1}{2}c^2h_{00,i} + h_{00,k}\dot{x}^k\dot{x}^i + (h_{ik,m} - \tfrac{1}{2}h_{km,i})\dot{x}^k\dot{x}^m + \tfrac{1}{2}ch_{00,0}\dot{x}^i + \\ + c(h_{0i,k} - h_{0k,i} + h_{ik,0})\dot{x}^k + c^{-1}(h_{0k,m} - \tfrac{1}{2}h_{km,0})\dot{x}^k\dot{x}^m\dot{x}^i + \mathrm{O}(c^{-2}).\end{aligned} \tag{5.25}$$

In the case of the constant field ($h_{\mu\nu,0} = 0$) these equations may be put into the Lagrange form with Lagrangian

$$L = \tfrac{1}{2}\dot{x}^s\dot{x}^s - \tfrac{1}{2}h_{00}\dot{x}^s\dot{x}^s - \tfrac{1}{2}h_{rs}\dot{x}^r\dot{x}^s - \tfrac{1}{2}ch_{0s}\dot{x}^s - \tfrac{1}{2}c^{-1}h_{0s}\dot{x}^s\dot{x}^r\dot{x}^r + \mathrm{O}(c^{-2}). \tag{5.26}$$

Infinitesimal time intervals and distances

As stated above the curvilinearity of the GRT space–time manifests itself differently in different coordinate systems demonstrating that generally time and spatial characteristics (including spatial distances) are coordinate–dependent quantities. The problem of determination of the coordinate–independent quantities from the coordinates x^0, x^1, x^2, x^3 of some given RS will be considered in subsequent lectures. For infinitesimal time intervals and distances the problem is comparatively simple. A set of relations useful in practice is given below.

By putting $dx^1 = dx^2 = dx^3 = 0$ in the expression of ds^2 one finds the proper time of a rest particle (a particle at rest with respect to a given system) or, in other words, the proper time of a RS at a given point

$$d\tau = c^{-1}g_{00}^{1/2}\,dx^0\,. \tag{5.27}$$

Similarly, the proper time of the moving particle (the time of a clock moving in a given system) is determined by

$$d\tau = c^{-1} \left(g_{00} + 2g_{0i}\frac{dx^i}{dx^0} + g_{ik}\frac{dx^i}{dx^0}\frac{dx^k}{dx^0} \right)^{1/2} dx^0 \tag{5.28}$$

generalizing (2.10).

The proper distance between points $A(x^0 + dx^0\,,\ x^i + dx^i)$ and $B(x^0\,,\ x^i)$ is defined as the multiplied by $c/2$ interval of the proper time of a system elapsed between emission a light signal from point A at moment $x^0 + dx^0_{(1)}$ and its back reception in A at moment $x^0 + dx^0_{(2)}$ after its reflection from B at moment x^0. Determining the values $dx^0_k\,,\ k = 1, 2$ from the condition $ds^2 = 0$ of the light propagation one finds the expression for the square of the infinitesimal space distance in GRT (proper distance)

$$dl^2 = \gamma_{ik} dx^i dx^k\,, \qquad \gamma_{ik} = \frac{1}{g_{00}} g_{0i} g_{0k} - g_{ik}\,. \tag{5.29}$$

Since this quadratic form generally changes in time the integral of dl depends on the world line between given space points. Only in a constant field with $g_{\mu\nu}$ independent of x^0 the integral of dl has quite definite sense determining the space distance between points.

The quantities γ_{ik} may be considered as components of the three–dimensional tensor determining the metric dl^2. They may be also used to facilitate operations in transforming from covariant components $g_{\mu\nu}$ to contravariant components $g^{\mu\nu}$. Indeed, if contravariant components γ^{ik} are known then

$$g^{0i} = \frac{1}{g_{00}}\, g_{0k}\gamma_{ik}\,, \qquad g^{ik} = -\gamma^{ik}\,, \qquad g = -g_{00}\gamma \tag{5.30}$$

with $\gamma = \det ||\gamma_{ik}||$.

In SRT time is different for moving clocks. In GRT it is different even for clocks at rest in different space points of one and the same RS. A comparison of clock readings at infinitely close points $A(x^0 + dx^0\,,\ x^i + dx^i)$ and $B(x^0\,,\ x^i)$, i.e. their synchronization, may be again performed by the previously considered light signals (Einstein synchronization). The moment $x^0 + \Delta x^0$ at point A representing the middle reading of clock A between emission and return of the light signal is adopted as being simultaneous with moment x^0 of clock B. There results

$$\Delta x^0 = \tfrac{1}{2}(dx^0_{(1)} + dx^0_{(2)}) = -\frac{g_{0i}}{g_{00}} dx^i\,. \tag{5.31}$$

In a RS with $g_{0i} \neq 0$ Einstein synchronization in the finite domain is impossible. Only under $g_{0i} = 0$ Einstein synchronization is possible in the whole space.

The relation (5.27) defines the proper time for events occurring at the same point of space. The coordinate time interval between the events occurring at different (infinitely close) points is

$$x^0 + dx^0 - (x^0 + \Delta x^0) = \frac{g_{0\mu}}{g_{00}} dx^\mu$$

resulting after multiplication in accordance with (5.27) by $c^{-1}\sqrt{g_{00}}$ in the proper time interval for the events occurring at different space points (proper time of RS)

$$d\tau = c^{-1}(g_{00})^{-1/2} g_{0\mu} dx^{\mu} \,. \tag{5.32}$$

At every point of the GRT space–time one may introduce locally, i.e. in the infinitesimal vicinity of the point, a pseudo–Cartesian RS. This system may be presented by means of a tetrad composed of four orthonormal vectors $\lambda^{\nu}_{(\mu)}$ where the lower index taken in parentheses means the number of the vector ($\mu = 0, 1, 2, 3$). All SRT relations are locally valid in such RS and all tensor operations wuth local tetrad indices are performed with the aid of the Minkowski tensor $\eta_{\mu\nu}$ as, for instance,

$$g_{\mu\nu}\lambda^{\mu}_{(\alpha)}\lambda^{\nu}_{(\beta)} = \eta_{\alpha\beta} \,, \tag{5.33}$$

$$\lambda^{(\beta)}_{\alpha} = \eta^{\beta\mu} g_{\alpha\nu}\lambda^{\nu}_{(\mu)} \,. \tag{5.34}$$

From this it follows that

$$g_{\mu\nu} = \lambda^{(\alpha)}_{\mu}\lambda^{(\beta)}_{\nu}\eta_{\alpha\beta} \tag{5.35}$$

and

$$ds^2 = \eta_{\mu\nu} dx^{(\mu)} dx^{(\nu)} \tag{5.36}$$

with

$$dx^{(\mu)} = \lambda^{(\mu)}_{\nu} dx^{\nu} \,, \qquad dx^{\mu} = \lambda^{\mu}_{(\nu)} dx^{(\nu)} \,. \tag{5.37}$$

The main merit of the tetrad formalism is the local Galilean form (5.36) for the field metric. All quantities being projected onto the tetrad have physical meaning. In particular, $dx^{(0)}$ and $dx^{(i)}$ are local measurable intervals of time and distance respectively. The tetrad formalism is one of the tools of the ‘physical’ approach mentioned above.

To illustrate the tetrad formalism techniques we consider two simple examples. The first example is related to the local 3+1 splitting of any GRT metric

$$ds^2 = c^2 d\tau^2 - dl^2 \tag{5.38}$$

where $d\tau$ and dl^2 are defined by (5.32) and (5.29), respectively. In the case of the weak field one has

$$c d\tau = (1 + \tfrac{1}{2}h_{00} - \tfrac{1}{8}h_{00}^2) c dt + (h_{0i} - \underline{\tfrac{1}{2}h_{00}h_{0i}}) dx^i \,, \tag{5.39}$$

$$\gamma_{ik} = \delta_{ik} - h_{ik} + \underline{h_{0i}h_{0k}} \,. \tag{5.40}$$

Here h_{00} , h_{ij} are $\mathrm{O}(c^{-2})$ quantities. As for h_{0i}, they are generally of the third order (with respect to c^{-1}). But in accelerated moving (for example, rotating) systems they are of the first order. Therefore, in (5.39), (5.40) and in the formulas given below the underlined terms are to be rejected if h_{0i} are of the third order. The tetrad associated with the splitting (5.38) is

$$\lambda^{(0)}_{0} = 1 \,, \ \lambda^{(0)}_{i} = 0 \,, \ \lambda^{(k)}_{0} = 0 \,, \ \lambda^{(k)}_{i} = \delta_{ik} - \tfrac{1}{2}h_{ik} + \underline{\tfrac{1}{2}h_{0i}h_{0k}} \,. \tag{5.41}$$

Using it one may compose the space triad

$$dx^{(i)} = \lambda^{(i)}_k dx^k = dx^i + \tfrac{1}{2}(-h_{ik} + \underline{h_{0i}h_{0k}})dx^k \tag{5.42}$$

enabling one to present the element of space distance in the form

$$dl^2 = \delta_{ik} dx^{(i)} dx^{(k)} \,. \tag{5.43}$$

$d\tau$ and $dx^{(i)}$ may be regarded as physically measurable infinitesimal intervals of time and distance, respectively.

The second example is related with rotation in a weak field determined by metric (3.71). A test particle characterized only by its mass moves in a given gravitational field on the geodesic. If the test particle represents a gyroscope then its axes are subjected to the Fermi–Walker transport described by equations (1.47). Mathematically, a gyroscope represents a particle with spin. The spin vector S^α is orthogonal to the 4–velocity vector of a particle. Therefore, the transport of S^α is described by the equations

$$\frac{DS^\alpha}{ds} = -A^\beta S_\beta u^\alpha \,, \qquad u^\alpha = \frac{dx^\alpha}{ds} \,. \tag{5.44}$$

The 4–acceleration of a particle $A^\beta = Du^\beta/ds$ is zero for the geodesic motion when there are no forces of non–gravitational origin. For a point moving in a given gravitational field the simplest relations may be obtained by using the co–moving tetrad. Its time–like vector coincides with the 4–velocity of the point

$$\lambda^\alpha_{(0)} = u^\alpha + dx^\alpha/ds \tag{5.45}$$

and the triad vectors correspond to coordinate directions. If $v^i = dx^i/dt$ are components of the coordinate 3–velocity of a moving point then

$$\begin{aligned} &\lambda^0_{(0)} = 1 - \tfrac{1}{2}h_{00} + \tfrac{1}{2}\frac{v^2}{c^2} + \ldots \,, \qquad \lambda^i_{(0)} = \frac{v^i}{c} + \ldots \,, \\ &\lambda^0_{(i)} = \frac{v^i}{c} + \ldots \,, \qquad \lambda^i_{(k)} = \delta_{ik} + \tfrac{1}{2}h_{ik} + \tfrac{1}{2}\frac{v^i v^k}{c^2} + \ldots \end{aligned} \tag{5.46}$$

(more accurate expressions for the co–moving tetrad are given in Brumberg, 1991). In the co–moving tetrad

$$u^{(0)} = 1 \,, \qquad u^{(i)} = 0 \,, \qquad A_{(0)} = u^\mu A_\mu = 0$$

and $S_{(0)} = u^\mu S_\mu = 0$ so that finally the equations of spin transport take the form

$$\frac{dS^{(i)}}{dt} = -\Gamma^{(i)}_{(0)(j)} S^{(j)} \,. \tag{5.47}$$

In three–dimensional vector notation

$$\mathbf{S} = (S^{(1)}, S^{(2)}, S^{(3)}), \qquad \Omega = (\Gamma^{(2)}_{(0)(3)}, \Gamma^{(3)}_{(0)(1)}, \Gamma^{(1)}_{(0)(2)})$$

these equations may rewritten as

$$\frac{d\mathbf{S}}{dt} = \Omega \times \mathbf{S}\,. \tag{5.48}$$

Computing the Christoffel symbols in the co–moving system by means of

$$\Gamma^{(\alpha)}_{(\beta)(\gamma)} = \lambda^{(\alpha)}_{\mu}\lambda^{\mu}_{(\gamma),(\beta)} + \lambda^{(\mu)}_{\beta}\lambda^{(\nu)}_{\gamma}\lambda^{(\alpha)}_{\sigma}\Gamma^{(\sigma)}_{(\mu)(\nu)}$$

and using the values (5.7)–(5.12) together with the Newtonian equations of motion

$$v^i_{,0} = -\tfrac{1}{2}ch_{00,i} + c^{-1}A^i$$

one finds (Lightman et al., 1975; Brumberg, 1991)

$$\begin{aligned}\Omega = &-\tfrac{1}{2}c^{-3}(\mathbf{v}\times\mathbf{A}) + \tfrac{3}{2}c^{-3}(\mathbf{v}\times\nabla U) + 2c^{-3}(\nabla\times\mathbf{U}) + \\ &+ \tfrac{1}{2}c^{-1}\mathbf{v}\nabla(\nabla\times\mathbf{a}) + \tfrac{1}{2}c^{-1}\frac{\partial}{\partial t}(\nabla\times\mathbf{a}).\end{aligned} \tag{5.49}$$

The three–dimensional vectors stand here for the triplets

$$\mathbf{A} = (A^i), \qquad \mathbf{U} = (U^i), \qquad \nabla = \left(\frac{\partial}{\partial x^i}\right), \qquad \mathbf{a} = (a_i).$$

Equation (5.48) describes the precession of spin relative to the co–moving system whose axes are assumed to be directed towards fixed distant celestial objects. If the RS associated with the gyroscope is considered as an analogue of the inertial dynamical RS and the co–ming system is treated as an analogue of the inertial kinematic RS then the space rotation of one system with respect to the other is determined by the angular velocity (5.49). The first term in (5.49) corresounds to the Thomas precession of SRT. The second term in (5.49) due to the velocity of the particle at hand is called geodesic precession or de Sitter–Fokker precession. If the vector potential $\mathbf{U}$ is caused by rotation of the central body determining the motion of a particle then the third term in (5.49) is called Lense–Thirring precession. The fourth and the fifth terms in (5.49) give a contribution from arbitrary coordinate functions entering into the weak field metric (3.71) (non–harmonic coordinates influence).

Lecture 6. One–body problem

This lecture deals with the exact solutions of the GRT field equations. Such solutions are applied in relativistic astrophysics and relativistic cosmology to model the processes of evolution of individual objects and the Universe as a whole. In relativistic celestial mechanics such solutions are related to the one–body problem, i.e. the Schwarzschild solution for the spherically symmetrical metric describing the field of a fixed spherical body, the Kerr solution for the axially symmetrical field due to a rotating spherical body, and the Weyl–Levi-Civita solution for the axially symmetric field due to a fixed spheroid. By its application the Schwarzschild solution is the most important one.

Schwarzschild metric

A fixed body of spherical structure produces a spherically symmetric gravitational field with a metric

$$ds^2 = p(r)c^2dt^2 + 2b(r)cdtdr - q(r)dr^2 - a^2(r)d\Omega^2\,, \tag{6.1}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2\,.$$

t is the coordinate time, r, φ, θ are spherical coordinates, p, q, a, b are functions of r to be determined from the field equations. As 'time' t and 'radial distance' r may be chosen arbitrary not violating the field spherical symmetry, two of these functions, a and b for example, may remain arbitrary. In terms of rectangular coordinates

$$x^1 = r\sin\theta\cos\varphi\,, \qquad x^2 = r\sin\theta\sin\varphi\,, \qquad x^3 = r\cos\theta$$

the metric (6.1) takes the form

$$ds^2 = p(r)c^2dt^2 + 2b(r)\frac{x^i}{r}\,cdt\,dx^i - \frac{1}{r^2}\left[a^2(r)\delta_{ik} + \Big(q(r) - \frac{a^2(r)}{r^2}\Big)x^ix^k\right]dx^idx^k\,. \tag{6.2}$$

Transformation

$$cdt^* = cdt + \frac{b(r)}{p(r)}dr \tag{6.3}$$

excludes the mixed term resulting to

$$ds^2 = p(r)c^2dt^{*2} - \left(q(r) + \frac{b^2(r)}{p(r)}\right)dr^2 - a^2(r)d\Omega^2. \tag{6.4}$$

The reduction

$$r^* = f(r) \tag{6.5}$$

to the isotropic form

$$ds^2 = A(r^*)c^2dt^{*2} - B(r^*)(dr^{*2} + r^{*2}d\Omega^2) \tag{6.6}$$

yields

$$A(r^*) = p(r)\,, \qquad B(r^*) = \frac{a^2(r)}{f^2(r)} = \frac{1}{f'^2(r)}\left(q(r) + \frac{b^2(r)}{p(r)}\right), \tag{6.7}$$

a prime denoting differentiation with respect to r. There results

$$f(r) = \exp \int \frac{1}{a(r)} \left(q(r) + \frac{b^2(r)}{p(r)} \right)^{1/2} dr\,. \tag{6.8}$$

In rectangular coordinates the metric (6.6) implies

$$g_{00} = A\,, \qquad g_{0i} = 0\,, \qquad g_{ik} = -B\delta_{ik}\,. \tag{6.9}$$

For the sake of more general applications of the isotropic metric it is reasonable to consider A and B as functions of all four coordinates. Ricci tensor for this case reads

$$R_{00} = -\frac{1}{2B} \left(A_{,ss} - \frac{1}{2A} A_{,s}A_{,s} + \frac{1}{2B} A_{,s}B_{,s} - 3B_{,00} + \frac{3}{2A} A_{,0}B_{,0} + \frac{3}{2B} B_{,0}B_{,0} \right), \tag{6.10}$$

$$R_{0i} = -\frac{1}{2B} \left(-2B_{,0i} + \frac{1}{A} A_{,i}B_{,0} + \frac{2}{B} B_{,i}B_{,0} \right), \tag{6.11}$$

$$\begin{aligned} R_{ik} = -\frac{1}{2B} \Bigg[&-B_{,ik} - \delta_{ik}B_{,ss} + \frac{1}{2A}(A_{,i}B_{,k} + A_{,k}B_{,i} - \delta_{ik}A_{,s}B_{,s} + \delta_{ik}B_{,0}B_{,0}) + \\ &+ \frac{1}{2B}(3B_{,i}B_{,k} + \delta_{ik}B_{,s}B_{,s}) - \frac{B}{A}\left(A_{,ik} - \delta_{ik}B_{,00} + \frac{\delta_{ik}}{2A}A_{,0}B_{,0} - \frac{1}{2A}A_{,i}A_{,k} \right) \Bigg]. \end{aligned} \tag{6.12}$$

Returning to the Schwarzschild problem one has $A_{,0} = B_{,0} = 0$). The field equations (3.10) for the empty space, i.e. for the external relative to the gravitating body space, yield

$$A_{,ss} - \frac{1}{2A} A_{,s}A_{,s} + \frac{1}{2B} A_{,s}B_{,s} = 0\,, \tag{6.13}$$

$$\begin{aligned} B_{,ik} + \delta_{ik}B_{,ss} + \frac{1}{2A}(+\delta_{ik}A_{,s}B_{,s} - A_{,i}B_{,k} - A_{,k}B_{,i}) - \\ - \frac{1}{2B}(3B_{,i}B_{,k} + \delta_{ik}B_{,s}B_{,s}) + \frac{B}{A}\left(A_{,ik} - \frac{1}{2A}A_{,i}A_{,k} \right) = 0\,. \end{aligned} \tag{6.14}$$

The solution of these equations may be found in the form

$$A = \left(\frac{1 - \psi/2}{1 + \psi/2} \right)^2, \qquad B = (1 + \tfrac{1}{2}\psi)^4 \tag{6.15}$$

with the function ψ of the spatial coordinates satisfying the relation

$$\psi\psi_{,ik} - 3\psi_{,i}\psi_{,k} + \delta_{ik}\psi_{,s}\psi_{,s} = 0\,. \tag{6.16}$$

By means of contraction $i = k$ it results to the Laplace equation $\psi_{,ss} = 0$ with the solution

$$\psi = \frac{m}{r^*}\,, \qquad m = \frac{GM}{c^2}\,. \tag{6.17}$$

G is the gravitational constant, M is the mass of the gravitating body, the constant m is chosen to provide the coincidence with the Newtonian limit for h_{00}. There results

$$a(r) = r^* \left(1 + \frac{m}{2r^*}\right)^2, \qquad r^* = \tfrac{1}{2}\{[a^2(r) - 2ma(r)]^{1/2} + a(r) - m\}, \tag{6.18}$$

$$p(r) = 1 - \frac{2m}{a(r)}\,, \qquad q(r) = \frac{a'^2(r) - b^2(r)}{p(r)}\,. \tag{6.19}$$

Depending on the choice of two arbitrary functions $a(r)$ and $b(r)$ one may obtain different coordinate forms of the Schwarzschild metric. The forms (in spherical or rectangular coordinates) most frequently employed are associated with the following six sets of $a(r)$ and $b(r)$:

1. Schwarzschild solution in standard coordinates

$$a(r) = r\,, \qquad b(r) = 0\,,$$

$$ds^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \dfrac{2m}{r}} - r^2 d\Omega^2\,, \tag{6.20}$$

$$ds^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(\delta_{ij} + \frac{2m}{r^3} \frac{x^i x^j}{1 - \dfrac{2m}{r}}\right) dx^i dx^j\,. \tag{6.21}$$

2. Schwarzschild solution in harmonic coordinates

$$a(r) = r + m\,, \qquad b(r) = 0\,,$$

$$ds^2 = \frac{r-m}{r+m}\, c^2 dt^2 - \frac{r+m}{r-m}\, dr^2 - (r+m)^2 d\Omega^2\,, \tag{6.22}$$

$$ds^2 = \frac{r-m}{r+m}\, c^2 dt^2 - \left[\left(1 + \frac{m}{r}\right)^2 \delta_{ij} + \frac{m^2}{r^4} \frac{r+m}{r-m}\, x^i x^j\right] dx^i dx^j\,. \tag{6.23}$$

3. Schwarzschild solution in isotropic coordinates

$$a(r) = r\left(1 + \frac{m}{2r}\right)^2, \qquad b(r) = 0\,,$$

$$ds^2 = \frac{\left(1 - \dfrac{m}{2r}\right)^2}{\left(1 + \dfrac{m}{2r}\right)^2} c^2 dt^2 - \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\Omega^2)\,, \tag{6.24}$$

$$ds^2 = \frac{\left(1-\dfrac{m}{2r}\right)^2}{\left(1+\dfrac{m}{2r}\right)^2}\,c^2dt^2 - \left(1+\frac{m}{2r}\right)^4 \delta_{ij}dx^i dx^j\,. \tag{6.25}$$

4. Schwarzschild solution in Painlevé coordinates

$$a^2(r) = r^2\Big[1-\frac{2m}{a(r)}\Big], \qquad b(r) = 0\,,$$

($a(r)$ being a root of the indicated cubic equation)

$$ds^2 = \Big[1-\frac{2m}{a(r)}\Big]c^2dt^2 - \frac{a'^2(r)}{1-\dfrac{2m}{a(r)}}\,dr^2 - \Big[1-\frac{2m}{a(r)}\Big]r^2d\Omega^2\,, \tag{6.26}$$

$$\begin{aligned} ds^2 =& \Big[1-\frac{2m}{a(r)}\Big]c^2dt^2 - \\ &- \left\{\Big[1-\frac{2m}{a(r)}\Big]\delta_{ij} + \frac{a'^2(r) - \Big[1-\dfrac{2m}{a(r)}\Big]^2}{r^2\Big[1-\dfrac{2m}{a(r)}\Big]}x^i x^j\right\} dx^i dx^j\,. \end{aligned} \tag{6.27}$$

5. Eddington metric

$$a(r) = r\,, \qquad b(r) = \frac{2m}{r}\,,$$

$$ds^2 = \left(1-\frac{2m}{r}\right)c^2dt^2 + \frac{4m}{r}\,cdt\,dr - \left(1+\frac{2m}{r}\right)dr^2 - r^2d\Omega^2\,, \tag{6.28}$$

$$ds^2 = \left(1-\frac{2m}{r}\right)c^2dt^2 + \frac{4m}{r}\,\frac{x^i}{r}\,cdt\,dx^i - \left(\delta_{ij} + \frac{2m}{r^3}x^i x^j\right)dx^i dx^j\,. \tag{6.29}$$

6. Painlevé metric

$$a(r) = r\,, \qquad b(r) = \sqrt{\frac{2m}{r}}\,,$$

$$ds^2 = \left(1-\frac{2m}{r}\right)c^2dt^2 + 2\sqrt{\frac{2m}{r}}\,cdt\,dr - dr^2 - r^2d\Omega^2\,, \tag{6.30}$$

$$ds^2 = \left(1-\frac{2m}{r}\right)c^2dt^2 + 2\sqrt{\frac{2m}{r}}\,\frac{x^i}{r}\,cdt\,dx^i - \delta_{ij}dx^i dx^j\,. \tag{6.31}$$

All these forms are related by a coordinate transformation. For example, if harmonic coordinates are denoted by a tilde then the transformation from the harmonic to arbitrary coordinates is represented by the relations

$$d\tilde{t} = dt + c^{-1}\frac{b(r)}{p(r)}\,dr \tag{6.32}$$

and

$$\tilde{r} = a(r) - m\,, \qquad \tilde{\theta} = \theta\,, \qquad \tilde{\varphi} = \varphi \tag{6.33}$$

or in rectangular coordinates

$$\tilde{x}^i = \frac{a(r) - m}{r}\, x^i\,. \tag{6.34}$$

Forms 5 and 6 introduced respectively by Eddington and Painlevé are examples of stationary metrics for the Schwarzschild problem. In actual celestial mechanics and astrometric applications mainly the static case $b(r) = 0$ of the Schwarzschild problem is used. In what follows only this case is considered. Then there remains in (6.1) or (6.2) only one arbitrary function $a(r)$ satisfying the condition of the Galilean metric at infinity: $a(r)/r \to 1,\ a'(r) \to 1$ with $r \to \infty$.

Solution (6.1) or (6.2) with (6.19) relates to the external Schwarzschild problem, i.e. the determination of the gravitational field outside a fixed spherical body. For cosmology and relativistic astrophysics the internal Schwarzschild problem, i.e. the determination of the gravitational field inside a fixed spherical body, is of no less importance.

The external Schwarzschild problem is valid as long as the component g_{00} is positive. The value of r which vanishes g_{00} is called the gravitational radius (radius of the Schwarzschild sphere inside which the external solution is not valid). This value determined by the equation $p(r) = 0$ is

$$r = \begin{cases} 2m\,, & \text{(in standard coordinates)}, \\ m\,, & \text{(in harmonic coordinates)}, \\ m/2\,, & \text{(in isotropic coordinates)}. \end{cases} \tag{6.35}$$

Motion of a test particle

The variational geodesic principle enables one to derive the exact equations of motion and light propagation in the Schwarzschild problem admitting the rigorous solution in elliptic functions. Taking the plane $\theta = \pi/2$ as a plane of motion one finds the Lagrange function from (6.1)

$$L = -p(r)c^2\Big(\frac{dt}{ds}\Big)^2 + q(r)\Big(\frac{dr}{ds}\Big)^2 + a^2(r)\Big(\frac{d\varphi}{ds}\Big)^2\,, \qquad \theta = \frac{\pi}{2}\,. \tag{6.36}$$

L being explicitly independent of ct, φ and s there exist three first integrals

$$p(r)c\frac{dt}{ds} = E\,, \qquad a^2(r)\frac{d\varphi}{ds} = K\,, \qquad L = I = \begin{cases} 1\,, & \text{material particle}, \\ 0\,, & \text{light particle}. \end{cases} \tag{6.37}$$

Therefore, the motion is described by the system with one degree of freedom with the Lagrangian

$$L = q(r)\Big(\frac{dr}{ds}\Big)^2 + K^2 W(r)\,, \qquad W(r) = \frac{1}{a^2(r)} - \frac{A}{p(r)}\,, \qquad A = (E/K)^2\,. \tag{6.38}$$

The circular solution r =constant is determined by the condition

$$W'(r) = 0 \tag{6.39}$$

resulting to

$$\frac{p^2(r)}{a(r)} = mA\,. \tag{6.40}$$

The stability condition of this solution reads

$$W''(r) \geq 0 \tag{6.41}$$

or explicitly

$$1 - \frac{6m}{a(r)} \geq 0\,. \tag{6.42}$$

In addition, the condition $L = I$ involves the restriction on a radius of the circular motion

$$\frac{1}{a(r)}\left(1 - \frac{3m}{a(r)}\right) = mB\,, \qquad B = \frac{I}{K^2}\,. \tag{6.43}$$

It follows that the smallest radius of the circular motion is determined by $a(r) = 3m$ (with the motion with the light velocity for this orbit) and the radius of the nearest stable circular orbit is determined by $a(r) = 6m$.

For the circular motion

$$\varphi = nt + \text{const}\,, \qquad n^2 a^3(r) = GM\,, \tag{6.44}$$

the mean motion n being related with $a(r)$ by the generalized Kepler third law. In accordance with (6.1) the proper time of a particle moving in a circular orbit is determined by

$$\left(\frac{d\tau}{dt}\right)^2 = p(r) - c^{-2} n^2 a^2(r)\,, \qquad \frac{d\tau}{dt} = \left(1 - \frac{3m}{a(r)}\right)^{1/2}\,. \tag{6.45}$$

The mean motion

$$n' = \frac{d\varphi}{d\tau} \tag{6.46}$$

referred to the proper time is evidently the measurable quantity since the sidereal period of revolution, expressed in the proper time,

$$T' = \frac{2\pi}{n'} \tag{6.47}$$

is directly obtained from astronomical observations. Introducing two auxiliary quantities r'_N and r_N by means of the classic third Kepler's law one has

$$n'^2 {r'_N}^3 = GM\,, \qquad n^2 {r_N}^3 = GM\,, \qquad r'_N = r_N \left(1 - \frac{3m}{r_N}\right)^{1/3}\,. \tag{6.48}$$

The physical constants GM and m are to be considered as the measurable quantities. Since n' is the measurable quantity then r'_N is an indirectly measurable quantity. The same is true for r_N and so for n. The angular coordinate φ in the Schwarzschild problem may be regarded as a measurable quantity as well. Therefore, the coordinate time t on the circular motion may also be treated as an indirectly measurable quantity (being not only one of the four coordinates of the GRT metric but an independent argument of the theory of motion subjected to observations). Only the radius r of the circular orbit determined by the equation

$$a(r) = r_N \tag{6.49}$$

represents a coordinate–dependent, unmeasurable quantity.

The terms 'measurable quantity', 'indirectly measurable quantity', and 'coordinate–dependent (unmeasurable quantity)' being conventional are used here in accordance with their meaning as given in (Brumberg, 1991, Sec. 3.1.2).

Returning to the general case one finds from (6.38) the differential equation of the trajectory

$$\left(\frac{d(1/a(r))}{d\varphi}\right)^2 = A - B + \frac{2mB}{a(r)} - \frac{1}{a^2(r)} + \frac{2m}{a^3(r)}\,. \tag{6.50}$$

This equation with respect to $a(r)$ may be rigorously solved in elliptic functions. For relativistic celestial mechanics it is sufficient to deal with an approximate solution within the post–Newtonian approximation.

Post–Newtonian approximation

For most practically employed quasi–Galilean reference systems one may represent the function $a(r)$ by the expansion

$$a(r) = r\Big(1 + (1-\alpha)\frac{m}{r} + \epsilon\frac{m^2}{r^2} + \ldots\Big), \tag{6.51}$$

α, $\varepsilon, \ldots$, being the coordinate parameters defining specific coordinate conditions, for instance:

$$\begin{array}{lll} \alpha = 1 & \epsilon = 0 & \text{standard coordinates,} \\ \alpha = 0 & \epsilon = 0 & \text{harmonic coordinates,} \\ \alpha = 0 & \epsilon = 1/4 & \text{isotropic coordinates,} \\ \alpha = 2 & \epsilon = -3/2 & \text{Painlevé coordinates.} \end{array}$$

In the post–Newtonian approximation (PNA) one may reject in (6.51) all terms of second and higher order in m/r. Usually in this approximation the metric (6.2) with (6.51) is considered taking into account the main parameters β and γ of the PPN (parametrized post–Newtonian) formalism as follows:

$$\begin{aligned} ds^2 = {} & \left(1 - \frac{2m}{r} + 2(\beta-\alpha)\frac{m^2}{r^2} + \ldots\right)c^2dt^2 - \\ & - \left[\delta_{ij} + \frac{2m}{r}\left((\gamma-\alpha)\delta_{ij} + \alpha\frac{x^ix^j}{r^2}\right) + \ldots\right]dx^idx^j\,. \end{aligned} \tag{6.52}$$

In dealing with (6.52) one should always remember two important matters. First of all, this metric represents the solution of the GRT field equations only for $\beta = \gamma = 1$. For β and γ different from 1 this metric is not a solution of these equations (for specific values of β and γ it may be a solution of the field equations of some alternative theory of gravitation). Not all GRT results and techniques are valid for this metric. This metric (called Eddington–Robertson metric in case of $\alpha = 0$) was introduced as an empiric generalization of GRT. By formal application of the geodesic principle one may derive the equations of motion of the test particle and light propagation and to determine β and γ from observations. Until now there is no evidence of violation of the GRT condition $\beta = \gamma = 1$. There are no objective reasons to introduce β, γ (and other PPN parameters). They are retained somewhere below just for the sake of comparison with other papers. It is tacitly always assumed that $\beta = \gamma = 1$. Then, it should be remembered from Lecture 3 that the metric (6.52) is superfluous for the post–Newtonian treatment of motion of a test particle. Indeed, by applying the Lagrangian (3.101) for the two–body problem in the post–Newtonian approximation of GRT and vanishing one of the masses one gets the post–Newtonian equations of a test particle in the Schwarzschild field just from the linearized metric, i.e. from the metric (6.52) ignoring the second–order term in g_{00} (involving parameter β).

Starting with (6.52) and taking the coordinate time t as an independent argument the Lagrangian of the equations of motion of the test particle reads

$$L = \tfrac{1}{2}\dot{\mathbf{r}}^2 + \frac{GM}{r} + \tfrac{1}{8}c^{-2}(\dot{\mathbf{r}}^2)^2 + \frac{m}{r}\Big[\Big(\tfrac{1}{2} + \gamma - \alpha\Big)\dot{\mathbf{r}}^2 + \Big(\tfrac{1}{2} - \beta + \alpha\Big)\frac{GM}{r} + \alpha\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\Big], \quad (6.53)$$

$\mathbf{r}$ denoting the triplet of coordinates x^i. This Lagrangian results in the equations of motion

$$\ddot{\mathbf{r}} + \frac{GM}{r^3}\mathbf{r} = \frac{m}{r^3}\Big[\Big(2(\beta+\gamma-\alpha)\frac{GM}{r} - (\gamma+\alpha)\dot{\mathbf{r}}^2 + 3\alpha\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\Big)\mathbf{r} + 2(1+\gamma-\alpha)(\mathbf{r}\dot{\mathbf{r}})\dot{\mathbf{r}}\Big]. \quad (6.54)$$

Analytical solution of these equations may be easily obtained by introducing polar coordinates r and u in the plane of motion

$$\mathbf{r} = X\mathbf{l} + Y\mathbf{m}\,, \qquad \dot{\mathbf{r}} = \dot{X}\mathbf{l} + \dot{Y}\mathbf{m}\,, \qquad X = r\cos u\,, \qquad Y = r\sin u\,, \quad (6.55)$$

$\mathbf{l}$ and $\mathbf{m}$ being unit vectors (4.14) determining the orientation of the plane of motion. There results

$$\ddot{r} - r\dot{u}^2 + \frac{GM}{r^2} = m\Big(2(\beta+\gamma-\alpha)\frac{GM}{r^3} - (\gamma+\alpha)\dot{u}^2 + (\gamma+2)\frac{\dot{r}^2}{r^2}\Big), \quad (6.56)$$

$$\frac{d}{dt}(r^2\dot{u}) = 2m(\gamma+1-\alpha)\dot{r}\dot{u}\,. \quad (6.57)$$

The particular circular solution of these equations reads

$$r = a\,, \qquad u = nt + \mathrm{const}\,, \qquad n = \Big(\frac{GM}{a^3}\Big)^{1/2}\Big[1 + \frac{m}{a}(-\tfrac{1}{2}\gamma - \beta + \tfrac{3}{2}\alpha)\Big]. \quad (6.58)$$

The general solution may be expressed as follows:

$$r = \frac{a(1-e^2)}{1+e\cos f}\,, \qquad f = \psi - \alpha\frac{m}{a(1-e^2)}\,e\sin\psi\,, \qquad \psi = \nu(u-\omega), \quad (6.59)$$

$$\nu = 1 - (2\gamma - \beta + 2)\frac{m}{a(1-e^2)}\,. \tag{6.60}$$

In the practically most important case of the quasi–elliptic motion the true anomaly changes from $f = 0$ to $f = 2\pi$ while the argument of latitude changes from $u = \omega$ to $u = \omega + 2\pi/\nu = \omega + 2\pi + \Delta\omega$ with the relativistic advancement of the pericentre

$$\Delta\omega = (2\gamma - \beta + 2)\frac{2\pi m}{a(1-e^2)} \tag{6.61}$$

reducing for $\beta = \gamma = 1$ to the famous formula of the Schwarzschild advancement of the pericentre for one revolution. The relation of the true (f), eccentric (E) and mean (l) anomaly with time is given by

$$n dt = \frac{r^2}{a^2(1-e^2)^{1/2}}\left[1 + \frac{m}{a}\Big(-\beta + \alpha + (2\gamma + 2 - \alpha)\frac{a}{r}\Big)\right]df\,, \tag{6.62}$$

$$E - \Big(1 + (-2\gamma - 2 + \alpha)\frac{m}{a}\Big)e\sin E = l\,, \tag{6.63}$$

and

$$\frac{dl}{dt} = n\Big(1 - (2\gamma + 2 - \beta)\frac{m}{a}\Big). \tag{6.64}$$

The relationship between the proper (τ) and coordinate (t) time of the moving particle is given by

$$\frac{d\tau}{dt} = 1 - \frac{2m}{r} + \frac{m}{2a} + \ldots \tag{6.65}$$

so that

$$n' d\tau = \frac{r^2}{a^2(1-e^2)^{1/2}}\left[1 + \frac{m}{a}\Big(2 - \beta + \alpha + (2\gamma - \alpha)\frac{a}{r}\Big)\right]df \tag{6.66}$$

with the mean motion n' for the proper time of the particle

$$n' = \left(\frac{GM}{a^3}\right)^{1/2}\left[1 + \frac{m}{a}(-\tfrac{1}{2}\gamma - \beta + \tfrac{3}{2}\alpha + \tfrac{3}{2})\right]. \tag{6.67}$$

The Kepler equation referred to the proper time is

$$E - \Big(1 + (-2\gamma + \alpha)\frac{m}{a}\Big)e\sin E = l'\,,$$

l' being the linear function of τ with the frequency

$$\frac{dl'}{d\tau} = n'\Big(1 - (2\gamma + 2 - \beta)\frac{m}{a}\Big). \tag{6.68}$$

The anomalistic period $\mathcal{T}_1$ defined as the proper time interval of the increase of f or E by 2π is

$$\mathcal{T}_1 = \frac{2\pi}{n'}\Big(1 + (2\gamma + 2 - \beta)\frac{m}{a}\Big). \tag{6.69}$$

The sidereal period $\mathcal{T}_2'$ defined as the proper time interval of the increase of u by 2π is

$$\mathcal{T}_2' = \frac{2\pi}{n'}\left[1 + (2\gamma + 2 - \beta)\frac{m}{a}\left(1 - \frac{(1-e^2)^{1/2}}{(1+e\cos f_0)^2}\right)\right], \tag{6.70}$$

f_0 being the initial value of the true anomaly. The mean value of the sidereal period in changing the initial position f_0 from 0 to 2π is determined by

$$\mathcal{T}_2 = \frac{2\pi}{n'}\left(1 - (2\gamma + 2 - \beta)\frac{m}{a}\frac{e^2}{1-e^2}\right). \tag{6.71}$$

Replacing n' by n results in the similar periods T_1, T_2', T_2 referred to the coordinate tome t.

For small eccentricity it is convenient to use expansions in powers of the eccentricity. These trigonometric expansions in terms of l and l/ν enable one to represent the coordinates of the particle as explicit functions of time t. The initial terms $\mathrm{O}(e^2)$ read

$$\frac{r}{a} = 1 + \tfrac{1}{2}\Big(1 + (-2\gamma - 2 + \alpha)\frac{m}{a}\Big)e^2 - e\cos l - \tfrac{1}{2}\Big(1 + (-2\gamma - 2 + \alpha)\frac{m}{a}\Big)e^2\cos 2l + \dots, \tag{6.72}$$

$$u = \omega + \frac{l}{\nu} + 2\Big(1 + (\gamma + 1 - \beta + \alpha)\frac{m}{a}\Big)e\sin l + \left[\tfrac{5}{4} + \Big(-\tfrac{1}{2}\gamma - \tfrac{1}{2} - \tfrac{5}{4}\beta + \tfrac{5}{2}\alpha\Big)\frac{m}{a}\right]e^2\sin 2l + \dots. \tag{6.73}$$

Let's introduce two new constants a^*, e^* and and two new trigonometric variables λ, π as follows:

$$\dot\lambda = n^* = \Big(1 + (2\gamma + 2 - \beta)\frac{m}{a}e^2\Big)n\,, \qquad \dot\pi = (2\gamma + 2 - \beta)\frac{m}{a(1-e^2)}n\,, \tag{6.74}$$

$$l = \lambda - \pi\,, \qquad \lambda = \omega + \Omega + l/\nu\,, \tag{6.75}$$

$$a^* = a\Big(1 + \tfrac{2}{3}(-2\gamma - 2 + \beta)\frac{m}{a}e^2\Big), \qquad e^* = \Big(1 + (\gamma + 1 - \beta + \alpha)\frac{m}{a}\Big)e\,. \tag{6.76}$$

It is to be noted that a^* is related to n^* just as a is related to n and e^* is one half of the leading trigonometric term in u. There results

$$\begin{aligned}\frac{r}{a^*} =& 1 + \tfrac{1}{2}\left[1 + \Big(-\tfrac{4}{3}\gamma - \tfrac{4}{3} + \tfrac{2}{3}\beta - \alpha\Big)\frac{m}{a^*}\right]e^{*2} - \Big(1 + (-\gamma - 1 + \beta - \alpha)\frac{m}{a^*}\Big)e^*\cos(\lambda - \pi) - \\ &- \tfrac{1}{2}\Big(1 + (-4\gamma - 4 + 2\beta - \alpha)\frac{m}{a^*}\Big)e^{*2}\cos 2(\lambda - \pi) + \dots,\end{aligned} \tag{6.77}$$

$$u = \lambda - \Omega + 2e^*\sin(\lambda - \pi) + \left[\tfrac{5}{4} + \Big(-3\gamma - 3 + \tfrac{5}{4}\beta\Big)\frac{m}{a^*}\right]e^{*2}\sin 2(\lambda - \pi) + \dots. \tag{6.78}$$

Comparing (6.73) and (6.78) and noting the disappearance of the coordinate parameter α in (6.78) it is seen that e as a parameter to characterize the form of the orbit is an unmeasurable quantity, and e^* as a half of a coefficient in $\sin(\lambda - \pi)$ in u is a measurable quantity.

Solution in osculating elements

There are many other ways to investigate the motion of a test particle in the Schwarzschild field. Within the classical celestial mechanics set of techniques the most traditional way is to present the solution in osculating elements. To extend the domain of application of the relevant results the equations of the perturbed two–body problem are considered in the form

$$\ddot{\mathbf{r}} + \frac{GM}{r^3}\mathbf{r} = \mathbf{F}\,, \qquad \mathbf{F} = m\Big[\Big(2\sigma\frac{GM}{r} - 2\epsilon\dot{\mathbf{r}}^2 + 3\alpha\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\Big)\frac{\mathbf{r}}{r^3} + 2\mu\frac{(\mathbf{r}\dot{\mathbf{r}})}{r^3}\dot{\mathbf{r}}\Big] \tag{6.79}$$

with particular values values

$$\sigma = \gamma + \beta - \alpha\,, \qquad 2\epsilon = \gamma + \alpha\,, \qquad \mu = \gamma + 1 - \alpha \tag{6.80}$$

for the Schwarzschild problem equations (6.54). Components of the perturbing acceleration (4.42) read

$$S = \frac{m}{r^2}\Big[2\sigma\frac{GM}{r} - 2\epsilon\dot{\mathbf{r}}^2 + (3\alpha + 2\mu)\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\Big],\; T = m\,2\mu\frac{\sqrt{GM}}{r^4}\,\sqrt{a}(1-e^2)^{1/2}(\mathbf{r}\dot{\mathbf{r}}),\; W = 0\,. \tag{6.81}$$

The Gauss equations (4.58)–(4/63) result in the first–order perturbations

$$\begin{aligned}\delta a =& \frac{me}{(1-e^2)^2}\{[4(\epsilon - \mu - \sigma) + e^2(-\tfrac{9}{2}\alpha + 4\epsilon - 4\mu)]\cos f + \\ &+ (2\epsilon - 2\mu - \sigma)e\cos 2f + \tfrac{1}{2}\alpha e^2\cos 3f\}|_{t_0}^{t}\,,\end{aligned} \tag{6.82}$$

$$\begin{aligned}\delta e =& \frac{m}{a(1-e^2)}\{[2(\epsilon - \sigma) + e^2(-\tfrac{9}{4}\alpha + 2\epsilon - 4\mu)]\cos f + \\ &+ (\epsilon - \mu - \tfrac{1}{2}\sigma)e\cos 2f + \tfrac{1}{4}\alpha e^2\cos 3f\}|_{t_0}^{t}\,,\end{aligned} \tag{6.83}$$

$$\begin{aligned}\delta\pi =& \frac{m}{a(1-e^2)}\Big\{(2\epsilon + 2\mu - \sigma)f + \frac{1}{e}[2(\epsilon - \sigma) + e^2(-\tfrac{3}{4}\alpha + 2\epsilon)]\sin f + \\ &+ (\epsilon - \mu - \tfrac{1}{2}\sigma)\sin 2f + \tfrac{1}{4}\alpha e\sin 3f\Big\}\Big|_{t_0}^{t}\,,\end{aligned} \tag{6.84}$$

$$\begin{aligned}\delta\epsilon =& [1 - (1-e^2)^{1/2}]\delta\pi + \frac{2m}{a(1-e^2)^{1/2}}[(3\alpha - 2\epsilon + 2\mu)(1-e^2)^{1/2}E + \\ &+ (-3\alpha + 4\epsilon - 2\mu - 2\sigma)f + (3\alpha + 2\mu)e\sin f]|_{t_0}^{t}\,,\end{aligned} \tag{6.85}$$

$$i = \text{const}\,, \qquad \Omega = \text{const}\,, \tag{6.86}$$

$$\Delta\lambda = n(t - t_0) + \delta\lambda\,, \qquad \delta\lambda = \int_{t_0}^{t}\delta n dt + \delta\epsilon\,, \qquad \delta n = -\frac{3}{2}\,\frac{n}{a}\,\delta a\,, \tag{6.87}$$

$$\int_{t_0}^{t} \delta n dt = 3\frac{m}{a}\Big\{(-3\alpha+2\epsilon-2\mu)E+(2\alpha-2\epsilon+2\mu+\sigma)\frac{f}{(1-e^2)^{1/2}}-\alpha\frac{e\sin f}{(1-e^2)^{1/2}}+$$
$$+\Big[a(1-e^2)\Big(\frac{a}{r_0}\Big)^3+(-3\alpha+2\epsilon-2\mu-\sigma)\Big(\frac{a}{r_0}\Big)^2+(3\alpha-2\epsilon+2\mu)\frac{a}{r_0}\Big]l\Big\}\Big|_{t_0}^{t}\,. \quad (6.88)$$

In particular, one gets again the secular advancement of the longitude of pericentre per one revolution

$$\delta\pi = \frac{2\pi m}{a(1-e^2)}(2\epsilon+2\mu-\sigma) = \frac{2\pi m}{a(1-e^2)}(2\gamma+2-\beta)_{(\mathrm{PPN})} = \frac{6\pi m}{a(1-e^2)}_{(\mathrm{GRT})}\,. \quad (6.89)$$

Light propagation in the Schwarzschild field in the post–Newtonian approximation

The Lagrangian of the post–Newtonian equations of the light propagation for metric (6.52) reads

$$L = \tfrac{1}{2}\dot{\mathbf{r}}^2 + \frac{m}{r}\Big[(\gamma+1-\alpha)\dot{\mathbf{r}}^2+\alpha\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\Big] \quad (6.90)$$

resulting to the equations

$$\ddot{\mathbf{r}} = \frac{m}{r^3}\Big[-(\gamma+1+\alpha)(\dot{\mathbf{r}}^2)\mathbf{r}+3\alpha\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\mathbf{r}+2(\gamma+1-\alpha)(\mathbf{r}\dot{\mathbf{r}})\dot{\mathbf{r}}\Big]. \quad (6.91)$$

General solution of these equations under conditions $\mathbf{r}(t_0)=\mathbf{r}_0\,,\ \boldsymbol{\sigma}=\dot{\mathbf{r}}(-\infty)/c\,,\ \boldsymbol{\sigma}^2=1$ (Cauchy problem) is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + c(t-t_0)\boldsymbol{\sigma} + m\Big[(\gamma+1)\Big(\frac{\boldsymbol{\sigma}\times(\mathbf{r}_0\times\boldsymbol{\sigma})}{r_0-\boldsymbol{\sigma}\mathbf{r}_0}-\frac{\boldsymbol{\sigma}\times(\mathbf{r}\times\boldsymbol{\sigma})}{r-\boldsymbol{\sigma}\mathbf{r}}-$$
$$-\boldsymbol{\sigma}\ln\frac{r+\boldsymbol{\sigma}\mathbf{r}}{r_0+\boldsymbol{\sigma}\mathbf{r}_0}\Big)+\alpha\Big(\frac{\mathbf{r}}{r}-\frac{\mathbf{r}_0}{r_0}\Big)\Big], \quad (6.92)$$

$$\frac{\dot{\mathbf{r}}}{c} = \boldsymbol{\sigma} - \frac{m}{r}\Big[(\gamma+1)\frac{\boldsymbol{\sigma}\times(\mathbf{r}\times\boldsymbol{\sigma})}{r-\boldsymbol{\sigma}\mathbf{r}}+(\gamma+1-\alpha)\boldsymbol{\sigma}+\alpha\frac{(\boldsymbol{\sigma}\mathbf{r})}{r^2}\mathbf{r}\Big]. \quad (6.93)$$

Needless to say, $\mathbf{r}$ should be replaced by its Newtonian expression $\mathbf{r}_N(t)=\mathbf{r}_0+c(t-t_0)\boldsymbol{\sigma}$ in the right–hand sides. As seen from (6.93) the magnitude of the coordinate light velocity is

$$\frac{|\dot{\mathbf{r}}(t)|}{c} = 1 - \frac{m}{r}\Big(\gamma+1-\alpha\frac{|\mathbf{r}\times\boldsymbol{\sigma}|^2}{r^2}\Big). \quad (6.94)$$

In the limit $t\to\infty$ the equation (6.93) yields

$$\frac{\dot{\mathbf{r}}(\infty)}{c} \equiv \boldsymbol{\nu} = \boldsymbol{\sigma} - 2m(\gamma+1)\frac{\boldsymbol{\sigma}\times(\mathbf{r}_0\times\boldsymbol{\sigma})}{|\mathbf{r}_0\times\boldsymbol{\sigma}|^2}\,. \quad (6.95)$$

The sine of the angle of the total deflection of light is

$$|\boldsymbol{\nu}\times\boldsymbol{\sigma}| = 2(\gamma+1)\,\frac{m}{|\mathbf{r}_0\times\boldsymbol{\sigma}|} \quad (6.96)$$

resulting to the value $1.75''$ for the solar limb. Another characteristic of the light propagation is the impact parameter

$$|\mathbf{r}_N \times \boldsymbol{\sigma}| = |\mathbf{r}_0 \times \boldsymbol{\sigma}| = d\,.$$

For particular case of the radial (solar) ray the expressions (6.92), (6.93) take the form

$$\mathbf{r}(t) = \mathbf{r}_0 + c(t - t_0)\frac{\mathbf{r}_0}{r_0} + m(\gamma + 1)\frac{\mathbf{r}_0}{r_0}\ln\frac{r_0}{r}\,, \tag{6.97}$$

$$\frac{\dot{\mathbf{r}}(t)}{c} = \frac{\mathbf{r}}{r}\Big[1 - (\gamma + 1)\frac{m}{r}\Big]. \tag{6.98}$$

The general expressions (6.92), (6.93) may be applied for deriving solution of the boundary value problem with $\mathbf{r}(t_0) = \mathbf{r}_0\,,\ \mathbf{r}(t) = \mathbf{r}\,,\ (t_0 < t)$. Denoting

$$\mathbf{D} = \mathbf{r}(t) - \mathbf{r}_0(t_0)\,, \qquad D = |\mathbf{D}|$$

one gets the light direction at $t \to -\infty$

$$\boldsymbol{\sigma} = \frac{\mathbf{D}}{D} + \frac{m}{D}\Big[(\gamma + 1)\frac{r - r_0 + D}{|\mathbf{r}_0 \times \mathbf{r}|^2} + \frac{\alpha}{D^2}\Big(\frac{1}{r_0} - \frac{1}{r}\Big)\Big][\mathbf{D} \times (\mathbf{r}_0 \times \mathbf{r})] \tag{6.99}$$

and the time of the light propagation (travel time)

$$t - t_0 = \frac{D}{c} + \frac{m}{c}\Big[(\gamma + 1)\ln\frac{r_0 + r + D}{r_0 + r - D} + \tfrac{1}{2}\alpha\frac{(r_0 + r)[(r_0 - r)^2 - D^2]}{r_0 r D}\Big]. \tag{6.100}$$

Therefore, the coordinate velocity at point $\mathbf{r}$ at moment t is

$$\begin{aligned}\frac{\dot{\mathbf{r}}}{c} =& \frac{\mathbf{D}}{D} + \frac{m}{D}\Big\{-\frac{\gamma + 1}{r}\mathbf{D} - \frac{\alpha}{r^3}[\mathbf{r} \times (\mathbf{r}_0 \times \mathbf{r})] + \\ &+ \Big[\frac{\alpha}{D^2}\Big(\frac{1}{r_0} - \frac{1}{r}\Big) - \frac{\gamma + 1}{r(r r_0 + \mathbf{r}\mathbf{r}_0)}\Big][\mathbf{D} \times (\mathbf{r}_0 \times \mathbf{r})]\Big\}.\end{aligned} \tag{6.101}$$

For $r \ll r_0$ the expressions (6.99), (6.100) may be replaced by their initial expansions

$$\boldsymbol{\sigma} = -\frac{\mathbf{r}_0}{r_0} + \frac{\mathbf{r}_0 \times (\mathbf{r} \times \mathbf{r}_0)}{r_0^3}\Big[1 + \frac{m}{r}\Big(\frac{(\gamma + 1) r r_0}{r r_0 + \mathbf{r}\mathbf{r}_0} - \alpha\Big)\Big] + \ldots\,, \tag{6.102}$$

$$t - t_0 = \frac{r_0}{c}\Big(1 - \frac{\mathbf{r}_0\mathbf{r}}{r_0^2}\Big) + \frac{m}{c}\Big[(\gamma + 1)\ln\frac{2r_0^2}{r r_0 + \mathbf{r}\mathbf{r}_0} + \alpha\Big(\frac{\mathbf{r}_0\mathbf{r}}{r_0 r} - 1\Big)\Big] + \ldots\,. \tag{6.103}$$

Doppler effect

Doppler effect, i.e. the displacement of the light frequency for a light signal emitted at t_0 with period δt_0 and received at t with period δt, may be calculated with the use of the formulas for light propagation. The proper time of the light emitter reads

$$\delta\tau_0 = c^{-1}\Big(\frac{ds}{dt}\Big)_{\mathbf{r}_0(t_0)}\delta t_0\,.$$

The proper time of the light receiver is

$$\delta\tau = c^{-1}\Big(\frac{ds}{dt}\Big)_{\mathbf{r}(t)}\delta t$$

with

$$\delta t = \frac{dt}{dt_0}\delta t_0\,.$$

Considering that the frequency is inversely proportional to the period in the proper time one may find the frequency ratio

$$\frac{\nu_0}{\nu} = \frac{\delta\tau}{\delta\tau_0} = \frac{(ds/dt)_{\mathbf{r}(t)}}{(ds/dt)_{\mathbf{r}_0(t_0)}}\frac{dt}{dt_0}\,. \tag{6.104}$$

For the Schwarzschild field in the first order post–Newtonian approximation one has

$$\frac{\nu_0}{\nu} = \frac{1-\dfrac{m}{r}-\dfrac{\dot{\mathbf{r}}^2}{2c^2}}{1-\dfrac{m}{r_0}-\dfrac{\dot{\mathbf{r}}_0^2}{2c^2}}\frac{dt}{dt_0}\,, \qquad \dot{\mathbf{r}}_0 = \frac{d\mathbf{r}_0}{dt_0}\,,\ \dot{r}_0 = \frac{\mathbf{r}_0\dot{\mathbf{r}}_0}{r_0}\,,\ \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}\,,\ \dot{r} = \frac{\mathbf{r}\dot{\mathbf{r}}}{r}\,, \tag{6.105}$$

$$\frac{dt}{dt_0} = \frac{1-(\mathbf{D}\dot{\mathbf{r}}_0/cD)[1+mS(r_0,r,D)] + m(\dot{r}_0/c)T(r_0,r,D)}{1-(\mathbf{D}\dot{\mathbf{r}}/cD)[1+mS(r_0,r,D)] - m(\dot{r}/c)T(r,r_0,D)} \tag{6.106}$$

with

$$S(r_0,r,D) = 2(\gamma+1)\frac{r_0+r}{(r_0+r)^2-D^2} - \tfrac{1}{2}\alpha(r_0+r)\frac{(r_0-r)^2+D^2}{r_0 r D^2} \tag{6.107}$$

and

$$T(r_0,r,D) = -2(\gamma+1)\frac{D}{(r_0+r)^2-D^2} + \alpha\frac{r^2-r_0^2+D^2}{2r^2D} + \alpha\frac{r}{D}\Big(\frac{1}{r_0}-\frac{1}{r}\Big)\,. \tag{6.108}$$

It may be reminded once again that the coordinate condition constant α enters into the solutions of the equations of motion of a test particle and light propagation and disappears in the expressions for measurable quantities provided that one and the same metric is used for dynamics (eqs. of motion) and kinematics (light propagation).

There exist two more closed (axial symmetric) solutions of the Einstein field equations of interest for celestial mechanics.

Kerr metric

This metric of a rotating spherical body reads

$$\begin{aligned} ds^2 =& p(r,\theta)c^2dt^2 + 2b(r,\theta)cdtdr + 2d(r,\theta)\sin^2\theta\, cdtd\varphi - q(r,\theta)dr^2 - \\ &- a^2(r,\theta)d\theta^2 - f^2(r,\theta)\sin^2\theta\, d\varphi^2 - 2g(r,\theta)\sin^2\theta\, drd\varphi\,, \end{aligned} \tag{6.109}$$

a, b, d, f, g, p, q being even functions of θ, $d \neq 0$, b, g may be zero. This metric is frequently used under one of three forms as follows:

$$\begin{array}{cccc} & \mathrm{I} & \mathrm{II} & \mathrm{III} \\ b(r,\theta) & \frac{2mr}{R^2} & 0 & \frac{2mr}{r^2+A^2} \\ q(r,\theta) & 1+\frac{2mr}{R^2} & \frac{r^2}{r^2-2mr+A^2} & \frac{R^2}{r^2+A^2}(1+\frac{2mr}{r^2+A^2}) \\ g(r,\theta) & A(1+\frac{2mr}{R^2}) & 0 & \frac{2mrA}{r^2+A^2} \end{array} \tag{6.110}$$

where

$$p(r,\theta) = 1 - \frac{2mr}{R^2}\,, \qquad d(r,\theta) = \frac{2mrA}{R^2}\,, \qquad a^2(r,\theta) = R^2$$

and

$$f^2(r,\theta) = r^2 + A^2 + \frac{2mr}{R^2}A^2\sin^2\theta\,, \qquad R^2 = r^2 + A^2\cos^2\theta\,.$$

The forms I and III reduce under $A = 0$ to Eddington form of the Schwarzschild metric whereas the form II reduces to the standard Schwarzschild form. By comparison with the case of the weak field one finds $A = C\omega/cM$, C being the moment of inertia (=$2ML^2/5$ for the homogeneous sphere of radius L).

Weyl–Levi-Civita metric

This axially symmetric metric of a fixed spheroid is of the form (6.109) with

$$b(r,\theta) = d(r,\theta) = g(r,\theta) = 0\,, \quad p(r,\theta) = \mathrm{e}^{2\psi}\,, \quad a^2(r,\theta) = (r^2 - 2mr + m^2\sin^2\theta)\,\mathrm{e}^{2\gamma-2\psi}\,,$$

$$f^2(r,\theta) = (r^2 - 2mr)\,\mathrm{e}^{-2\psi}\,, \qquad q(r,\theta) = \frac{a^2(r,\theta)}{r^2 - 2mr}$$

and the closed–form expressions for ψ, γ. In practice, these closed–form expressions are replaced by their expansions

$$\psi = \tfrac{1}{2}\ln\Big(1 - \frac{2m}{r}\Big) - \tfrac{1}{2}q(3\cos^2\theta - 1)\Big(\frac{2}{15}\frac{m^3}{r^3} + \frac{2}{5}\frac{m^4}{r^4} + \ldots\Big), \tag{6.111}$$

$$\begin{aligned}\gamma =& \tfrac{1}{2}\ln\frac{r^2 - 2mr}{r^2 - 2mr + m^2\sin^2\theta} + q\Big[\Big(-\frac{2}{5}\sin^2\theta + \frac{1}{2}\sin^4\theta\Big)\frac{m^4}{r^4} + \ldots\Big] + \\ &+ q^2\Big[\Big(-\frac{2}{25}\sin^2\theta + \frac{6}{25}\sin^4\theta - \frac{1}{6}\sin^6\theta\Big)\frac{m^6}{r^6} + \ldots\Big].\end{aligned} \tag{6.112}$$

Comparison with the Newtonian potential of the spheroid

$$U = \frac{GM}{r} - \frac{Q}{2r^3}(1 - 3\cos^2\theta) + \ldots \qquad Q = G(A - C) \tag{6.113}$$

results in

$$q = \frac{15}{2}\frac{Q}{c^2m^3}\,. \tag{6.114}$$

The Weyl–Levi-Civita metric is rarely used as an exact solution. In practice it is incorporated in the approximate (post–Newtonian) metric of Lecture 3 for a slowly rotating spheroid. The components of the spheroid vector potential U^i are determined by

$$v^i = \varepsilon_{ijk}\omega^j x^k\,, \qquad \varepsilon_{ijk} = \tfrac{1}{2}(i-j)(j-k)(k-i), \qquad U^i = G\varepsilon_{ijk}\omega^j I^{km}\frac{x^m}{r^3}\,. \tag{6.115}$$

Taking the plane $x^3 = z = 0$ as the equatorial plane one may assume

$$\omega^i = \omega s^i\,, \qquad s^1 = s^2 = 0\,,\ s^3 = 1\,, \qquad I^{11} = I^{22} = \tfrac{1}{2}C\,, \qquad I^{33} = A - \tfrac{1}{2}C\,.$$

Therefore,

$$U^1 = -\tfrac{1}{2}GC\omega\frac{y}{r^3}\,, \qquad U^2 = \tfrac{1}{2}GC\omega\frac{x}{r^3}\,, \qquad U^3 = 0\,. \tag{6.116}$$

Choosing the arbitrary coordinate functions as

$$a_0 = 0\,, \qquad a_i = \alpha\frac{m}{r}x^i$$

one gets the metric as follows:

$$h_{00} = -\frac{2m}{r} + \frac{Q}{c^2r^3}\Big(1-\frac{3z^2}{r^2}\Big) + 2(1-\alpha)\frac{m^2}{r^2} + (-2+3\alpha)\frac{mQ}{c^2r^4}\Big(1-\frac{3z^2}{r^2}\Big),$$

$$h_{ik} = -2(1-\alpha)\frac{m}{r}\delta_{ik} - 2\alpha\frac{m}{r^3}x^ix^k + \frac{Q}{c^2r^3}\Big(1-\frac{3z^2}{r^2}\Big)\delta_{ik}\,, \tag{6.117}$$

$$h_{0i} = -\frac{2GC\omega}{c^3r^3}\,\varepsilon_{ijk}x^js^k$$

or explicitly

$$h_{01} = -\frac{2GC\omega}{c^3}\frac{y}{r^3}\,, \qquad h_{02} = \frac{2GC\omega}{c^3}\frac{x}{r^3}\,, \qquad h_{03} = 0\,.$$

This metric involves the Lagrangian

$$L = \tfrac{1}{2}\dot{\mathbf{r}}^2 + \frac{GM}{r} + R_1 + R_2 + R_3 + R_4 \tag{6.118}$$

with perturbing contributions

$$R_1 = -\frac{Q}{r^3}\Big(1-\frac{3z^2}{r^2}\Big) \qquad \text{(Newtonian oblateness)},$$

$$R_2 = \frac{1}{8c^2}(\dot{\mathbf{r}}^2)^2 + \frac{m}{r}\Big[(\tfrac{3}{2}-\alpha)\dot{\mathbf{r}}^2 + (-\tfrac{1}{2}+\alpha)\frac{GM}{r} + \alpha\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\Big] \qquad \text{(Schwarzschild part)},$$

$$R_3 = -\frac{2GC\omega}{c^2r^3}(x\dot{y} - y\dot{x}) \qquad \text{(Lense} - \text{Thirring part)},$$

$$R_4 = \frac{Q}{2c^2r^3}\Big(1-\frac{3z^2}{r^2}\Big)\Big[(1-3\alpha)\frac{GM}{r} - \tfrac{3}{2}\dot{\mathbf{r}}^2\Big] \qquad \text{(GRT oblateness).}$$

Then the equations of motion take the form

$$\ddot{\mathbf{r}} + \frac{GM}{r^3}\mathbf{r} = \mathbf{F}\,, \qquad \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 \tag{6.119}$$

with the perturbing components

$$\mathbf{F}_1 = \frac{3Q}{r^5}\Big[\tfrac{1}{2}\Big(1-5\frac{z^2}{r^2}\Big)\mathbf{r} + z\mathbf{s}\Big], \qquad \mathbf{s} = (0\,,0\,,1),$$

$$\mathbf{F}_2 = \frac{m}{r^3}\Big[\Big((4-2\alpha)\frac{GM}{r} - (1+\alpha)\dot{\mathbf{r}}^2 + 3\alpha\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\Big)\mathbf{r} + (4-2\alpha)(\mathbf{r}\dot{\mathbf{r}})\dot{\mathbf{r}}\Big],$$

$$\mathbf{F}_3 = \frac{2GC\omega}{c^2r^3}\Big(\frac{3z}{r^2}(\mathbf{r}\times\dot{\mathbf{r}}) + (\dot{\mathbf{r}}\times\mathbf{s})\Big),$$

$$\mathbf{F}_4 = \frac{Q}{c^2r^5}\Big\{\Big[\Big(-8+6\alpha+3(12-11\alpha)\frac{z^2}{r^2}\Big)\frac{GM}{r} + \tfrac{3}{2}\Big(1-5\frac{z^2}{r^2}\Big)\dot{\mathbf{r}}^2\Big]\mathbf{r} + \\ + 3\Big((-4+5\alpha)\frac{GM}{r} + \dot{\mathbf{r}}^2\Big)z\mathbf{s} - 6\Big[\Big(1-5\frac{z^2}{r^2}\Big)(\mathbf{r}\dot{\mathbf{r}}) + 2z\dot{z}\Big]\dot{\mathbf{r}}\Big\}.$$

It should be noted that the GRT direct oblateness perturbations $\mathbf{F}_4$ are of the same order as the indirect second–order perturbations caused by the interaction of $\mathbf{F}_1$ and $\mathbf{F}_2$. Ignorance of such mixture of the perturbations may lead to confusion.

It is easy to show that for a spherical body ($Q = 0$) the equations of motion in vectorial elements reduce to

$$\dot{\mathbf{c}} = \Omega\times\mathbf{c}\,, \qquad \dot{\mathbf{f}} = \Omega\times\mathbf{f}\,, \tag{6.120}$$

$$\Omega = \frac{3mn}{a(1-e^2)}\mathbf{k} + \frac{2GC\omega}{c^2}\frac{1}{a^3(1-e^2)^{3/2}}[\mathbf{s} - 3(\mathbf{s}\mathbf{k})\mathbf{k}] \tag{6.121}$$

resulting to the Schwarzschild and Lense–Thirring secular advances of the node and the argument of pericentre

$$\dot{\Omega} = \frac{2GC\omega}{c^2a^3(1-e^2)^{3/2}}\,, \tag{6.122}$$

$$\dot{\omega} = \frac{3mn}{a(1-e^2)} - \frac{6GC\omega\cos i}{c^2a^3(1-e^2)^{3/2}}\,. \tag{6.123}$$

Lecture 7. Relativistic hierarchy of the reference systems

One of the most commonly used tools of present relativistic celestial mechanics and astrometry is to apply an adequate hierarchy of reference systems. Such an adequate hierarchy may significantly facilitate the solution of a problem under consideration. It is true that there are no physically privileged coordinates in GRT and in this respect all coordinates are the same. But it is true also that the solution of a specific problem may be simplified by using some appropriate coordinates.

First of all, one should note the terminology dualism of using attributes 'coordinate' and 'reference' with subjects 'system' and 'frame' in different combinations. Besides, as mentioned in Lecture 5 the concept of reference frame is used in a different sense in physics and astronomy. In accordance with the operational definition used in astronomy the reference (coordinate) system (RS) is the primary mathematical construction to be given in GRT by a metric form. For any specific astronomical RS its main characteristics involved in the underlying metric form are as follows:

(1) metric coordinate conditions,

(2) the type of solution of the field equations,

(3) the world line of the origin of RS,

(4) the angular velocity rotation of the spatial axes.

An astronomical reference frame (RF) results from the matching of the RS to some reference astronomical objects ('materialization' of a RS).

The main astronomical reference systems and frames now used in practice and provided by IERS (International Earth Rotation Service) are ICRS/ICRF and ITRS/ITRF, C standing for 'celestial' and T for 'terrestrial'. ICRS is an example of global (solar system barycentric) system and ITRS represents a local (geocentric) rotating (with the Earth) system. One of the practical goals of the present hierarchy of relativistic systems is to describe these two systems with their interrelation. In constructing this hierarchy in quasi-Galilean coordinates the four above-indicated conditions are specified as follows:

(1) harmonic coordinate conditions,

(2) physically adequate type of solution (avoiding non–physically meaningful terms as much as possible and manifesting the external mass influence only in form of the tidal terms in conformity with the EEP, Einstein equivalence principle),

(3) BRS, barycentric RS, with the origin at the solar system barycenter; GRS, geocentric RS, with the origin at the geocenter; TRS, topocentric RS, with the origin at the topocenter (point of observation on the surface of the Earth); SRS, satellite RS, with the origin at the Earth artificial satellite; this hierarchy may be extended beyond the solar system (GalRS, galactic RS) or inside the solar system (Earth–Moon barycenter, Moon, other solar system bodoes); each specific origin, especially within the GRT framework, should be strictly defined),

(4) with respect to the Newtonian rotation of the spatial axes (characterized by the Coriolis term of the type $c^{-1}\varepsilon_{ijk}\omega^j x^k$ in g_{0i}) all systems are divided into rotating (e.g., ITRS) and non–rotating (e.g., ICRS) systems; all non–rotating (at the Newtonian level) systems with respect to the relativistic–order rotation of the spatial axes (characterized by the Coriolis term of the type $c^{-3}\varepsilon_{ijk}\omega^j x^k$ in g_{0i}) are divided into dynamically non–rotating (e.g., DGRS, no Coriolis term in g_{0i}) and kinematically non–rotating (e.g., KGRS, a relativistic–order Coriolis term in g_{0i}) systems; D non–rotating systems are preferable for celestial mechanics (no rotation terms in the equations of motion) whereas K non–rotating systems are preferable for astrometry (no kinematical rotation with respect to distant astronomical objects).

For celestial mechanics problems where the solar system is regarded as isolated and gravitationally non–radiating system of bodies the four BRS coordinates play the role of the global coordinates. Generally speaking, it is possible to use only this system. But to have more compact solution and to describe the physical characteristics of celestial non–point bodies (their characteristics of figure, rotation, etc.) in physically more adequate form one makes use of local coordinates related to GRS, TRS, or any other specific local RS. Many definitions of classical astronomy (including the system of astronomical constants) may be interpreted in GRT consistent manner just using the hierarchy of relativistic systems. But the use of such hierarchy is a convenient but in no way the only one possible way of research. Irrespective of employed global or local coordinates it is of primary importance that the same RS is used both for dynamics (equations of motion of bodies) and kinematics (observation analysis procedures based on light propagation).

To illustrate the difference between physically adequate and inadequate systems let us consider a quasi–Galilean BRS

$$ds^2 = (1 + h_{00})c^2 dt^2 + 2h_{0i} c dt\, dx^i + (-\delta_{ik} + h_{ik}) dx^i dx^k \,,$$

with

$$h_{00} \sim c^{-2}\,,\ h_{0i} \sim c^{-3}\,,\ h_{ik} \sim c^{-2}$$

and perform the three–dimensional Galilean transformation of Newtonian mechanics

$$x^i = R^i(t) + \xi^i\,, \qquad dx^i = \dot{R}^i dt + d\xi^i\,,$$

$R^i(t)$ being the BRS spatial coordinates of a point of observation. If this point coincides with the geocenter E ($R^i(t) = x^i_E(t)$) then ξ^i represent the formally geocentric spatial coordinates, i.e. just the differences of the BRS coordinates of the point of observation and the geocenter. In such coordinates the metric takes the form

$$ds^2 = (1 + h_{00} - c^{-2}\dot{\mathbf{R}}^2 + 2c^{-1}h_{0k}\dot{R}^k + c^{-2}h_{km}\dot{R}^k\dot{R}^m)c^2dt^2 +$$
$$+ 2(-c^{-1}\dot{R}^i + h_{0i} + c^{-1}h_{ik}\dot{R}^k) c dt\, d\xi^i + (-\delta_{ik} + h_{ik}) d\xi^i d\xi^k \,.$$

This metric contains explicitly the terms of kinematic origin (non–physical terms), i.e. the terms involving (in the Newtonian order) the velocity $\dot{R}^i$ and the potential of the external masses (by means of h_{00}) as well as the c^{-1} terms (in g_{0i}) caused by the motion of the RS origin. The dynamical theory constructed in such RS would involve large terms of non–physical origin. Although these terms would vanish in deriving observational effects (using the solution for the light propagation in the same RS) it is more effective to use an adequate RS where such terms of the non–physical origin are absent (at the Newtonian level at least).

Let's describe the key point in solving any astronomical problem in the GRT framework once again. Having found the metric form of the problem at hand as a solution of the GRT field equations in some specific coordinates (in principle one can use any coordinates one likes) one should find the solution of the dynamical equations (of motion, of rotation, etc.) and the solution of the kinematical equations (of light propagation, of observational procedure, etc.) in these coordinates. Combining these two solutions one gets the measurable, coordinate independent effects peculiar to the problem ar hand. In doing so one may play with the arbitrary choice of the coordinates choosing the coordinates mostly facilitating the treatment of the dynamical part, or the treatment of the kinematical part, or else presenting a reasonable compromise in solving both parts. At present, the motions of the Sun and major planets are investigated in BRS, a solar system GRT barycentric RS (although for the major planets the GRT heliocentric RS might be more adequate in some respects). The motion of the Moon, Earth's artificial satellites and Earth's rotation are treated most effectively in GRS, a GRT geocentric RS. There are two advantages in using GRS in these problems. First, the 'internal' characteristics of the Earth (its form, e.g. sphericity, non–sphericity parameters, rotation velocity, e.g. rigid–body rotation, etc.) are described much more adequately than in BRS. Second, for most practical applications these problems may be treated in GRS within the Newtonian dynamical equations because the main relativistic effects are taken into account therewith in the space–time transformation BRS→GRS. Considering that all technical details may be found in the corresponding references we reproduce here only the final results.

BRS metric (in harmonic quasi–Galilean coordinates):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \qquad x^0 = ct \,, \qquad t = TCB \,, \tag{7.1}$$

$$g_{00} = 1 + h_{00} \,, \qquad g_{0i} = h_{0i} \,, \qquad g_{ij} = -\delta_{ij} + h_{ij} \,, \tag{7.2}$$

$$h_{00} = -2c^{-2} U(t, \mathbf{x}) \,, \qquad h_{0i} = 4c^{-3} U^i(t, \mathbf{x}) \,, \qquad h_{ij} = h_{00}\delta_{ij} \,. \tag{7.3}$$

U is the Newtonian potential and U^i the vector–potential with splitting into internal and external parts with respect to the Earth E (as one of the bodies generating the solar system gravitational field)

$$U(t, \mathbf{x}) = U_E(t, \mathbf{x}) + \bar{U}_E(t, \mathbf{x}), \qquad U^i(t, \mathbf{x}) = U^i_E(t, \mathbf{x}) + \bar{U}^i_E(t, \mathbf{x}) \,. \tag{7.4}$$

In evaluating in the geocenter $\mathbf{x} = \mathbf{x}_E$ for the nonrotating point mass model there results

$$\bar{U}_E(t, \mathbf{x}_E) = \sum_{A\neq E} \frac{GM_A}{r_{EA}}\,, \qquad \bar{U}^i_E(t, \mathbf{x}_E) = \sum_{A\neq E} \frac{GM_A}{r_{EA}} v^i_A \tag{7.5}$$

with summation over solar system bodies A (the Sun, the Moon, major planets) and

$$\mathbf{x}_E = \mathbf{x}_E(t), \qquad \mathbf{v}_E = \dot{\mathbf{x}}_E(t), \qquad \mathbf{r}_E = \mathbf{x} - \mathbf{x}_E\,, \qquad \mathbf{r}_{EA} = \mathbf{x}_E - \mathbf{x}_A\,, \tag{7.6}$$

$\mathbf{x}_E$, $\mathbf{v}_E$ being the BRS position vector and velocity of the Earth E. It should be reminded once again that vector designations are used here rather conventionally just as triplets of the corresponding spatial components.

GRS metric (with spatial origin at the geocenter E):

$$ds^2 = \hat{g}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu\,, \qquad \hat{x}^0 = c\hat{t}\,, \qquad \hat{t} = u = TCG\,, \tag{7.7}$$

$$\hat{h}_{00} = -2c^{-2}\left(\hat{U}_E + Q_j \hat{x}^j + T(\hat{\mathbf{x}})\right),$$
$$\hat{h}_{0i} = c^{-3}\Big[(q-1)\dot{F}^{ij}\hat{x}^j + 4\big(\hat{U}^i_E + T^i(\hat{\mathbf{x}}) - v^i_E T(\hat{\mathbf{x}})\big) + \tfrac{2}{5}\big(3\dot{a}^j_E \hat{x}^j \hat{x}^i - \dot{a}^i_E \hat{x}^j \hat{x}^j\big)\Big], \tag{7.8}$$
$$\hat{h}_{ij} = \hat{h}_{00}\delta_{ij}\,.$$

$\hat{U}_E$ is the GRS geopotential, $\hat{U}^i_E$ the GRS vector–potential, $T(\hat{\mathbf{x}})$ is the tidal potential,

$$T(\hat{\mathbf{x}}) = \bar{U}_E(\mathbf{x}_E + \hat{\mathbf{x}}) - \bar{U}_E(\mathbf{x}_E) - \bar{U}_{E,j}(\mathbf{x}_E)\hat{x}^j\,, \tag{7.9}$$

$T^i(\hat{\mathbf{x}})$ is the tidal vector–potential,

$$T^i(\hat{\mathbf{x}}) = \bar{U}^i_E(\mathbf{x}_E + \hat{\mathbf{x}}) - \bar{U}^i_E(\mathbf{x}_E) - \bar{U}^i_{E,j}(\mathbf{x}_E)\hat{x}^j\,, \tag{7.10}$$

Q_i is the nongeodesic acceleration in the BRS motion of the Earth:

$$a^i_E = \bar{U}_{E,i}(t,\, \mathbf{x}_E) - Q_i + \mathrm{O}(c^{-2})\,. \tag{7.11}$$

Antisymmetric matrix F^{ij} (geodesic rotation in the broad sense):

$$\dot{F}^{ij} = \tfrac{3}{2}\big(v^i_E a^j_E - v^j_E a^i_E\big) - 2\big(\bar{U}^i_{E,j}(t,\, \mathbf{x}_E) - \bar{U}^j_{E,i}(t,\, \mathbf{x}_E)\big) + 2\big(v^i_E Q_j - v^j_E Q_i\big)\,. \tag{7.12}$$

Representation of F^{ij} with the aid of the corresponding triplet F^i:

$$F^i = \tfrac{1}{2}\varepsilon_{ijk}F^{jk}\,, \qquad F^{ij} = \varepsilon_{ijk}F^k\,, \qquad \varepsilon_{ijk} = \tfrac{1}{2}(i-j)(j-k)(k-i). \tag{7.13}$$

For any vector a^j

$$F^{ij}a^j = -\varepsilon_{ijk}F^j a^k = -(\mathbf{F}\times\mathbf{a})^i\,.$$

Components F^{ij} satisfy the relations

$$F^{ij} = \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{mnk}F^{mn}\,, \qquad \varepsilon_{ijm}F^{km} + \varepsilon_{jkm}F^{im} + \varepsilon_{kim}F^{jm} = 0\,. \tag{7.14}$$

In dealing with ε_{ijk} one often makes use of

$$\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\,. \tag{7.15}$$

The geodesic rotation vector is determined by the differential relation

$$\begin{aligned}\dot{F}^i &= \varepsilon_{ijk}\left[\tfrac{3}{2}v^j_E a^k_E - 2\bar{U}^j_{E,k}(t,\mathbf{x}_E) + 2v^j_E Q_k\right] = \\ &= \varepsilon_{ijk}\left[\tfrac{3}{2}v^j_E \bar{U}_{E,k}(t,\mathbf{x}_E) - 2\bar{U}^j_{E,k}(t,\mathbf{x}_E) + \tfrac{1}{2}v^j_E Q_k\right]\end{aligned} \tag{7.16}$$

combining the effects of de Sitter precession, Lense–Thirring precession and Thomas precession.

Main terms (assuming in particular the nonrotating point mass model) are as follows:

$$\dot{F}^i = \sum_{A\neq E}\frac{GM_A}{r^3_{EA}}[(-\tfrac{3}{2}\mathbf{v}_E + 2\mathbf{v}_A)\times\mathbf{r}_{EA}]^i\,. \tag{7.17}$$

Considering that vector F^i is determined by its time–derivative it is necessary to fix an additive constant (e.g., by implying the condition $F^i = 0$ for J2000.0). Until now it is not done by any IAU resolutions.

For the two–body problem F^i may be expressed in closed form:

$$\dot{F}^i = -\tfrac{3}{2}\frac{GM}{r^3}(\mathbf{v}\times\mathbf{r})^i = \tfrac{3}{2}\frac{n^3a^5}{r^3}\sqrt{1-e^2}k^i\,,$$

$$r^2\frac{df}{dt} = na^2\sqrt{1-e^2}\,, \qquad r = \frac{a(1-e^2)}{1+e\cos f}\,,$$

$$F^i = \tfrac{3}{2}\frac{n^2a^2}{1-e^2}\,k^i\int(1+e\cos f)df = \tfrac{3}{2}\frac{n^2a^2}{1-e^2}\,k^i(f+e\sin f)\,. \tag{7.18}$$

Using the two–body problem trigonometric expansions in mean anomaly l one gets the expressions of the geodesic precession (secular term in l) and the geodesic nutation (trigonometric terms):

$$F^i = \tfrac{3}{2}\frac{n^2a^2}{1-e^2}\,k^i(l + 3e\sin l + \tfrac{9}{4}e^2\sin 2l + \ldots). \tag{7.19}$$

These expressions describe geodesic precession and geodesic nutation in narrow sense, i.e. in the framework of the point–mass two–body problem.

The four–dimensional BRS→ GRS transformation results from the matching procedure interrelating BRS and GRS:

$$g_{\alpha\beta}(t,\mathbf{x}) = \hat{g}_{\mu\nu}(\hat{t},\hat{\mathbf{x}})\frac{\partial\hat{x}^\mu}{\partial x^\alpha}\frac{\partial\hat{x}^\nu}{\partial x^\beta}\,.$$

The relationship between t and u reads

$$\hat{t} = t - c^{-2}\left[A(t) + v_E^j r_E^j\right], \tag{7.20}$$

Function $A(t)$ satisfies the time equation (TCG→ TCB transformation in the geocenter)

$$\dot{A}(t) = \frac{1}{2}\mathbf{v}_E^2 + \bar{U}_E(t, \mathbf{x}_E). \tag{7.21}$$

Its solution is presented with separating a linear secular term from all other (polynomial, trigonometric and mixed) terms

$$A(t) = c^2 L_C t + A_p(t)\,, \qquad L_C t = L_C(J - 2443144.5)86400s \tag{7.22}$$

with

$$c^{-2}A_p(t) = P = \sum_\alpha t^\alpha \left[\sum_k A_k^\alpha \cos(\psi_k^\alpha + \nu_k^\alpha t)\right] \tag{7.23}$$

and condition $P = 0$ on Jan. 1, 1977 0h 0m 0s TAI (J=2443144.5 TAI). Theoretically, t and $\hat{t}$ are supposed to be TCB and TCG, coordinate time scales of BRS and GRS, respectively. But in practice t and $\hat{t}$ are often used as the time scales TDB and TT differing by scalar factors from TCB and TCG, respectively,

$$\mathrm{TDB} = (1 - L_B)\mathrm{TCB}\,, \qquad \mathrm{TT} = (1 - L_G)\mathrm{TCG}\,. \tag{7.24}$$

These three scalar factors satisfy the relation

$$1 - L_B = (1 - L_C)(1 - L_G), \qquad (L_B = L_C + L_G - L_C L_G). \tag{7.25}$$

According IAU Resolution B1 (2000) their values read

$$L_C = 1.48082686741 \times 10^{-8}\,, \qquad L_B = 1.55051976772 \times 10^{-8}\,, \tag{7.26}$$

$$L_G = 6.969290134 \times 10^{-10}\,. \tag{7.27}$$

Contrary to values (7.26) dependent on the O–C analysis of the planetary–lunar motions L_G is a defining constant. In terms of TDB and TT the equation (7.20) takes the form

$$\mathrm{TT} = \mathrm{TDB} - c^{-2}[A_p(t) + \mathbf{v}_E \mathbf{r}_E] + \ldots\,. \tag{7.28}$$

The use of TDB and TT involves the scale factors for spatial coordinates and mass coefficients

$$(\mathbf{x})_{\mathrm{TDB}} = (1 - L_B)\mathbf{x}\,, \qquad (GM)_{\mathrm{TDB}} = (1 - L_B)GM\,, \tag{7.29}$$

$$(\hat{\mathbf{x}})_{\mathrm{TT}} = (1 - L_G)\hat{\mathbf{x}}\,, \qquad (G\hat{M})_{\mathrm{TT}} = (1 - L_G)(G\hat{M})\,, \tag{7.30}$$

so that the velocity components and the equations of motion remain the same. By introducing the scalar parameters

$$\mu = \begin{cases} 1\,, & t = \mathrm{TCB}\,, \\ 0\,, & t = \mathrm{TDB}\,, \end{cases} \qquad \nu = \begin{cases} 1\,, & \hat{t} = \mathrm{TCG}\,, \\ 0\,, & \hat{t} = \mathrm{TT} \end{cases} \tag{7.31}$$

one may write the direct BRS↔GRS transformation for any possible combination of the time scales as follows:

$$\hat{t} = (1 - \mu L_B + \nu L_G)t - c^{-2}(A_p + \mathbf{v}_E\mathbf{r}_E), \tag{7.32}$$

$$\hat{x}^i = [1 + (1-\mu)L_B - (1-\nu)L_G]r_E^i + c^{-2}\Lambda^i(t,\, \mathbf{r}_E), \tag{7.33}$$

$$\Lambda^i(t,\, \mathbf{r}_E) = \tfrac{1}{2}\mathbf{v}_E\mathbf{r}_E v_E^i - q\varepsilon_{ijk}F^j r_E^k + \bar{U}_E(t, \mathbf{x}_E)r_E^i + \mathbf{a}_E\mathbf{r}_E r_E^i - \tfrac{1}{2}\mathbf{r}_E^2 a_E^i\,, \tag{7.34}$$

where $\mathbf{a}_E$ is the BRS acceleration of the Earth, F^j is the vector of geodesic rotation and q is a numerical parameter to distinguish between kinematically non–rotating ($q = 0$) or dynamically non–rotating ($q = 1$) GRS (e.g., Bretagnon and Brumberg, 2003). Inverse transformation reads

$$t = (1 + \mu L_B - \nu L_G)\hat{t} + c^{-2}(A_p + \mathbf{v}_E\hat{\mathbf{x}}), \tag{7.35}$$

$$x^i = [1 - (1-\mu)L_B + (1-\nu)L_G](\hat{x}^i + z_E^i) + c^{-2}\Gamma^i(\hat{t},\, \hat{\mathbf{x}}), \tag{7.36}$$

$$\Gamma^i(\hat{t},\, \hat{\mathbf{x}}) = \tfrac{1}{2}\mathbf{v}_E\hat{\mathbf{x}}v_E^i + q\varepsilon_{ijk}F^j\hat{x}^k - \bar{U}_E(t, \mathbf{x}_E)\hat{x}^i - \mathbf{a}_E\hat{\mathbf{x}}\hat{x}^i + \tfrac{1}{2}\hat{\mathbf{x}}^2 a_E^i\,. \tag{7.37}$$

Function $z_E^i = z_E^i(\hat{t})$ representing the Earth's motion referred to the GRS coordinate time can be computed from the BRS–time representation of the Earth's motion by means of

$$z_E^i(\hat{t}) = [1 + (1-\mu)L_B - (1-\nu)L_G]x_E^i(t^*), \tag{7.38}$$

where

$$t^* = (1 + \mu L_B - \nu L_G)\hat{t} + c^{-2}A_p\,. \tag{7.39}$$

Instead of determining t^* one can just use

$$x_E^i(t^*) = x_E^i[(1 + \mu L_B - \nu L_G)\hat{t}] + c^{-2}A_p v_E^i\,. \tag{7.40}$$

By substituting (7.38) into (7.36) one can use also

$$x^i = [1 - (1-\mu)L_B + (1-\nu)L_G]\hat{x}^i + x_E^i(t^*) + c^{-2}\Gamma^i(\hat{t},\, \hat{\mathbf{x}}). \tag{7.41}$$

The right–hand members of the Newtonian equations of the Earth's satellite motion and the Earth's rotation involve the geocentric position vectors of the disturbing bodies $\hat{x}_A^i(\hat{t})$. Their expressions in terms of the BRS quantities result from the BRS↔GRS transformation as follows:

$$\hat{x}_A^i(\hat{t}) = z_A^i(\hat{t}) - z_E^i(\hat{t}) + c^{-2}\left[\Lambda^i(t^*,\, \mathbf{r}_{AE}) + \mathbf{v}_E\mathbf{r}_{AE}v_{AE}^i\right], \tag{7.42}$$

$$z_A^i(\hat{t}) - z_E^i(\hat{t}) = [1 + (1-\mu)L_B - (1-\nu)L_G][x_A^i(t^*) - x_E^i(t^*)] \tag{7.43}$$

with $\mathbf{v}_{AE} = \mathbf{v}_A - \mathbf{v}_E$. These expressions contain relativistic terms leading to the indirect relativistic perturbations in the formally Newtonian right–hand members of the GRS equations of motion and rotation. The direct relativistic terms come from the relativistic parts of the complete GRS equations of motion and rotation. But as a rule these direct relativistic terms are smaller than the indirect terms coming from the Newtonian parts of the right–hand members. In many problems one may just neglect such direct relativistic terms. The possibility to use only the Newtonian parts in the GRS equations is due to the adequate choice of GRS. The extension of this technique for other planets of the solar system is straightforward.

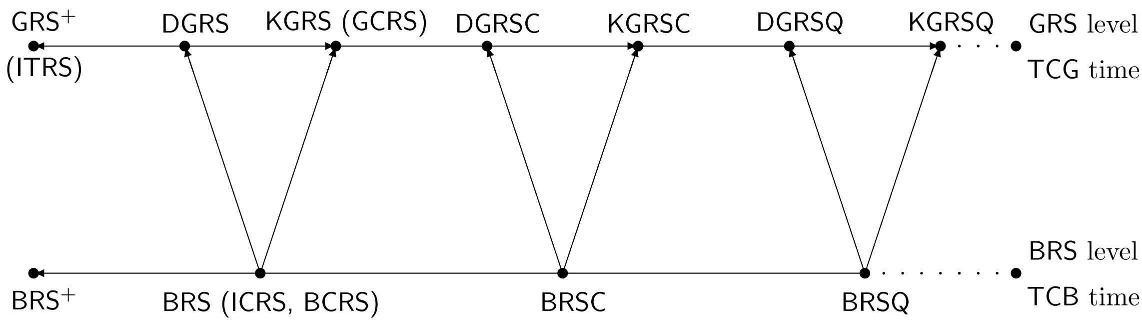
Recent IAU (2000) Resolution B1 on reference systems and time scales involves much more profound use of general relativity theory in ephemeris astronomy compared to the IAU (1991) Resolution A4. However, there still exists some confusion in using the time scales TDB and TCB, TT and TCG, and in GRT–consistent interpretation of ICRS and ITRS, the two main reference systems.

For astronomical practice it is sufficient to have only the reference systems ICRS and ITRS with their physical realizations given by the reference frames ICRF and ITRF, respectively. In the GRT framework, ICRS represents a four–dimensional Barycentric Celestial Reference System (BCRS) whereas ITRS represents a local four–dimensional geocentric system rotating with the Earth. For the relationship between BCRS and ITRS it is necessary to introduce one more local geocentric system with the same time scale, TCG, as used for ITRS and the same directions of the spatial axes as for BCRS. Such a Geocentric Celestial Reference System (GCRS) is introduced by the IAU Resolution B1.3 (2000). GCRS represents a system kinematically non-rotating with respect to BCRS. This system rotates dynamically with respect to BCRS. The angular velocity of this rotation has a relativistic order of smallness and is caused by the motion of the geocenter around the solar system barycenter. Indeed, the four–dimensional transformation from the barycenter to the geocenter can transform any barycentric system BRS either into DGRS, a geocentric system dynamically non–rotating with respect to BRS, or into KGRS, a geocentric system kinematically non–rotating with respect to BRS. GCRS represents just such a KGRS suitable for astrometric purposes. The systems of DGRS type are more preferable for celestial mechanics enabling one to deal with the geocentric equations of motion without the terms caused by the rotation of the system. One should note also that the classical concepts of ephemeris astronomy and the system of astronomical constants are based on Newtonian mechanics with its absolute time and absolute space. In Newtonian astronomy such concepts are invariant under the transformation from the barycenter to the geocenter and vice versa. Such invariance does not take place in relativistic astronomy. To avoid possible confusion one has to go beyond the framework of ICRS and ITRS systems.

One of the possibilities (not unique, of course) is to consider several reference systems at the barycentric and geocentric levels. At the barycentric level, along with ICRS (BCRS) one may consider ecliptical BRSC and equatorial BRSQ systems. Their main planes are chosen to coincide with planes of the ecliptic and equator fixed for J2000.0 . The coordinate time scale of all these four–

dimensional systems is t =TCB. Their spatial coordinates are designated in this paper by $\mathbf{x} = (x^i)$, $\mathbf{x}_C = (x^i_C)$ and $\mathbf{x}_Q = (x^i_Q)$, respectively. At the geocentric level each of these three system involves two geocentric systems, dynamically (D) or kinematically (K) non–rotating with respect to the corresponding barycentric system. As a result one has at the geocentric level six geocentric systems DGRS, KGRS, DGRSC, KGRSC, DGRSQ and KGRSQ. They all have (within the post–Newtonian approximation) the coordinate time scale $\hat{t}$ =TCG. Their spatial coordinates are designated below by $\hat{\mathbf{x}} = (\hat{x}^i)$, $\hat{\mathbf{x}}_C = (\hat{x}^i_C)$ and $\hat{\mathbf{x}}_Q = (\hat{x}^i_Q)$, respectively. If necessary, the spatial coordinates for dynamically (D) or kinematically (K) non–rotating systems may be distinguished explicitly by indicating $\underset{q}{\hat{x}^i}$ with $q = 1$ for version D and $q = 0$ for version K. The GCRS system as defined by the IAU Resolution B1.3 (2000) corresponds in these notations to KGRS. VSOP theories of planetary motion are constructed in BRSC. SMART theory of the Earth's rotation is to be considered in DGRSC. Since the ecliptic may be defined reasonably only at the barycentric level, GRSC (in any version) should be regarded just as another intermediate system between ICRS and ITRS. Quite similarly, since the equator is defined reasonably only at the geocentric level, BRSQ should be regarded as one more intermediary between ICRS and ITRS.

This two–level (wide) RS hierarchy is shown below:



Two–level RS hierarchy. Barycentric and Geocentric Reference Systems (RSs) B — barycentric, G — geocentric, C — ecliptical, Q — equatorial, D — dynamical, K — kinematical, $^+$ — rotating; ICRS — International Celestial RS, ITRS — International Terrestrial RS (IERS); BCRS — Barycentric Celestial RS, GCRS — Geocentric Celestial RS (IAU 2000)

The basic relationships between these systems are as follows:

G level:

$$[\mathrm{GRS}^+] = \underset{1}{\hat{P}}(\hat{t})[\mathrm{DGRSC}] = \underset{1}{\hat{P}}(\hat{t})P_C[\mathrm{DGRS}] = \underset{1}{\hat{P}}(\hat{t})P_C P_Q^T[\mathrm{DGRSQ}] \qquad (7.44).$$

$$[\mathrm{GRS}^+] = \underset{0}{\hat{P}}(\hat{t})[\mathrm{KGRSC}] = \underset{0}{\hat{P}}(\hat{t})P_C[\mathrm{KGRS}] = \underset{0}{\hat{P}}(\hat{t})P_C P_Q^T[\mathrm{KGRSQ}] \qquad (7.45).$$

B level:

$$[\mathrm{BRS}^+] = P(t)[\mathrm{BRSC}] = P(t)P_C[\mathrm{BRS}] = P(t)P_C P_Q^T[\mathrm{BRSQ}]. \qquad (7.46)$$

K and D versions:

$$[\mathrm{KGRS}] = (E - c^{-2}F)[\mathrm{DGRS}]\,, \qquad \underset{1}{\hat{P}}(\hat{t}) = \underset{0}{\hat{P}}(\hat{t})(E - c^{-2}F_C) \qquad (7.47)$$

$$[\mathrm{KGRSC}] = (E - c^{-2}F_C)[\mathrm{DGRSC}], \qquad [\mathrm{KGRSQ}] = (E - c^{-2}F_Q)[\mathrm{DGRSQ}] \qquad (7.48)$$

$$F_C = P_C F P_C^T, \qquad F_Q = P_Q F P_Q^T\,, \qquad P_C = D_1(\varepsilon)D_3(\chi), \qquad P_Q = D_3(\chi) \qquad (7.49)$$

$$\varepsilon = 23^\circ 26' 21.408800'' = 0.409092614174\,, \qquad \chi = -0.053727'' = -0.000000260476 \qquad (7.50)$$

$$\underset{0}{\hat{P}}(\hat{t}) = P(t^*) \qquad (7.51)$$

$$D_1(\alpha)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix},\ D_2(\alpha)=\begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix},\ D_3(\alpha)=\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

P and D_i being the Earth's rotation matrix and elementary rotation matrices, respectively.

Two–level hierarchy is sufficient to treat most dynamical problems related with the motion and rotation of celestial bodies. To describe the observational procedure one needs a RS of actual observer, i.e. a topocentric RS (TRS) or Earth's satellite RS (SRS). These systems may be regarded as RS of a massless observer with gravitating masses acting only as external masses (by means of tidal forces). These systems transform the two–level hierarchy of BRS and GRS into three–lever hierarchy by adding TRS or SRS. We will consider this hierarchy with minimal versions.

A single RS at the barycenric level with some given orientation of the spatial axes (BRS) generates at the geocentric level two different systems, dynamically nonrotating system (DGRS) and kinematically nonrotating system (KGRS). One may treat these systems as one system supplied by numerical parameter q taking values 1 or 0, correspondingly. In its turn each of this system generates at the satellite level (related to a satellite orbiting the Earth) two systems, dynamically (DSRS) or kinematically (KSRS) nonrotating with respect to the generating GRS. One may again distinguish these systems by values 1 or 0 of numerical parameter $\hat{q}$. Transformations BRS $\rightarrow$ GRS and GRS $\rightarrow$ SRS represent generalized Lorentz transformations. Even in case of special relativity two consequent Lorentz transformations without rotation (BRS $\rightarrow$ KGRS and KGRS $\rightarrow$ SRS4) result in spatial rotation of the final system (SRS4) with respect to the initial one (BRS). GRT treatment results in more accurate estimation of this effect. Consideration of this rotation leads to the fifth system at the satellite level, SRS5, kinematically nonrotating with respect to BRS. To describe all five satellite systems as one system one has to introduce additive numerical parameter $\tilde{q}$ equal to 1 for all four preceding systems and vanishing for SRS5. Replacing SRS by TRS changes nothing in this hierarchy.

Three–level (deep) RS hierarchy

$$\mathrm{BRS} \longrightarrow \begin{cases} \mathrm{DGRS} \quad (q=1) \longrightarrow \begin{cases} \mathrm{SRS1} & (\hat{q}=1, \quad \tilde{q}=1) \\ \mathrm{SRS2} & (\hat{q}=0, \quad \tilde{q}=1) \end{cases} \\ \mathrm{KGRS} \quad (q=0) \longrightarrow \begin{cases} \mathrm{SRS3} & (\hat{q}=1, \quad \tilde{q}=1) \\ \mathrm{SRS4} & (\hat{q}=0, \quad \tilde{q}=1) \\ \mathrm{SRS5} & (\hat{q}=0, \quad \tilde{q}=0) \end{cases} \end{cases} \tag{7.52}$$

STRS (TRS) metric (with spatial origin at the satellite location or at the topocenter T):

$$ds^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu \,, \qquad \tilde{x}^0 = c\tilde{t}\,, \qquad \tilde{t} = \tau \text{ (proper time at the topocenter)} \tag{7.53}$$

$$\tilde{h}_{00} = -2c^{-2}\left[E_j \tilde{x}^j + \mathcal{T}(\tilde{\mathbf{x}})\right] \tag{7.54}$$

$$\tilde{h}_{0i} = c^{-3}\Big[(q-1)\dot{F}^{ij}\tilde{x}^j + (\hat{q}-1)\dot{R}^{ij}\tilde{x}^j + (\tilde{q}-1)\dot{K}^{ij}\tilde{x}^j + 4\mathcal{T}^i(\tilde{\mathbf{x}}) - \\ -4\big(v_E^i + \hat{v}_T^i\big)\mathcal{T}(\tilde{\mathbf{x}}) + \tfrac{6}{5}\big(\dot{a}_E^j + \dot{\hat{a}}_T^j\big)\tilde{x}^j\tilde{x}^i - \tfrac{2}{5}\big(\dot{a}_E^i + \dot{\hat{a}}_T^i\big)\tilde{x}^j\tilde{x}^j\Big] \tag{7.55}$$

$$\tilde{h}_{ij} = \tilde{h}_{00}\delta_{ij} \tag{7.56}$$

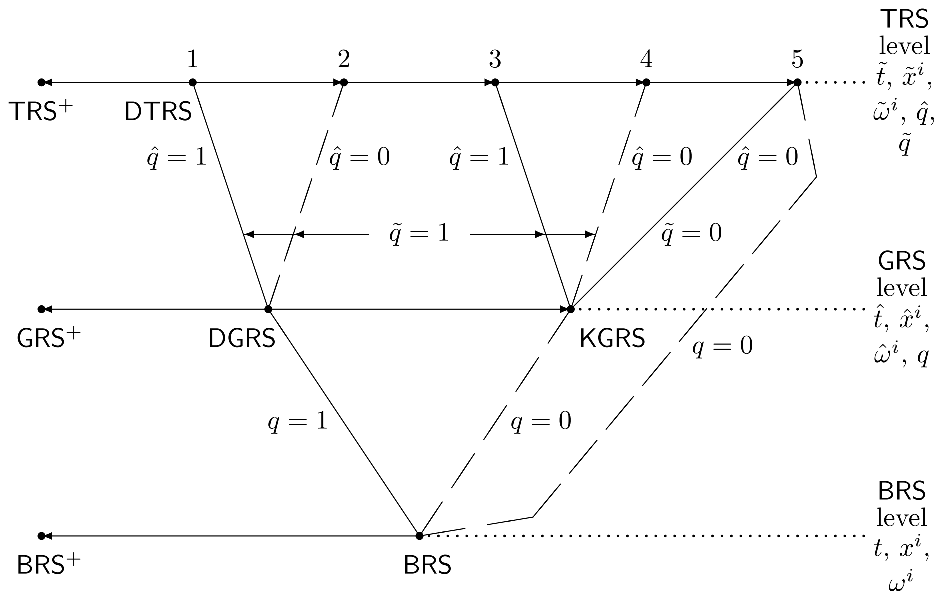
$\mathcal{T}(\tilde{\mathbf{x}})$ and $\mathcal{T}^i(\tilde{\mathbf{x}})$ are the tidal potentials constructed with the total potentials U and U^i

$$\mathcal{T}(\tilde{\mathbf{x}}) = U(\mathbf{x}_E + \hat{\mathbf{x}}_T + \tilde{\mathbf{x}}) - U(\mathbf{x}_E + \hat{\mathbf{x}}_T) - U_{,j}(\mathbf{x}_E + \hat{\mathbf{x}}_T)\tilde{x}^j \tag{7.57}$$

$$\mathcal{T}^i(\tilde{\mathbf{x}}) = U^i(\mathbf{x}_E + \hat{\mathbf{x}}_T + \tilde{\mathbf{x}}) - U^i(\mathbf{x}_E + \hat{\mathbf{x}}_T) - U^i_{,j}(\mathbf{x}_E + \hat{\mathbf{x}}_T)\tilde{x}^j \,. \tag{7.58}$$

E_i stands for the nongeodesic acceleration of the topocenter. In case of SRS one has $E_i = 0$ resulting to the equations of an Earth satellite:

$$E_i = -\hat{a}_T^i + \hat{U}_{E,i}(\hat{\mathbf{x}}_T) + Q_i + \bar{U}_{E,i}(\mathbf{x}_E + \hat{\mathbf{x}}_T) - \bar{U}_{E,i}(\mathbf{x}_E) + \mathrm{O}(c^{-2}) \tag{7.59}$$



Three–level RS hierarchy. BRS→GRS(s) →TRS(s)

(Reference systems kinematically nonrotating one with respect to another are connected by dashed lines.)

Antisymmetric matrix R^{ij} (topocentric precession):

$$\dot{R}^{ij} = \tfrac{3}{2}\left(\hat{v}^i_T\hat{a}^j_T - \hat{v}^j_T\hat{a}^i_T\right) + \left(\dot{a}^i_E\hat{x}^j_T - \dot{a}^j_E\hat{x}^i_T\right) - 2\left(\hat{U}^i_{E,j}(\hat{\mathbf{x}}_T) - \hat{U}^j_{E,i}(\hat{\mathbf{x}}_T)\right) +$$

$$+2\left[v^i_E\bar{U}_{E,jk}(\mathbf{x}_E) - v^j_E\bar{U}_{E,ik}(\mathbf{x}_E) - \bar{U}^i_{E,jk}(\mathbf{x}_E) + \bar{U}^j_{E,ik}(\mathbf{x}_E)\right]\hat{x}^k_T +$$

$$+2\left(\hat{v}^i_T E_j - \hat{v}^j_T E_i\right). \tag{7.60}$$

K^{ij} precession for SRS (TRS) kinematically nonrotating with respect to BRS:

$$K^{ij} = \hat{x}^i_T a^j_E - \hat{x}^j_T a^i_E + \tfrac{1}{2}\left(\hat{v}^i_T v^j_E - \hat{v}^j_T v^i_E\right). \tag{7.61}$$

GRS→TRS transformation:

$$\tilde{t} = \hat{t} - c^{-2}\left[V(\hat{t}) + \hat{v}^j_T\hat{r}^j_T\right], \qquad \hat{r}^i_T = \hat{x}^i - \hat{x}^i_T \tag{7.62}$$

$$\tilde{x}^i = \hat{r}^i_T + c^{-2}\left\{\left[\tfrac{1}{2}\hat{v}^i_T\hat{v}^j_T + \hat{q}R^{ij}(\hat{t}) + (\tilde{q}-1)K^{ij}(\hat{t}) + \mathcal{D}^{ij}(\hat{t})\right]\hat{r}^j_T + \mathcal{D}^{ijk}(\hat{t})\hat{r}^j_T\hat{r}^k_T\right\}. \tag{7.63}$$

with

$$\mathcal{D}^{ij}(\hat{t}) = \left[\hat{U}_E(\hat{\mathbf{x}}_T) + Q_j \hat{x}_T^j + T(\hat{\mathbf{x}}_T)\right] \delta_{ij} \tag{7.64}$$

$$\mathcal{D}^{ijk}(\hat{t}) = \tfrac{1}{2}\left(\delta_{ij}\hat{a}_T^k + \delta_{ik}\hat{a}_T^j - \delta_{jk}\hat{a}_T^i\right). \tag{7.65}$$

In dependence on the values of parameters q, $\hat{q}$ and $\tilde{q}$ (q enters in (7.63) implicitly by means of the GRS coordinates $\hat{x}_T^i$) one gets all five SRS systems of the three–level hierarchy.

One should add to it the relationship of the proper time of an observer located in T with TCG

$$\dot{V} = \tfrac{1}{2}\hat{v}_T^2 + \hat{U}_E(\hat{\mathbf{x}}_T) + Q_j \hat{x}_T^j + T(\hat{\mathbf{x}}_T). \tag{7.66}$$

The three–level hierarchy can be applied, for instance, to study the main effect of the presence of the Galaxy (motion of the solar system barycenter in the Galaxy). In this case one has the hierarchy

GalRS→(DBRS/KBRS)→GRS(s)

consisting of galactic RS, two solar system barycentric RS (dynamically or kinematically nonrotating with respect to the galactic RS) and five geocentric systems. The galactic precession (similar to the geodesic precession) amounts to $0.85'' \cdot 10^{-6}$ per century.

Two–level or three–level hierarchies with the corresponding time scales can be developed in the same way for any solar system bodies.

To conclude this lecture let's say once again that the IAU resolutions 2000 (IAU, 2001) with their positive aspects in using unambiguous concepts such as reference systems, time scales, astronomical constants and units, etc. should not be considered as the panacea in treating problems of relativistic celestial mechanics and astrometry (Brumberg and Groten, 2001). Fundamental distinction of RF (reference frame) notion in astronomy and physics, the use of Newtonian three–dimensional rotation in defining rotating reference systems (violating harmonic coordinate conditions for rotating RS), the validity of many concepts only within the post–Newtonian approximation, and many other problems lie indirectly beyond the scope of these resolutions. In any case one should not be constrained by these resolutions in solving in optimal way specific astronomical problems.

Lecture 8. Equations of motion of bodies and gravitational radiation

The N–body problem in the solar system is characterized by small velocities $v \ll c$, weak gravitation $U \ll c^2$ and quasi–point body structure $L \ll R$, v being the characteristic velocity of the bodies, U standing for Newtonian potential, L and R being the lengths characterizing the linear seizes of the bodies (their diameters) and their mutual distances, respectively. As already mentioned in Lecture 3 the PNA (post–Newtonian approximation) expansions of the metric tensor components in quasi–Galilean coordinates read

$$h_{00} = c^{-2} \underset{2}{h_{00}} + c^{-4} \underset{4}{h_{00}} + c^{-5} \underset{5}{A_{00}} + c^{-6} \underset{6}{h_{00}} + c^{-7} \underset{7}{A_{00}} + \mathrm{O}(c^{-8}), \tag{8.1}$$

$$h_{0i} = c^{-3} \underset{3}{h_{0i}} + c^{-5} \underset{5}{h_{0i}} + c^{-6} \underset{6}{A_{0i}} + \mathrm{O}(c^{-7}), \tag{8.2}$$

$$h_{ij} = c^{-2} \underset{2}{h_{ij}} + c^{-4} \underset{4}{h_{ij}} + c^{-5} \underset{5}{A_{ij}} + \mathrm{O}(c^{-6}), \tag{8.3}$$

with $A_{\mu\nu}$ standing for the gravitational radiation terms.

The history of development of the GRT problem of motion goes back to the first years since GRT foundation. The early history is covered in the encyclopedic paper by Kottler (1922). The present techniques of this problem were elaborated by Einstein, Infeld, Fock and then advanced by Thorne, Damour, Kopeikin, Schäfer and many others.

As first indicated by Infeld and exposed in Lecture 3 the post–Newtonian equations of motion may be derived from the field variational principle just using the linearized metric with $\underset{2}{h_{00}}$, $\underset{3}{h_{0i}}$, $\underset{2}{h_{ij}}$ alone resulting to the Lagrangian (3.100). The famous EIH (Einstein–Infeld–Hoffman) BRS equations of motion of the Sun and major planets considered as point masses result just from this linearized metric. This metric contains only Newtonian potential and vector–potential. In present practice the non–point structure of the bodies is taken into account only at the Newtonian level in the Newtonian parts of the post–Newtonian equations of motion. To take into account the non–point structure at the post–Newtonian level (mainly for advanced research study) it is to be reminded that such body characteristics (rotation velocity, multipole moments, etc.) should be considered in a body reference system. In so doing it is sufficient to use the post–Newtonian theory of reference system transformations in the equations obtained in a global reference system (Lecture 7) not demanding $\underset{4}{h_{00}}$ (both in a global system and a local body–related system). It involves relativistic contributions to the parameters occurring in U and U^i. The post–Newtonian equations of rotation of bodies also may be derived from the variational principle.

Considering that most results in the problem of motion were obtained by the PNA techniques with using $\underset{4}{h_{00}}$ term we reproduce below the main points of that approach. As was indicated earlier the N–body field metric (in harmonic coordinates)reads

$$\begin{aligned} g_{00} &= 1 - c^{-2} 2U + c^{-4}(U^2 - W) + \dots, \\ g_{0i} &= c^{-3} 4U^i + \dots, \\ g_{ik} &= -\delta_{ik} - c^{-2} 2U\delta_{ik} + \dots \end{aligned} \tag{8.4}$$

with

$$W = \tfrac{3}{2}\Phi_1 - \Phi_2 + \Phi_3 + 3\Phi_4 + c^2\chi_{,00}\,. \tag{8.5}$$

The occurring potential functions satisfy the equations

$$U_{,ss} = -4\pi G\rho\,, \qquad U^i_{,ss} = -4\pi G\rho v^i\,, \qquad \chi, ss = U\,,$$

$$\Phi_{1,ss} = -4\pi G\rho v^2\,, \quad \Phi_{2,ss} = -4\pi G\rho U\,, \quad \Phi_{3,ss} = -4\pi G\rho\Pi\,, \quad \Phi_{4,ss} = -4\pi G\rho$$

(under designations of Lecture 3).

Equations of motion of a test particle in the N–body field (8.4) (the restricted problem of $N+1$ bodies) may be presented in the form

$$\ddot{x}^i = U_{,i} + c^{-2}G^i \tag{8.6}$$

with

$$G^i = -4UU_{,i} - U_{,k}\dot{x}^k\dot{x}^i + U_{,i}\dot{x}^k\dot{x}^k - 3\dot{U}\dot{x}^i + 4\dot{U}^i - 4U^k_{,i}\dot{x}^k + W_{,i}\,. \tag{8.7}$$

As usually, the dot means here the total time derivative, e.g.

$$\dot{U} = cU_{,0} + U_{,k}\dot{x}^k\,.$$

These equations may be presented in the Lagrangian form with the Lagrangian

$$L = \tfrac{1}{2}\dot{x}^k\dot{x}^k + U + c^{-2}\big[\tfrac{1}{8}(\dot{x}^k\dot{x}^k)^2 + \tfrac{3}{2}U\dot{x}^k\dot{x}^k - 4U^k\dot{x}^k - \tfrac{1}{2}U^2 + W\big]. \tag{8.8}$$

Geodesic principle is applied rigorously only to the test (massless) particles. However, the equations of the N–body problem still may be presented in the form close to the geodesic equations of the test particles. Splitting the potential functions into internal and external (regular with respect to the substitution $\mathbf{x} = \mathbf{x}_E$) parts

$$U = U_E + \bar{U}_E\,,\ U^i = U^i_E + \bar{U}^i_E\,,\ W = W_E + \bar{W}_E\,,\ \chi = \chi_E + \bar{\chi}_E \tag{8.9}$$

one may write the BRS equations of the motion of the Earth in the form

$$a^i_E = \bar{U}_{E,i}(\mathbf{x}_E) + H^i_E + c^{-2}\bar{G}^i_E \tag{8.10}$$

with

$$\begin{aligned}\bar{G}^i_E =& -4\bar{U}_E(\mathbf{x}_E)\bar{U}_{E,i}(\mathbf{x}_E) - \bar{U}_{E,k}(\mathbf{x}_E)v^k_E v^i_E + \bar{U}_{E,i}(\mathbf{x}_E)v^2_E - \\ &- 3\dot{\bar{U}}_E(\mathbf{x}_E)v^i_E + 4\dot{\bar{U}}^i_E(\mathbf{x}_E) - 4\bar{U}^k_{E,i}(\mathbf{x}_E)v^k_E + \bar{W}_{E,i}(\mathbf{x}_E)\end{aligned} \tag{8.11}$$

and non–geodesic acceleration designated in (7.11) as $-Q_i$

$$H^i_E \equiv -Q_i = \tfrac{1}{2}M_E^{-1}I^{km}_E\bar{U}_{E,ikm}(\mathbf{x}_E) + \dots\,, \tag{8.12}$$

I^{km}_E being quadrupole moments of inertia of the Earth

$$I^{km}_E = \int_{(E)} \rho'(x'^k - x^k_E)(x'^m - x^m_E)d^3x'\,. \tag{8.13}$$

BRS equations for any other solar system body (the Sun, major planets, etc.) are written in the similar manner.

The classical multipole expansions for the Newtonian potential reads

$$U = \sum_A G \int_{(A)} \frac{\rho' d^3x'}{|\mathbf{x}-\mathbf{x}'|} = \sum_A G\Big[\frac{M_A}{r_A} + I_A^k \frac{r_A^k}{r_A^3} + \frac{1}{2r_A^3} I_A^{km}\Big(-\delta_{km} + \frac{3}{r_A^2} r_A^k r_A^m\Big) + \ldots\Big]. \quad (8.14)$$

Here and below we use previous designations $r_A^k = x^k - x_A^k$, $r_{AB}^k = x_A^k - x_B^k$. The monopole expansion is based on

$$|\mathbf{x}-\mathbf{x}'|^n = (r_A)^n \Big[1 - \frac{n}{r_A^2} r_A^k r_A'^k + \frac{n}{2r_A^2}\Big(r_A'^k r_A'^k + \frac{n-2}{r_A^2}(r_A^k r_A'^k)^2\Big) + \ldots\Big]$$

and involves mass

$$M_A = \int_{(A)} \rho' d^3x' \quad (8.15)$$

and monopole moments starting with the dipole moments

$$I_A^k = \int_{(A)} \rho'(x'^k - x_A^k) d^3x' \, . \quad (8.16)$$

Velocity distribution inside body A in Newtonian approximation is assumed to correspond to the rigid–body rotation (to be used in the relativistic terms)

$$v^i = v_A^i + \varepsilon_{ijk}\omega_A^j r_A^k \, , \quad (8.17)$$

with BRS angular rotation velocity ω_A^j. Time–derivatives of the inertia moments satisfy in the Newtonian approximation the relations

$$\dot{I}_A^k = \varepsilon_{kjn}\omega_A^j I_A^n \, , \qquad \dot{I}_A^{km} = (\varepsilon_{kjn} I_A^{mn} + \varepsilon_{mjn} I_A^{kn})\omega_A^J \, . \quad (8.18)$$

Newtonian vector–potential is given by

$$U^i = \sum_A G \int_{(A)} \frac{\rho' v'^i d^3x'}{|\mathbf{x}-\mathbf{x}'|} = \sum_A G\Big[\frac{M_A}{r_A} v_A^i + \varepsilon_{ijk}\omega_A^j I_A^{km} \frac{r_A^m}{r_A^3} + \\ + \frac{1}{2r_A^3} I_A^{km}\Big(-\delta_{km} + \frac{3}{r_A^2} r_A^k r_A^m\Big) v_A^i + \ldots\Big]. \quad (8.19)$$

Retaining only monopoles and rotation terms the additive potential will be

$$W = \sum_A G \int_{(A)} (\tfrac{3}{2}\rho v^2 - \rho U + \rho\Pi + 3p)' \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|} + \frac{\partial^2\chi}{\partial t^2} = \\ = \sum_A G\Big[\frac{3}{2}\frac{M_A}{r_A} v_A^2 + 3\varepsilon_{kjn}\omega_A^j v_A^k I_A^{mn} \frac{r_A^m}{r_A^3} - \frac{M_A}{r_A} \sum_{B\neq A} \frac{GM_B}{r_{AB}}\Big] + \frac{\partial^2\chi}{\partial t^2} + \ldots \quad (8.20)$$

with

$$\chi = \tfrac{1}{2}\sum_A G\int_{(A)} \rho' |\mathbf{x}-\mathbf{x}'| d^3x' = \tfrac{1}{2}\sum_A GM_A r_A + \dots . \tag{8.21}$$

Presently, the BRS N–body relativistic equations of motion are in use in form of the EIH equations with some additive terms in the Newtonian parts to produce numerical ephemerides of the major planets and the Moon. It does not mean that in the planetary ephemerides the planetary relativistic terms are indeed relevant. In producing numerical ephemerides it is simpler to use the complete EIH equations available rather than to make estimates of different relativistic contributions. In any case the EIH equations involving the non–rotating point–mass model are more than sufficient for practical purposes and there is no need to introduce more complicated model in relativistic terms to take into account the structure and proper rotation of the bodies.

For the non–rotating point–mass model the integral of the motion of the center of mass is

$$\sum_A \tilde{M}_A x^i_A = K^i t + N^i \tag{8.22}$$

with the Tolman mass of body A

$$\tilde{M}_A = M_A\Big(1 + \tfrac{1}{2}c^{-2}v_A^2 - \tfrac{1}{2}c^{-2}\sum_{B\neq A}\frac{GM_B}{r_{AB}}\Big). \tag{8.23}$$

By introducing the Tolman mass (conserving) of the system of bodies

$$\tilde{M} = \sum_A \tilde{M}_A\,, \qquad \dot{\tilde{M}} = 0 \tag{8.24}$$

the coordinates of the center of mass will be

$$\tilde{M}\tilde{X}^i = \sum_A \tilde{M}_A x^i_A = K^i t + N^i\,. \tag{8.25}$$

The BRS system is defined by conditions $K^i = N^i = 0$.

The BRS equations of motion involving Newtonian potentials and vector–potentials contain multipole moments of inertia and angular rotation velocities of the bodies. Being referred to BRS these quantities have no physical meaning and should be considered just as coordinate quantities. It does not affect the applicability of the BRS equations of motion but to construct a GRT–consistent system of astronomical constants one should consider the physical characteristics of a body in a body centered RS. It enables one to get the body characteristics as the coordinate–independent quantities (at least within the post–Newtonian approximation). This may be done by using DSX (Damour–Soffel–XU) approach in constructing relativistic RSs. Even earlier this problem was treated just by RS theory exposed in Lecture 7. To illustrate this approach we will consider BRSC and GRSC quantities relating to rotation and multipole structure of the Earth.

Assuming the Earth's rigid–body rotation in DGRS

$$\hat{v}^i = \varepsilon_{ijk}\hat{\omega}^j \hat{x}^k \tag{8.26}$$

the corresponding rotation in BRS may be presented in the form

$$v^i = v^i_E + \varepsilon_{ijk}\omega^j (x^k - x^k_E) + c^{-2} f^i \,, \tag{8.27}$$

where corrections f^i are determined from the differentials of BRS↔GRS transformation (Brumberg, 1995) and vanish only in the geocenter. Generally, $v^i \neq v^i_E$ for $\omega^i = 0$ demonstrating that the rigid–body rotation depends on RS. Returning to the relations of the previous lecture involving the Earth's rotation matrix ($P(t)$ in BRS, $\hat{P}(\hat{t})$ in GRS) one has

$$\hat{P}(\hat{t}) = P(t^*) - c^{-2} q P F_C \,, \qquad [\mathrm{GRS}^+] = \underset{q}{\hat{P}}(\hat{t})[\mathrm{GRSC}], \qquad [\mathrm{BRS}^+] = P(t)[\mathrm{BRSC}].$$

These relations result in the corresponding relations for angular rotation velocities as follows:

$$\omega^i = \tfrac{1}{2}\varepsilon_{ijk} P_{mj} \dot{P}_{mk} \,, \qquad \hat{\omega}^i = \tfrac{1}{2}\varepsilon_{ijk} \hat{P}_{mj} \dot{\hat{P}}_{mk} \,, \tag{8.28}$$

$$t^* = \hat{t} + c^{-2} A(t) \,, \tag{8.29}$$

$$\hat{\omega}^i(\hat{t}) = \omega^i(t^*) + c^{-2}(\dot{A}\omega^i - q\dot{F}^i_C + q\varepsilon_{ijk}\omega^j F^k_C) \tag{8.30}$$

(the dot denotes here the derivative with respect to the corresponding time argument).

In virtue of $\rho' d^3x' = \hat{\rho}' d^3\hat{x}'$ one has the mass relation

$$M_E = \hat{M}_E \,. \tag{8.31}$$

BRS and GRS multipole moments involve volume integrals over hypersurfaces t=const and $\hat{t}$=const, respectively. Any quantity A defined on t=const takes value $\tilde{A}$ on $\hat{t}$=const in result of the Lie transfer

$$\tilde{A} = A + v'^i A_{,i}\Delta t \,, \qquad \Delta t = t' - t = c^{-2} v^n_E (x'^n - x^n) \,, \qquad v'^i = \varepsilon_{ikm}\omega^k_E r'^m_E \,. \tag{8.32}$$

Therefore,

$$I^i_E = \hat{I}^i_E + c^{-2}(\tfrac{1}{2} a^i_E \hat{I}^{kk}_E - a^k_E \hat{I}^{ik}_E - \varepsilon_{ikm}\omega^k_E v^n_E \hat{I}^{mn}_E) \,, \tag{8.33}$$

$$\begin{aligned} I^{ik}_E(t^*) = \left[1 - 2c^{-2}\bar{U}(t, \mathbf{x}_E)\right] \hat{I}^{ik}_E(u) - \tfrac{1}{2} c^{-2} v^m_E (v^i_E \hat{I}^{km}_E + v^k_E \hat{I}^{im}_E) - \\ - c^{-2} q(\varepsilon_{imn}\hat{I}^{km}_E + \varepsilon_{kmn}\hat{I}^{im}_E) F^n_C \end{aligned} \tag{8.34}$$

The relation $\hat{I}^i_E = 0$ is used for the definition of the geocenter as a spatial origin of GRS. It involves the dipole term of the relativistic order of smallness in the BRS multipole expansion of the Newtonian potential. Since vector–potential and additive potential enter only in the relativistic terms such dipole term is omitted there.

The concept of sphericity also depends on RS. It is reasonable to define a spherical Earth in GRS by relation

$$\hat{I}_E^{km} = \delta_{km}\hat{I}_E \, . \tag{8.35}$$

From this it follows

$$I_E^i = c^{-2}(\tfrac{1}{2}a_E^i - \varepsilon_{ikm}\omega_E^k v_E^m)\hat{I}_E \, , \tag{8.36}$$

$$I_E^{ik} = [1 - 2c^{-2}\bar{U}(t, \mathbf{x}_E)]\hat{I}_E\delta_{ik} - c^{-2}v_E^i v_E^k \hat{I}_E \, , \tag{8.37}$$

demonstrating that the BRS quadrupole matrix is not diagonal.

The GRS quadrupole moments are similar to BD (Blanchet–Damour) moments used in the DSX approach and recommended for use by IAU (2000) Resolutions. In fact, it is possible to use the BRS equations with Newtonian potentials taking into account the relations like (8.34), (8.36), (8.37).

Gravitational radiation and motion in a binary pulsar

Virtually, the post–Newtonian ($\mathrm{O}(c^{-2})$) and post–post–Newtonian ($\mathrm{O}(c^{-4})$) equations of motion of the N–body problem do not differ qualitatively from the Newtonian equations. In all cases the system of equations may be presented in the Lagrange form with corresponding consequences (long–term evolution, stability questions, etc.). Qualitative difference starts with ($\mathrm{O}(c^{-5})$) equations of motion involving dissipative terms due to gravitational radiation and breaking the Lagrange form of the equations. But even before these ($\mathrm{O}(c^{-5})$) equations were derived and studied (by Damour, and Grishchuk and Kopeikin) it was stated that the system of gravitating masses loses energy by radiating gravitating waves. The loss of energy determined from the linearized field equations is given by the expression

$$\frac{dW}{dt} = -\frac{G}{5c^5}B_{ik}B_{ik} \, , \qquad B_{ik} = \frac{d^3}{dt^3}D_{ik}(t), \tag{8.38}$$

$D_{ik}(t)$ being the quadrupole moments of the system or, for the system of point masses,

$$D_{ik}(t) = \sum_A M_A(x_A^i x_A^k - \tfrac{1}{3}\delta_{ik}x_A^m x_A^m). \tag{8.39}$$

This formula has been applied to calculate the loss of energy in the binary pulsar system PSR 1913+16. Assuming that the motion is performed in the plane $z = 0$ one has for the coordinates of masses M_1 and M_2

$$x_1 + \mathrm{i}\, y_1 = -\frac{M_2}{M} r \exp \mathrm{i}\, f \, , \qquad x_2 + \mathrm{i}\, y_2 = \frac{M_1}{M} r \exp \mathrm{i}\, f \, , \qquad M = M_1 + M_2 \, , \tag{8.40}$$

so that

$$D_{11} = \frac{M_1 M_2}{M} r^2 (\tfrac{1}{6} + \tfrac{1}{2}\cos 2f), \qquad D_{22} = \frac{M_1 M_2}{M} r^2 (\tfrac{1}{6} - \tfrac{1}{2}\cos 2f),$$

$$D_{33} = -\frac{M_1 M_2}{3M} r^2 \, , \qquad D_{12} = \frac{M_1 M_2}{2M} r^2 \sin 2f \, ,$$

f being the true anomaly. Therefore,

$$\begin{aligned}\frac{dW}{dt} &= -\frac{G}{5c^5}(B_{11}^2 + B_{22}^2 + B_{33}^2 + 2B_{12}^2) = \\ &= -\frac{G}{5c^5}\left(\frac{M_1 M_2}{M}\right)^2 \frac{n^6 a^8}{r^4(1-e^2)}(32 + \tfrac{52}{3}e^2 + 64e\cos f + \tfrac{44}{3}e^2\cos 2f). \qquad (8.41)\end{aligned}$$

Using the Hansen coefficients

$$X_0^{-4,0} = \frac{1+\frac{1}{2}e^2}{(1-e^2)^{5/2}}\,, \qquad X_0^{-4,1} = \frac{e}{(1-e^2)^{5/2}}\,, \qquad X_0^{-4,2} = \frac{e^2}{4(1-e^2)^{5/2}}$$

one gets for the averaged value of the loss of energy in motion on the elliptic orbit the well–known formula

$$\left[\frac{dW}{dt}\right] = -\frac{32G}{5c^5}\left(\frac{M_1 M_2}{M}\right)^2 \frac{n^6 a^4}{(1-e^2)^{7/2}}\left(1 + \tfrac{73}{24}e^2 + \tfrac{37}{96}e^4\right). \qquad (8.42)$$

Since the total energy of the two–body problem is related to the semi–major axis by

$$W = -\frac{GM_1M_2}{2a}\,, \qquad n^2a^3 = GM \qquad (8.43)$$

one obtains the rate of the decrease of the semi–major axis

$$\dot{a} = -\frac{64}{5}\frac{G^3 M_1 M_2 M}{c^5 a^3 (1-e^2)^{7/2}}\left(1 + \tfrac{73}{24}e^2 + \tfrac{37}{96}e^4\right). \qquad (8.44)$$

This involves the time–quadratic term in the mean longitude and a secular decrease of the period of motion confirmed by observations.

This result is based on the linearized field equations involving the quadrupole formula. No wonder that this approach caused doubt in two aspects, i.e. 1) the validity of the quadrupole formula for compact objects (with U/c^2 being not small), 2) correctness of the value $\dot{a}$. These doubts were completely removed when the analysis of the equations of motion within radiation approximation (c^{-5}) derived by the authors indicated above had led to the same results.

The equations of motion of two compact bodies with consideration of gravitational radiation are originally of the form

$$\begin{aligned}a_1^i = &\underset{0}{F_1^i}(\mathbf{x}_1 - \mathbf{x}_2) + c^{-2}\underset{2}{F_1^i}(\mathbf{x}_1 - \mathbf{x}_2,\, \mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{a}_1,\, \mathbf{a}_2) + \\ &+ c^{-4}\underset{4}{F_1^i}(\mathbf{x}_1 - \mathbf{x}_2,\, \mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{a}_1,\, \mathbf{a}_2,\, \dot{\mathbf{a}}_1,\, \dot{\mathbf{a}}_2) + \\ &+ c^{-5}\underset{5}{F_1^i}(\mathbf{x}_1 - \mathbf{x}_2,\, \mathbf{v}_1 - \mathbf{v}_2,\, \mathbf{a}_1 - \mathbf{a}_2,\, \dot{\mathbf{a}}_1 - \dot{\mathbf{a}}_2,\, \ddot{\mathbf{a}}_1 - \ddot{\mathbf{a}}_2,\, \dddot{\mathbf{a}}_1 - \dddot{\mathbf{a}}_2) + \ldots . \qquad (8.45)\end{aligned}$$

By eliminating in the right–hand members the higher order derivatives be means of Newtonian and post–Newtonian equations one gets the reduced equations with the right–hand members dependent only on the coordinates and velocities

$$\ddot{x}_1^i = \underset{0}{A_1^i}(\mathbf{x}_1 - \mathbf{x}_2) + c^{-2}\,\underset{2}{A_1^i}(\mathbf{x}_1 - \mathbf{x}_2,\, \mathbf{v}_1,\, \mathbf{v}_2) + c^{-4}\,\underset{4}{A_1^i}(\mathbf{x}_1 - \mathbf{x}_2,\, \mathbf{v}_1,\, \mathbf{v}_2) + \\ + c^{-5}\,\underset{5}{A_1^i}(\mathbf{x}_1 - \mathbf{x}_2,\, \mathbf{v}_1 - \mathbf{v}_2) + \dots\,. \tag{8.46}$$

Using the integral of the center of mass motion one may obtain the equations of relative motion

$$\ddot{x}^i = \underset{0}{B^i} + c^{-2}\,\underset{2}{B^i} + c^{-4}\,\underset{4}{B^i} + c^{-5}\,\underset{5}{B^i} + \dots\,. \tag{8.47}$$

With designations

$$N^i = (x_1^i - x_2^i)r^{-1}\,, \qquad v^i = v_1^i - v_2^i\,, \qquad r = [(\mathbf{x}_1 - \mathbf{x}_2)^2]^{1/2}\,, \qquad \mu = \frac{M_1 M_2}{M^2}$$

the right–hand members are as follows:

Newtonian terms:

$$\underset{0}{B^i} = -\frac{GM}{r^3}x^i\,, \tag{8.48}$$

Post–Newtonian terms:

$$\underset{2}{B^i} = \frac{GM}{r^3}\left\{\Big[(4+2\mu)\frac{GM}{r} + \tfrac{3}{2}\mu(\mathbf{N}\mathbf{v})^2 - (1+3\mu)\mathbf{v}^2\Big]x^i + (4-2\mu)(\mathbf{x}\mathbf{v})v^i\right\}, \tag{8.49}$$

Post–post–Newtonian terms:

$$\underset{4}{B^i} = \frac{GM}{r^3}\left\{\Big[\mu(-3+4\mu)(\mathbf{v}^2)^2 + \tfrac{15}{8}\mu(-1+3\mu)(\mathbf{N}\mathbf{v})^4 + \mu(\tfrac{9}{2} - 6\mu)(\mathbf{N}\mathbf{v})^2\mathbf{v}^2 + \right. \\ + \mu(\tfrac{13}{2} - 2\mu)\frac{GM}{r}\mathbf{v}^2 + (2 + 25\mu + 2\mu^2)\frac{GM}{r}(\mathbf{N}\mathbf{v})^2 - (9 + \tfrac{87}{4}\mu)\frac{G^2M^2}{r^2}\Big]x^i + \\ \left. + \Big[\mu(\tfrac{15}{2} + 2\mu)\mathbf{v}^2 - \mu(\tfrac{9}{2} + 3\mu)(\mathbf{N}\mathbf{v})^2 - (2 + \tfrac{41}{2}\mu + 4\mu^2)\frac{GM}{r}\Big](\mathbf{x}\mathbf{v})v^i\right\}, \tag{8.50}$$

Gravitational radiation terms:

$$\underset{5}{B^i} = \frac{8}{5}\frac{G^2M^2}{r^3}\mu\Big[\Big(3\mathbf{v}^2 + \tfrac{17}{3}\frac{GM}{r}\Big)(\mathbf{N}\mathbf{v})N^i - \Big(\mathbf{v}^2 + 3\frac{GM}{r}\Big)v^i\Big]. \tag{8.51}$$

Solution of equations (8.47) in all details is rather time–consuming. But the post–post–Newtonian solution caused by B_4^i and the quadratic contribution by B_2^i does not differ essentially from the known post–Newtonian solution. The gravitational radiation is caused by B_5^i terms. By (4.40), (4.41) applied to $F^i = B_5^i$ one finds the radiation perturbations

$$\dot{\mathbf{c}} = -\frac{8}{5}\frac{G^2M_1M_2}{c^5r^3}\Big(\mathbf{v}^2 + 3\frac{GM}{r}\Big)\mathbf{c}\,, \tag{8.52}$$

$$\dot{\mathbf{f}} = \frac{8}{5}\frac{G^2 M_1 M_2}{c^5 r^3}\Big\{\Big[\Big(3\mathbf{v}^2 + \frac{17}{3}\frac{GM}{r}\Big)(\mathbf{N}\mathbf{v})^2 - 2\Big(\mathbf{v}^2 + 3\frac{GM}{r}\Big)\mathbf{v}^2\Big]\mathbf{r} + \\ + \Big(-\mathbf{v}^2 + \frac{1}{3}\frac{GM}{r}\Big)(\mathbf{r}\mathbf{v})\mathbf{v}\Big\}. \tag{8.53}$$

Substituting the Keplerian values for the coordinates and velocities and performing averaging with the aid of the Hansen coefficients one obtains the equations for the secular perturbations

$$\dot{\mathbf{c}} = -\frac{4}{5}\frac{G^3 M M_1 M_2}{c^5 a^4}\frac{8 + 7e^2}{(1-e^2)^{5/2}}\mathbf{c}\,, \tag{8.54}$$

$$\dot{\mathbf{f}} = -\frac{1}{15}\frac{G^3 M M_1 M_2}{c^5 a^4}\frac{304 + 121e^2}{(1-e^2)^{5/2}}\mathbf{f}\,. \tag{8.55}$$

The corresponding expressions for the Keplerian elements will be

$$\frac{di}{dt} = 0\,,\ \frac{d\Omega}{dt} = 0\,,\ \frac{dp}{dt} = -\frac{8}{5}\frac{G^3 M M_1 M_2}{c^5 a^3}\frac{8 + 7e^2}{(1-e^2)^{3/2}}\,,\ \frac{d\omega}{dt} = 0\,, \tag{8.56}$$

$$\frac{de}{dt} = -\frac{1}{15}\frac{G^3 M M_1 M_2}{c^5 a^4}\frac{304 + 121e^2}{(1-e^2)^{5/2}}\,e\,. \tag{8.57}$$

Combination of these values leads again to (8.44) confirming all results of the quadrupole formula approach.

In no way minimizing the importance of advanced research in GRT problem of motion one cannot help to note that two practically used GRT solutions, i.e. EIH equations and binary pulsar motion presentation, result just from the linearized GRT field equations.

Irrespective of practical applications the motion of two compact masses with gravitational radiation taken into account is of great interest revealing new types of motion in celestial mechanics. On the basis of this problem one may consider, for example, a relativistic restricted three–body problem in the gravitational radiation approximation (the motion of a test particle in the binary pulsar field). To simplify the problem the motion of the massive binary components may be assumed to be quasi–circular. In terms of time these orbits have linearly changing radii and quadratically changing phase angles. By substituting this motion into the Newtonian–like equations of motion of a test particle one gets the quasi–Newtonian restricted quasi–circular three–body problem sufficient to take into account the main indirect perturbations caused by the binary radiation terms (Brumberg, 2003). In contrast to the evolution of motion in the classical restricted three–body problem the Lagrange libration points undergo secular trends whereas the quasi–circular orbits lying at large distance from the binary have linearly changing radii and quadratically changing phase angles.

There are still many open questions in the GRT problem of motion, e.g. the structure of the equations of motion with respect to the higher–order derivatives, Cauchy problem (what is the GRT statement for the Laplace determinism, i.e. the ability to predict the motion of all bodies of the Universe provided that their initial positions and velocities are known?), evolution of motion under gravitational radiation, celestial mechanics with gravitational waves, etc.

Lecture 9. Motion and rotation in the geocentric reference system

As indicated above the most adequate RS for the post–Newtonian consideration of the motion of the Moon, Earth's satellites and Earth's rotation is GRS. Needless to say, it is possible to use BRS for these objects as well but consideration in GRS is more compact and more straightforward. In so doing one may choose several options, e.g.:

1) starting with the GRT BRS equations and performing [BRS↔GRS] transformation to get GRT GRS equations of the problem under consideration,

2) to deal from the beginning with GRT GRS metric and GRT GRS equations,

3)starting with Newtonian GRS equations (complemented by relativistic perturbations from the Earth, if necessary) and performing [BRS↔GRS] transformation to derive indirect relativistic perturbations (main relativistic perturbations) of the problem at hand.

In this lecture we will consider just the third option.

Earth's satellite problem in GRSQ

Denoting GRS time and spatial coordinates by $u \equiv \hat{t}$, $w^i \equiv \hat{x}^i$, $\rho^2 = w^i w^i$ the equations of motion of the Earth's satellite problem in a geocentric equatorial RS may be presented in form (Brumberg, 2004)

$$\frac{d^2 w^i}{du^2} = \hat{U}_{E,i}(u,\, \mathbf{w}) + Q_i + R_{,i}(u,\, \mathbf{w}) + c^{-2}\Phi^i \tag{9.1}$$

with the right–hand members

$$\Phi^i = \sum_{k=0}^{3} \Phi^i_k\,. \tag{9.2}$$

Here R stands for the perturbing function of classical celestial mechanics

$$R = \sum_{A \neq E} R_A\,, \tag{9.3}$$

$$R_A = GM_A\left(\frac{1}{|\mathbf{w}-\mathbf{w}_A|} - \frac{1}{|\mathbf{w}_A|} - \frac{\mathbf{w}\mathbf{w}_A}{|\mathbf{w}_A|^3}\right), \tag{9.4}$$

(the second term not depending on the satellite coordinates and not affecting the equations of motion converts the perturbing function into the tidal potential)

$$R_{A,i} = -GM_A\left(\frac{w^i - w^i_A}{|\mathbf{w}-\mathbf{w}_A|^3} + \frac{w^i_A}{|\mathbf{w}_A|^3}\right). \tag{9.5}$$

To avoid the loss of accuracy the Encke transformation is often applied

$$R_{A,i} = -\frac{GM_A}{|\mathbf{w}_A|^3}[w^i + (w^i_A - w^i)f(\alpha_A)], \tag{9.6}$$

$$\alpha_A = \frac{1}{|\mathbf{w}_A|^2}(2w^k w^k_A - w^k w^k), \tag{9.7}$$

$$f(\alpha) = 1 - (1-\alpha)^{3/2} = \frac{\alpha(2-\alpha+\sqrt{1-\alpha})}{1+\sqrt{1-\alpha}}. \tag{9.8}$$

Newtonian terms due to the Earth are as follows:

$$\begin{aligned} U_{,i} = &-\frac{GM}{\rho^3}w^i + \frac{3Q}{\rho^5}\left[\frac{1}{2}\left(1 - 5\frac{{w^3}^2}{\rho^2}\right)w^i + w^3\delta_{3i}\right] + \\ &+ \frac{3G(A-B)}{4\rho^5}\left[\left(5\frac{{w^1}^2 - {w^2}^2}{\rho^2}w^i - 2w^1\delta_{1i} + 2w^2\delta_{2i}\right)\cos 2\varphi + \right. \\ &\left. + \left(10\frac{w^1w^2}{\rho^2}w^i - 2w^2\delta_{1i} - 2w^1\delta_{2i}\right)\sin 2\varphi\right], \end{aligned} \tag{9.9}$$

$$Q = G\Big(\frac{A+B}{2} - C\Big),$$

A, B, C being the Earth's principal moments of inertia and φ standing for the linear function of time u with the frequency equal to the Earth's angular velocity. Q_i in (9.1) is the non–geodesic Earth BRS acceleration

$$Q_i = -\frac{1}{2}M_E^{-1}I_E^{km}\bar{U}_{E,ikm}(t,\,\mathbf{x}_E)$$

or

$$\begin{aligned} Q_i = &-\frac{1}{2}(GM_E)^{-1}Q\bar{U}_{E,i33}(t,\,\mathbf{x}_E) + \frac{1}{2}M_E^{-1}(A-B)\Big[\bar{U}_{E,i12}(t,\,\mathbf{x}_E)\,\sin 2\varphi + \\ &+ \frac{1}{2}\Big(\bar{U}_{E,i11}(t,\,\mathbf{x}_E) - \bar{U}_{E,i22}(t,\,\mathbf{x}_E)\Big)\cos 2\varphi\Big]. \end{aligned} \tag{9.10}$$

Direct relativistin (one–body) perturbations involve

Geodesic rotation in GRSQ:

$$\delta\hat{h}_{0i} = c^{-3}(1-q)\varepsilon_{ijk}\dot{F}_Q^j w^k$$

resulting in

$$\Phi_0^i = (1-q)\varepsilon_{ijk}(2\dot{F}_Q^j\dot{w}^k + \ddot{F}_Q^j w^k) \tag{9.11}$$

(as previously parameter q is introduced to distinguish between K ($q=0$) and D ($q=1$) versions of GRS).

Schwarzschild perturbations:

$$\Phi_1^i = \frac{GM}{c^2\rho^3}\Big[\Big(4\frac{GM}{\rho} - \dot{w}^k\dot{w}^k\Big)w^i + 4(w^k\dot{w}^k)\dot{w}^i\Big]. \tag{9.12}$$

Lense–Thirring perturbations:

$$\begin{aligned} \Phi_2^i = &\frac{2G\omega}{c^2\rho^3}C\Big(\varepsilon_{ijk}\dot{w}^j s^k + \frac{3}{\rho^2}w^3\varepsilon_{ijk}w^j\dot{w}^k\Big) + \\ &+ \frac{2G\omega}{c^2\rho^5}(A-B)\big\{[3w^i(w^1\dot{w}^1 - w^2\dot{w}^2) + 3(w^2\delta_{i2} - w^1\delta_{i1})w^k\dot{w}^k + \\ &+ 2\rho^2\omega(w^1\delta_{i2} + w^2\delta_{i1})]\sin 2\varphi + [-3w^i(w^2\dot{w}^1 + w^1\dot{w}^2) + \\ &+ 3(w^1\delta_{i2} + w^2\delta_{i1})w^k\dot{w}^k + 2\rho^2\omega(w^1\delta_{i1} - w^2\delta_{i2})]\cos 2\varphi\big\}. \end{aligned} \tag{9.13}$$

Relativistic figure (non–sphericity) perturbations:

$$\Phi_3^i = \frac{Q}{c^2\rho^5}\left\{\left[4\left(-2+9\frac{w^{3\,2}}{\rho^2}\right)\frac{GM}{\rho}+\frac{3}{2}\left(1-5\frac{w^{3\,2}}{\rho^2}\right)\dot{w}^k\dot{w}^k\right]w^i+\right.$$
$$\left.+3\left(-4\frac{GM}{\rho}+\dot{w}^k\dot{w}^k\right)w^3\delta_{3i}-6\left[\left(1-5\frac{w^{3\,2}}{\rho^2}\right)w^k\dot{w}^k+2w^3\dot{w}^3\right]\dot{w}^i\right\}. \qquad (9.14)$$

Absent functions Φ_k^i for $k=4,5,6$ represent relativistic tidal third–body perturbations in the GRS representation of the satellite motion. They are divided into coupling Earth – third–body perturbations ($k=4$), gravimagnetic third–body perturbations depending on the satellite velocity ($k=5$) and gravielectric third–body perturbations not depending on the satellite velocity ($k=6$).

Indirect relativistic third–body perturbations result from applying (7.42), i.e.

$$w_A^i(u)=z_A^i(u)-z_E^i(u)+c^{-2}\left[\Lambda^i(t^*,\,\mathbf{r}_{AE})+\mathbf{v}_E\mathbf{r}_{AE}(v_A^i-v_E^i)\right], \qquad z_A^i(u)=x_A^i(t^*). \quad (9.15)$$

Hence, substituting into R_A only the 'Newtonian' part $w_A^i(u)=z_A^i(u)-z_E^i(u)$ and using the expression for Φ_k^i with $k=0,1,...,6$ one gets the complete post–Newtonian GRS equations of satellite motion. Substituting into R_A the complete expression (9.15) and using expression for Φ_k^i only with $k=0,1,2,3$ one gets GRS equations of satellite motion with taking into account the indirect relativistic third–body perturbations. This might be quite sufficient for most practical applications related to the Earth's satellite motion.

Earth's rotation problem in DGRSC:
SMART97 (Bretagnon et al., 1997, 1998) represents the most accurate semi–analytical theory of rotation of the rigid Earth constructed so far. SMART97 is a purely Newtonian theory. Its conversion into relativistic theory was envisaged in (Bretagnon and Brumberg, 2003) and was continued after the death of Pierre Bretagnon in (Brumberg and Simon, 2003, 2007).

One may find in literature the relativistic equations of the Earth's equations of different type in dependence on adopted Earth's model. Instead of dealing with such complicated equations it is reasonable to start by taking into account in SMART97 the relativistic indirect third–body perturbations. In doing so, one neglects by very small direct relativistic third–body perturbations. It enables one to retain the formally Newtonian differential equations of the Earth's rotation and to get the relativistic extension of SMART97 solution by applying in the right–hand members of these equations the four–dimensional transformation between geocentric and barycentric quantities. It leads to the main relativistic terms in the Earth's rotation problem called in Lecture 7 the relativistic indirect third–body perturbations.

This relativistic extension of SMART97 is based completely on the two–level (wide) RS hierarchy of Lecture 7. Since SMART97 is supposed to be constructed in DGRSC three Euler angles, $\underset{1}{\psi}$, $\underset{1}{\theta}$, $\underset{1}{\varphi}$, of matrix $\underset{1}{\hat{P}}(\hat{t})$ may be regarded as dynamical Earth orientation parameters (EOP). The analogous Euler angles $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$, of matrix $\underset{0}{\hat{P}}(\hat{t})$ relating KGRSC and ITRS may be regarded as kinematical EOP. One has

$$\hat{\underset{q}{P}}(\hat{t}) = D_3(\underset{q}{\varphi})D_1(-\underset{q}{\theta})D_3(-\underset{q}{\psi}) \qquad (q = 0,\, 1), \tag{9.16}$$

D_i being the elementary rotation matrices. The dynamical and kinematical Euler angles are related by the formulae

$$\underset{1}{\varphi} - \underset{0}{\varphi} = -\frac{c^{-2}}{\sin\theta}\left(F_C^1 \sin\psi + F_C^2 \cos\psi\right), \tag{9.17}$$

$$\underset{1}{\theta} - \underset{0}{\theta} = c^{-2}\left(F_C^1 \cos\psi - F_C^2 \sin\psi\right), \tag{9.18}$$

$$\underset{1}{\psi} - \underset{0}{\psi} = c^{-2}\left[F_C^3 - \frac{\cos\theta}{\sin\theta}\left(F_C^1 \sin\psi + F_C^2 \cos\psi\right)\right] \tag{9.19}$$

(in the post–Newtonian approximation there is no need to distinguish between Newtonian and relativistic values in the relativistic right–hand members). These relationships have been actually used in SMART. In taking into account only the geodesic precession and nutation in narrow sense, one has $F_C^1 = F_C^2 = 0$ and, hence, $\underset{1}{\varphi} = \underset{0}{\varphi}$, $\underset{1}{\theta} = \underset{0}{\theta}$.

Note that to get the designations of the original papers on SMART (Bretagnon et al., 1997, 1998) one should put $\psi = -\psi$ and $\theta = -\omega$.

Repeating for the sake of convenoence the discussion of the RS hierarchy of Lecture 7 let us note that the Earth's rotation matrix relating GCRS and ITRS is determined in our notation as $T = \hat{\underset{0}{P}}(\hat{t})P_C$. The Earth's rotation in BRS may be described by the rotation matrix

$$P(t^*) = \hat{\underset{0}{P}}(\hat{t}) \tag{9.20}$$

where t^* is the solution of the relativistic time equation

$$\hat{t} = t^* - c^{-2}A(t^*) \tag{9.21}$$

with the time function determined by

$$\dot{A}(t) = \tfrac{1}{2}\mathbf{v}_E^2 + \bar{U}_E(t, \mathbf{x}_E), \qquad \bar{U}_E(t, \mathbf{x}_E) = \sum_{A\neq E} \frac{GM_A}{r_{EA}}. \tag{9.22}$$

However, rotating system BRS$^+$ is not used in practice.

Within the RS hierarchy the VSOP theories are constructed in BRSC with TDB as a time argument while SMART97 is considered in DGRSC with TT as a time argument. Therefore, the first thing in treating SMART in the relativistic framework is to adjust the values of masses taking into account that $(GM)_{\mathrm{TDB}}$ coefficients in VSOP and $(GM)_{\mathrm{TT}}$ coefficients in SMART are related by

$$(GM)_{\mathrm{TT}} = (1 + L_C)(GM)_{\mathrm{TDB}} \tag{9.23}$$

with the value of L_C obtained with the VSOP solution (Bretagnon and Brumberg, 2003). But this mass–adjustment may not be made in the actual work. Indeed, the main perturbation factors in the right–hand members of the DGRSC equations of the Earth's rotation, are due to the action of the Sun (S) and the Moon (L). Inially, these right–hand members contain geocentric position vectors $\mathbf{w}_A$ for $A = S,\, L$. These geocentric vectors are to be expressed by virtue of BRSC↔DGRSC transformation in terms of BRSC quantities as follows:

$$w^i_A(u) = z^i_A(u) - z^i_E(u) + c^{-2}\big[\Lambda^i(t^*,\, \mathbf{r}_{AE}) + \mathbf{v}_E\mathbf{r}_{AE}v^i_{AE}\big], \tag{9.24}$$

with $\mathbf{x}_E$, $\mathbf{v}_E$, $\mathbf{x}_A$, $\mathbf{v}_A$ denoting BRSC coordinates and velocities of the Earth and the disturbing body, respectively, $\mathbf{r}_{AE} = \mathbf{x}_A - \mathbf{x}_E$, $\mathbf{v}_{AE} = \mathbf{v}_A - \mathbf{v}_E$ and

$$\Lambda^i(t,\, \mathbf{r}_{AE}) = \tfrac{1}{2}\mathbf{v}_E\mathbf{r}_{AE}v^i_E - q\varepsilon_{ijk}F^j r^k_{AE} + \bar{U}_E(t, \mathbf{x}_E)r^i_{AE} + \mathbf{a}_E\mathbf{r}_{AE}r^i_{AE} - \tfrac{1}{2}\mathbf{r}^2_{AE}a^i_E\,, \tag{9.25}$$

$\mathbf{a}_E$ being BRSC acceleration of the Earth. We use here again more simple designations for GRS quantities $u = \hat{t}$ and $w^i = \hat{x}^i$. The moment t^* means here

$$\mathrm{TDB}^* = \mathrm{TT} + c^{-2}A_p \tag{9.26}$$

if time function $A(t)$ is represented in TDB as

$$A(t) = c^2 L_C t + A_p(t). \tag{9.27}$$

The function z^i_E representing the BRSC position of the Earth in terms of TCG or TT is given in our case by

$$(1 - L_C)z^i_E(\mathrm{TT}) = x^i_E(\mathrm{TDB}^*) = x^i_E(\mathrm{TT}) + c^{-2}A_p v^i_E + \ldots\,. \tag{9.28}$$

The function z^i_A is determined by the same formula by replacing E for A.

Bretagnon and Brumberg (2003) have used the VSOP theories to compute the initial terms of the series for $\mathbf{x}_E$, $\mathbf{v}_E$, $c^{-2}\mathbf{a}_E$, $c^{-2}\bar{U}_E(t, \mathbf{x}_E)$, $c^{-2}A_p$, $c^{-2}A_p\mathbf{v}_E$, and $c^{-2}\mathbf{F}$ in function of t =TDB. These series are to be used to evaluate explicitly the coefficients in BRS↔GRS] transformation. For the sake of completeness the series for $c^{-2}\dot{\mathbf{F}}$ needed when considering the equations of motion in GCRS were computed as well. All values have been computed using the astronomical unit as the unit of length and 1000 Julian years (365250 Julian days) as the unit of time. The value of L_C obtained with the VSOP solution reads

$$L_C = 1.480826855667 \times 10^{-8}\,. \tag{9.29}$$

In integrating the equations for $\dot{A}$ and $\dot{\mathbf{F}}$ the additive constants were introduced to annul these quantities for the 1977 Origin and J2000.0, respectively.

All series are presented in the form

$$x^i_A(t) = \sum_\alpha t^\alpha \left[\sum_k X^\alpha_{ik}\cos(\psi^\alpha_k + \nu^\alpha_k\, t)\right] \tag{9.30}$$

in Tables 1–20 of Appendix B containing the ordinal number of the term, components of the trigonometric argument (mean longitudes of eight major planets from Mercury to Neptune and arguments D, F, l of the lunar theory given to show the physical meaning of the term, coefficient X, the phase angle ψ of the argument, the frequency ν of the argument and exposant α of power of t. The negative components are underlined.

Functions $x^i_E(\mathrm{TT})$, $x^i_A(\mathrm{TT})$ represent just VSOP series of the argument TDB taken for the moment TT. Therefore, they are expressed in terms of 11 fundamental arguments representing now linear functions of TT. In such a way, the right–hand members of the DGRSC equation of the Earth's rotation become functions of TT and may be solved by iterations just as in Newtonian case (Bretagnon et al., 1997, 1998). In result one gets the solution taking into account relativistic indirect third–body perturbations.

Having got the dynamical solution for $\underset{1}{\psi}$, $\underset{1}{\theta}$, $\underset{1}{\varphi}$ one finds the kinematical solution $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$ and then the astrometric Earth's rotation matrix $T = \underset{0}{\hat{P}}(u)P_C$ including now the main relativistic corrections.

Computation of the right–hand members of the Earth's rotation equations in the SMART theory (Bretagnon et al., 1997, 1998) is based on the VSOP series for $\mathbf{x}_{(C)A}(\mathrm{TDB})$ where (C) indicates the ecliptical system BRSC and A stands for the body A ($A = E$ for the Earth, $A = S$ for the Sun, $A = L$ for the Moon, etc.). In the original SMART theory referred to TDB the geocentric coordinates of the Sun and the Moon in DGRSC are treated just as the differences of the corresponding BRSC coordinates $\mathbf{x}_{(C)S} - \mathbf{x}_{(C)E}$ and $\mathbf{x}_{(C)L} - \mathbf{x}_{(C)E}$ referred to TDB. In the present work the equations of the Earth's rotation are referred to TT with using $\mathbf{w}_{(C)A}(\mathrm{TT})$ for the geocentric coordinates of the Sun ($A = S$) and the Moon ($A = L$). We give the initial terms of all these series in Appendix B. The term by term comparison of these series enables one to see the difference between the Newtonian and relativistic values of the third–body coordinates. As useful intermediate results we reproduce also the series for the ICRS coordinates of the Earth $\mathbf{z}_E(\mathrm{TT})$, for the solar BRSC coordinates $\mathbf{x}_{(C)S}(\mathrm{TDB})$ and $\mathbf{z}_{(C)S}(\mathrm{TT})$, for the lunar BRSC coordinates $\mathbf{x}_{(C)L}(\mathrm{TDB}) - \mathbf{x}_{(C)E}(\mathrm{TDB})$ and $\mathbf{z}_{(C)L}(\mathrm{TT}) - \mathbf{z}_{(C)E}(\mathrm{TT})$. All series are presented in the compact form (9.30) adopted presently in VSOP with one more fundamental argument ϕ (the linear part of the expression for the Euler angle φ) specific for SMART solution.

Dependent on application of (9.30) the time argument t therewith is in fact either TDB or TT. The fundamental trigonometric arguments of the semi–analytical SMART series are given in Appendix A.

The final solution may be compared with the Newtonian SMART solution based on the Newtonian luni–solar coordinates $\mathbf{x}_{(C)A}(\mathrm{TDB}) - \mathbf{x}_{(C)E}(\mathrm{TDB})$ ($A = S$, L). The differences between the dynamical Euler angles $\underset{1}{\psi}$, $\underset{1}{\theta}$, $\underset{1}{\varphi}$ (relating ITRS and DGRSC) in the Newtonian (N) and relativistic solutions demonstrate the influence of the indirect relativistic third–body perturbations (Tables (45)–(47) of Appendix C). The dynamical Euler angles for both (Newtonian and relativistic) versions are converted by means of (9.17)–(9.19) into the kinematical Euler angles $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$ (relating ITRS and KGRSC) also for the Newtonian (N) and relativistic solutions. The differences between the dynamical and kinematical Euler angles for the relativistic solution (evidently, within the post–Newtonian approximation the similar differences for the Newtonian version are practically the same) exposed in Appendix

C (Tables (48)–(50)) improve the corresponding values given in (Bretagnon et al. 1997). The differences between the kinematical Euler angles in the Newtonian and relativistic solutions (Tables (51)–(53) of Appendix C) differ only slightly from the corresponding differences between the dynamical angles (this discrepancy reveals only in terms of the third and higher power of time).

Let us note once again that the Newtonian and relativistic SMART solutions are distinguished just with respect to the employed luni–solar coordinates as stated above. When converting from DGRSC to KGRSC both these solutions are transformed practically in the same manner as prescribed by the geodesic rotation. Newtonian solutions in DGRSC and KGRSC are differ by relativistic terms caused by the mutual rotation of reference systems not affecting the Newtonian nature of the solution itself. The final expansions show that the differences in the Euler angles for the Newtonian and relativistic solutions are of the order of 35 μas over 20 yrs (cf. the precision of SMART97 of 2 μas) and 150 μas over 100 yrs (cf. the precision of SMART97 of 12 μas). Therefore, the relativistic indirect third–body perturbations are within the accuracy of SMART97 theory and may be used to improve this theory.

Relativistic contributions in the GCRS↔ITRS transformation

Using the formalism of Bretagnon and Brumberg (2003) it is possible to compute the rotation vector $\mathbf{A}$ of GCRS→ITRS transformation for the Newtonian and relativistic SMART solutions and to find explicitly the relativistic contributions in the components of this vector. Denoting the triplet of the ITRS spatial coordinates by $\mathbf{y}$ we may represent the GCRS→ITRS transformation in form

$$\mathbf{y} = T\,\underset{0}{\mathbf{w}} \tag{9.31}$$

$$T = \underset{0}{\hat{P}}(u) P_C\,. \tag{9.32}$$

or

$$T = D_3(\underset{0}{\varphi}) D_1(-\underset{0}{\theta}) D_3(-\underset{0}{\psi}) D_1(\varepsilon) D_3(\chi)\,. \tag{9.33}$$

Introducing the rotation vector A one may use the rotation formula

$$T = R(\mathbf{A}), \qquad R(\mathbf{A})\mathbf{x} = \mathbf{x} - \sin a(\hat{\mathbf{A}} \times \mathbf{x}) + (1 - \cos a)[\hat{\mathbf{A}} \times (\hat{\mathbf{A}} \times \mathbf{x})] \tag{9.34}$$

where $a = |\mathbf{A}|$ is the rotation angle, $\hat{\mathbf{A}} = \mathbf{A}/a$ is the unit vector along the rotation axis, $\mathbf{x}$ is an arbitrary coordinate vector. Vector A is given in Bretagnon and Brumberg (2003) in three forms corresponding to ‘dynamical’ representation with three Euler angles, ‘classical kinematical’ representation (precession/nutation, diurnal rotation and polar motion) and modern ‘kinematical’ representation involving the non–rotating origin. The first representation in terms of $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$ is most closely related with the SMART solution. Evaluating the variation $\delta\mathbf{A}$ between the relativistic and Newtonian values of $\mathbf{A}$ one may find the influence of the relativistic terms on the GCRS→ITRS transformation (see (A.24), (A.25) in Bretagnon and Brumberg, 2003).

For simple evaluation of relativistic contributions in the GCRS↔ITRS transformation one may use just the representation (9.33). Originally SMART97 results in purely Newtonian (N) values for dynamical Euler angles. The corresponding kinematical Euler angles of (9.33) are to be considered also as Newtonian quantities (the differences of these angles are of relativistic order of smallness due to the geodesic rotation vector $\mathbf{F}_C$ of mutual rotation DGRSC and KGRSC). The differences $\delta\underset{0}{\psi}$, $\delta\underset{0}{\theta}$ and $\delta\underset{0}{\varphi}$ between relativistic and Newtonian values of these angles may be found from Tables 51–53 (Tables 4–6 of Brumberg and Simon, 2003) in form of polynomial–trigonometric series in terms of terrestrial time TT. In the linear approximation the relativistic contributions in the GCRS↔ITRS transformation are determined just by variation of (9.33), i.e.

$$\delta T = [\delta D_3(\underset{0}{\varphi})\, D_1(-\underset{0}{\theta})\, D_3(-\underset{0}{\psi}) + D_3(\underset{0}{\varphi})\, \delta D_1(-\underset{0}{\theta})\, D_3(-\underset{0}{\psi}) + D_3(\underset{0}{\varphi})\, D_1(-\underset{0}{\theta})\, \delta D_3(-\underset{0}{\psi})] P_C\,. \tag{9.35}$$

Directly from the definitions of $D_k(\alpha)$ $(k = 1, 2, 3)$ one may derive the first–order relations

$$D_k(\delta\alpha) D_k(\alpha) = D_k(\alpha) D_k(\delta\alpha) \tag{9.36}$$

and

$$\delta D_k(\alpha) = [D_k(\delta\alpha) - E] D_k(\alpha) = D_k(\alpha)[D_k(\delta\alpha) - E], \tag{9.37}$$

E being a unit matrix. It is easy to see that

$$D_k(\delta\alpha) - E = \nabla_k \delta\alpha \tag{9.38}$$

where matrix ∇_k consists of the elements

$$(\nabla_k)_{ij} = \varepsilon_{ijk} \tag{9.39}$$

with fully antisymmetric Levi–Civita symbols ε_{ijk}. By using (9.37) and (9.38) one may present the variation (9.35) in the form

$$\begin{aligned}\delta T = [&\nabla_3\, D_3(\underset{0}{\varphi})\, D_1(-\underset{0}{\theta})\, D_3(-\underset{0}{\psi})\delta\underset{0}{\varphi} - D_3(\underset{0}{\varphi})\, \nabla_1\, D_1(-\underset{0}{\theta})\, D_3(-\underset{0}{\psi})\delta\underset{0}{\theta} - \\ &- D_3(\underset{0}{\varphi})\, D_1(-\underset{0}{\theta})\, \nabla_3\, D_3(-\underset{0}{\psi})\delta\underset{0}{\psi}] P_C\,.\end{aligned} \tag{9.40}$$

With known functions $\underset{0}{\psi}$, $\underset{0}{\theta}$, $\underset{0}{\varphi}$, $\delta\underset{0}{\psi}$, $\delta\underset{0}{\theta}$ and $\delta\underset{0}{\varphi}$ it is easy to compute the relativistic variation δT for any moment TT and compare it with the Newtonian value T from (9.33).

Appendix A

Fundamental arguments

As stated above, the expansions of the present paper have 12 trigonometrical arguments as follows:

$$\lambda_1(t) = 4.40260867435 + 26087.9031415742\,t\,,$$

$$\lambda_2(t) = 3.17614652884 + 10213.2855462110\,t\,,$$

$$\lambda_3(t) = 1.75347029148 + 6283.0758511455\,t\,,$$

$$\lambda_4(t) = 6.20347594486 + 3340.6124266998\,t\,,$$

$$\lambda_5(t) = 0.59954632934 + 529.6909650946\,t\,,$$

$$\lambda_6(t) = 0.87401658845 + 213.2990954380\,t\,,$$

$$\lambda_7(t) = 5.48129370354 + 74.7815985673\,t\,,$$

$$\lambda_8(t) = 5.31188611871 + 38.1330356378\,t\,,$$

$$D(t) = 5.19846640063 + 77713.7714481804\,t\,,$$

$$F(t) = 1.62790513602 + 84334.6615717837\,t\,,$$

$$l(t) = 2.35555563875 + 83286.9142477147\,t\,,$$

$$\phi(u) = 4.89496121282 + 2301216.7536515365\,u\,.$$

The mean longitudes of eight major planets $\lambda_i(t)$ are referred to ICRS (to the reference system of of DE403 in practice). Therefore, their constant parts are those given in (Bretagnon et al. 1998). Their frequencies are also taken from (Bretagnon et al. 1998) but without taking into account the precession.

The Delaunay arguments $D(t)$, $F(t)$, $l(t)$ of the lunar theory are taken from (Bretagnon et al. 1998).

Originally, these arguments are functions of t =TDB but in accordance with (9.36) they are used here just as the linear functions of u =TT.

The last argument, $\phi(u)$, representing the linear part of the Euler angle φ of the Earth's rotation is taken from (Bretagnon et al. 1998) as well.

All values are given using the astronomical unit as the unit of length and 1000 Julian years (365250 Julian days) as the unit of time (tjy).

Appendix B

Right–hand members

This Appendix contains the initial terms of the series needed to compute the right–hand members of the Earth's rotation equation of the SMART theory (Bretagnon et al. 1997, 1998). All values are given using the astronomical unit as the unit of length and 1000 Julian years (365250 Julian days) as the unit of time. All series are presented in the previous form

$$x^i_A(t) = \sum_\alpha t^\alpha \left[\sum_k X^\alpha_{ik} \cos(\psi^\alpha_k + \nu^\alpha_k\, t) \right].$$

The data in the 6–column Tables 21–44 read (we numerate the tables from No. 21 taking into account Tables 1–20 of Bretagnon and Brumberg (2003) underlying the present work): ordinal number of the term, components of the trigonometric argument (mean longitudes of eight major planets from Mercury to Neptune, arguments D, F, l of the lunar theory and the Earth's rotation angle ϕ) given to show the physical meaning of the term (for the series of the right–hand members the last argument is absent), coefficient X, the phase angle ψ of the argument, the frequency ν of the argument and exposant α of t.

The time argument t is either TDB or TT as indicated explicitly. The negative components of the trigonometric arguments are underlined.

These tables contain the initial terms of the series for
Earth ICRS coordinates $\mathbf{z}_E$(TT) (Tables 21–23),
Sun BRSC coordinates $\mathbf{x}_{(C)S}$(TDB) (Tables 24–26),
Sun BRSC coordinates $\mathbf{z}_{(C)S}$(TT) (Tables 27–29),
Sun BRSC geocentric coordinates $\mathbf{x}_{(C)S}(\mathrm{TDB}) - \mathbf{x}_{(C)E}(\mathrm{TDB})$ (Tables 30–32),
Sun DGRSC geocentric coordinates $\mathbf{w}_{(C)S}$(TT) (Tables 33–35),
Moon BRSC geocentric coordinates $\mathbf{x}_{(C)L}(\mathrm{TDB}) - \mathbf{x}_{(C)E}(\mathrm{TDB})$ (Tables 36–38),
Moon BRSC geocentric coordinates $\mathbf{z}_{(C)L}(\mathrm{TT}) - \mathbf{x}_{(C)E}(\mathrm{TT})$ (Tables 39–41),
Moon DGRSC geocentric coordinates $\mathbf{w}_{(C)L}$(TT) (Tables 42–44).

The resulting series to perform SMART iterations are those of Tables (33)–(35) and (42)–(44).

Table 1. x_E^1

1	0 0 1 0 0 0 0 0 0 0 0	$.999826249 + 00$	$.175348528 + 01$	$.628307585 + 04$	0
2	0 0 2 0 0 0 0 0 0 0 0	$.835254761 - 02$	$.171034482 + 01$	$.125661517 + 05$	0
3	0 0 0 0 0 0 0 0 0 0 0	$.590519376 - 02$	$.000000000 + 00$	$.000000000 + 00$	0
4	0 0 0 0 1 0 0 0 0 0 0	$.493123608 - 02$	$.374115795 + 01$	$.529690965 + 03$	0
5	0 0 0 0 0 1 0 0 0 0 0	$.271697270 - 02$	$.401601399 + 01$	$.213299095 + 03$	0
6	0 0 0 0 0 0 0 1 0 0 0	$.154647690 - 02$	$.217052025 + 01$	$.381330356 + 02$	0
7	0 0 0 0 0 0 1 0 0 0 0	$.836647696 - 03$	$.233967709 + 01$	$.747815986 + 02$	0
8	0 0 0 0 0 0 0 0 0 0 0	$.122106593 - 02$	$.000000000 + 00$	$.000000000 + 00$	1
9	0 0 2 0 0 0 0 0 0 0 0	$.514999998 - 03$	$.600266210 + 01$	$.125661517 + 05$	1

Table 2. x_E^2

1	0 0 1 0 0 0 0 0 0 0 0	$.917380354 + 00$	$.182658535 + 00$	$.628307585 + 04$	0
2	0 0 0 0 0 0 0 0 0 0 0	$.220960391 - 01$	$.314159265 + 01$	$.000000000 + 00$	0
3	0 0 2 0 0 0 0 0 0 0 0	$.766363606 - 02$	$.139528008 + 00$	$.125661517 + 05$	0
4	0 0 0 0 1 0 0 0 0 0 0	$.453127029 - 02$	$.218025318 + 01$	$.529690965 + 03$	0
5	0 0 0 0 0 1 0 0 0 0 0	$.251550284 - 02$	$.246154598 + 01$	$.213299095 + 03$	0
6	0 0 0 0 0 0 0 1 0 0 0	$.143167358 - 02$	$.609170505 + 00$	$.381330356 + 02$	0
7	0 0 0 0 0 0 1 0 0 0 0	$.766049684 - 03$	$.774439757 + 00$	$.747815986 + 02$	0
8	0 0 1 0 0 0 0 0 0 0 0	$.909408579 - 03$	$.269013703 + 00$	$.628307585 + 04$	1
9	0 0 0 0 0 0 0 0 0 0 0	$.831610134 - 03$	$.000000000 + 00$	$.000000000 + 00$	1

Table 3. x_E^3

1	0 0 1 0 0 0 0 0 0 0 0	$.397732873 + 00$	$.182658320 + 00$	$.628307585 + 04$	0
2	0 0 0 0 0 0 0 0 0 0 0	$.959237327 - 02$	$.314159265 + 01$	$.000000000 + 00$	0
3	0 0 2 0 0 0 0 0 0 0 0	$.332258679 - 02$	$.139529338 + 00$	$.125661517 + 05$	0
4	0 0 0 0 1 0 0 0 0 0 0	$.194477279 - 02$	$.211821614 + 01$	$.529690965 + 03$	0
5	0 0 0 0 0 1 0 0 0 0 0	$.104351977 - 02$	$.234916313 + 01$	$.213299095 + 03$	0
6	0 0 0 0 0 0 0 1 0 0 0	$.586898449 - 03$	$.543535503 + 00$	$.381330356 + 02$	0
7	0 0 1 0 0 0 0 0 0 0 0	$.208884480 - 02$	$.341431350 + 01$	$.628307585 + 04$	1

Table 4. v_E^1

1	0 0 1 0 0 0 0 0 0 0 0	$.628198416 + 04$	$.332428161 + 01$	$.628307585 + 04$	0
2	0 0 2 0 0 0 0 0 0 0 0	$.104958910 + 03$	$.328114315 + 01$	$.125661517 + 05$	0
3	0 0 1 0 0 0 0 0 1 0 0	$.261300614 + 01$	$.223954744 + 01$	$.839968473 + 05$	0
4	0 0 0 0 1 0 0 0 0 0 0	$.261203107 + 01$	$.531195439 + 01$	$.529690965 + 03$	0
5	0 0 3 0 0 0 0 0 0 0 0	$.197286075 + 01$	$.323802471 + 01$	$.188492276 + 05$	0
6	0 0 0 0 0 1 0 0 0 0 0	$.579527948 + 00$	$.558680935 + 01$	$.213299095 + 03$	0
7	0 0 2 0 0 0 0 0 0 0 0	$.647152462 + 01$	$.129027334 + 01$	$.125661517 + 05$	1

Table 5. v_E^2

1	0 0 1 0 0 0 0 0 0 0 0	$.576397043 + 04$	$.175345470 + 01$	$.628307585 + 04$	0
2	0 0 2 0 0 0 0 0 0 0 0	$.963019826 + 02$	$.171032626 + 01$	$.125661517 + 05$	0
3	0 0 0 0 1 0 0 0 0 0 0	$.240017211 + 01$	$.375104918 + 01$	$.529690965 + 03$	0
4	0 0 1 0 0 0 0 0 1 0 0	$.239738605 + 01$	$.668751135 + 00$	$.839968473 + 05$	0
5	0 0 3 0 0 0 0 0 0 0 0	$.181013049 + 01$	$.166721271 + 01$	$.188492276 + 05$	0
6	0 0 0 0 0 1 0 0 0 0 0	$.536555672 + 00$	$.403234472 + 01$	$.213299095 + 03$	0
7	0 0 2 0 0 0 0 0 0 0 0	$.589242923 + 01$	$.601669452 + 01$	$.125661517 + 05$	1
8	0 0 1 0 0 0 0 0 0 0 0	$.571394167 + 01$	$.183981344 + 01$	$.628307585 + 04$	1

Table 6. v_E^3

1	0 0 1 0 0 0 0 0 0 0 0	$.249898562 + 04$	$.175345548 + 01$	$.628307585 + 04$	0
2	0 0 2 0 0 0 0 0 0 0 0	$.417519410 + 02$	$.171032808 + 01$	$.125661517 + 05$	0
3	0 0 1 0 0 0 0 0 1 0 0	$.103939352 + 01$	$.668751488 + 00$	$.839968473 + 05$	0
4	0 0 0 0 1 0 0 0 0 0 0	$.103013061 + 01$	$.368901547 + 01$	$.529690965 + 03$	0
5	0 0 3 0 0 0 0 0 0 0 0	$.784766043 + 00$	$.166722822 + 01$	$.188492276 + 05$	0
6	0 0 1 0 0 0 0 0 0 0 0	$.131245539 + 02$	$.498511188 + 01$	$.628307585 + 04$	1
7	0 0 2 0 0 0 0 0 0 0 0	$.268841949 + 01$	$.593129433 + 01$	$.125661517 + 05$	1
8	0 0 1 0 0 0 0 0 0 0 0	$.582773985 + 00$	$.418397765 + 00$	$.628307585 + 04$	2

Table 7. $c^{-2}a_E^1$

1	0 0 1 0 0 0 0 0 0 0 0	.986893894 − 08	.489507792 + 01	.628307585 + 04	0
2	0 0 2 0 0 0 0 0 0 0 0	.329776944 − 09	.485194202 + 01	.125661517 + 05	0
3	0 0 1 0 0 0 0 0 1 0 0	.551783697 − 10	.668751463 + 00	.839968473 + 05	0
4	0 0 3 0 0 0 0 0 0 0 0	.929792837 − 11	.480881552 + 01	.188492276 + 05	0
5	0 0 1 0 0 0 0 0 1 0 1	.601681591 − 11	.302430741 + 01	.167283762 + 06	0
6	0 0 1 0 0 0 0 0 3 0 $\underline{1}$	.115298468 − 11	.242694156 + 01	.156137476 + 06	0
7	0 0 1 0 0 0 0 0 3 0 0	.883536302 − 12	.478249888 + 01	.239424390 + 06	0
8	0 0 1 0 0 0 0 0 $\underline{1}$ 0 0	.584587289 − 12	.344424694 + 01	.714306956 + 05	0
9	0 0 1 0 0 0 0 0 1 0 2	.552934674 − 12	.537986069 + 01	.250570676 + 06	0
10	0 0 2 0 0 0 0 0 0 0 0	.203332122 − 10	.286106952 + 01	.125661517 + 05	1
11	0 0 3 0 0 0 0 0 0 0 0	.114660466 − 11	.281783826 + 01	.188492276 + 05	1
12	0 0 2 0 0 0 0 0 0 0 0	.858984433 − 12	.125839666 + 01	.125661517 + 05	2

Table 8. $c^{-2}a_E^2$

1	0 0 1 0 0 0 0 0 0 0 0	.905514020 − 08	.332425118 + 01	.628307585 + 04	0
2	0 0 2 0 0 0 0 0 0 0 0	.302576832 − 09	.328112536 + 01	.125661517 + 05	0
3	0 0 3 0 0 0 0 0 0 0 0	.853078687 − 11	.323797957 + 01	.188492276 + 05	0
4	0 0 1 0 0 0 0 0 $\underline{1}$ 0 0	.421138453 − 12	.501588270 + 01	.714306956 + 05	0
5	0 0 2 0 0 0 0 0 0 0 0	.185136843 − 10	.130430489 + 01	.125661517 + 05	1
6	0 0 1 0 0 0 0 0 0 0 0	.897641470 − 11	.341060482 + 01	.628307585 + 04	1
7	0 0 3 0 0 0 0 0 0 0 0	.104798335 − 11	.125407561 + 01	.188492276 + 05	1
8	0 0 2 0 0 0 0 0 0 0 0	.795069151 − 12	.598199237 + 01	.125661517 + 05	2

Table 9. $c^{-2}a_E^3$

1	0 0 1 0 0 0 0 0 0 0 0	.392588190 − 08	.332425096 + 01	.628307585 + 04	0
2	0 0 2 0 0 0 0 0 0 0 0	.131183005 − 09	.328112554 + 01	.125661517 + 05	0
3	0 0 3 0 0 0 0 0 0 0 0	.369853480 − 11	.323798141 + 01	.188492276 + 05	0
4	0 0 1 0 0 0 0 0 0 0 0	.206182711 − 10	.272721140 + 00	.628307585 + 04	1
5	0 0 2 0 0 0 0 0 0 0 0	.844685906 − 11	.121890327 + 01	.125661517 + 05	1
6	0 0 3 0 0 0 0 0 0 0 0	.465920918 − 12	.121063123 + 01	.188492276 + 05	1
7	0 0 1 0 0 0 0 0 0 0 0	.915520132 − 12	.198919397 + 01	.628307585 + 04	2

Table 10. $c^{-2}\bar{U}_E(t, \mathbf{x}_E)$

1	0 0 0 0 0 0 0 0 0 0 0	$.987296507-08$	$.000000000+00$	$.000000000+00$	0
2	0 0 1 0 0 0 0 0 0 0 0	$.164919613-09$	$.624005793+01$	$.628307585+04$	0
3	0 0 2 0 0 0 0 0 0 0 0	$.275546329-11$	$.619690753+01$	$.125661517+05$	0
4	0 0 0 0 0 0 0 0 1 0 0	$.304463527-12$	$.205687375+01$	$.777137714+05$	0
5	0 0 1 0 $\underline{1}$ 0 0 0 0 0 0	$.191888080-12$	$.113799397+01$	$.575338489+04$	0
6	0 0 1 0 0 0 0 0 0 0 0	$.101695147-10$	$.424903231+01$	$.628307585+04$	1
7	0 0 2 0 0 0 0 0 0 0 0	$.339821531-12$	$.420590851+01$	$.125661517+05$	1
8	0 0 1 0 0 0 0 0 0 0 0	$.430338848-12$	$.264288755+01$	$.628307585+04$	2

Table 11. $c^{-2}A_p$

1	0 0 1 0 0 0 0 0 0 0 0	$.524968495-13$	$.466925770+01$	$.628307585+04$	0
2	0 0 0 0 0 0 0 0 0 0 0	$.207544788-14$	$.000000000+00$	$.000000000+00$	0
3	0 0 1 0 $\underline{1}$ 0 0 0 0 0 0	$.710370763-15$	$.272618109+01$	$.575338489+04$	0
4	0 0 2 0 0 0 0 0 0 0 0	$.438556542-15$	$.462610773+01$	$.125661517+05$	0
5	0 0 0 0 1 0 0 0 0 0 0	$.151155817-15$	$.515678954+01$	$.529690965+03$	0
6	0 0 1 0 0 $\underline{1}$ 0 0 0 0 0	$.148221791-15$	$.245039882+01$	$.606977676+04$	0
7	0 0 0 0 0 1 0 0 0 0 0	$.715230974-16$	$.397232448+01$	$.213299095+03$	0
8	0 0 4 $\underline{8}$ 3 0 0 0 0 0 0	$.536956529-16$	$.282864247+01$	$.352311373+01$	0
9	0 0 1 0 0 0 0 0 0 0 0	$.323715127-14$	$.267823553+01$	$.628307585+04$	1
10	0 0 2 0 0 0 0 0 0 0 0	$.540855083-16$	$.263510709+01$	$.125661517+05$	1
11	0 0 1 0 0 0 0 0 0 0 0	$.136987235-15$	$.107209724+01$	$.628307585+04$	2

Table 12. $c^{-2}A_p v_E^1$

1	0 0 2 0 0 0 0 0 0 0 0	$.164856072-09$	$.171162969+01$	$.125661517+05$	0
2	0 0 0 0 0 0 0 0 0 0 0	$.369256889-10$	$.000000000+00$	$.000000000+00$	0
3	0 0 1 0 0 0 0 0 0 0 0	$.125285537-10$	$.344550232+01$	$.628307585+04$	0
4	0 0 3 0 0 0 0 0 0 0 0	$.413171537-11$	$.166817571+01$	$.188492276+05$	0
5	0 0 2 0 $\underline{1}$ 0 0 0 0 0 0	$.223146167-11$	$.604732071+01$	$.120364607+05$	0
6	0 0 0 0 1 0 0 0 0 0 0	$.223066484-11$	$.595994882+00$	$.529690965+03$	0
7	0 0 2 0 0 0 0 0 0 0 0	$.101653838-10$	$.600406989+01$	$.125661517+05$	1
8	0 0 0 0 0 0 0 0 0 0 0	$.812044446-11$	$.000000000+00$	$.000000000+00$	1

Table 13. $c^{-2}A_pv_E^2$

1	0 0 2 0 0 0 0 0 0 0 0	.151233515 − 09	.140894440 + 00	.125661517 + 05	0
2	0 0 0 0 0 0 0 0 0 0 0	.147475447 − 09	.314159265 + 01	.000000000 + 00	0
3	0 0 1 0 0 0 0 0 0 0 0	.130400536 − 10	.203833678 + 01	.628307585 + 04	0
4	0 0 3 0 0 0 0 0 0 0 0	.379053829 − 11	.974098084 − 01	.188492276 + 05	0
5	0 0 0 0 1 0 0 0 0 0 0	.204827491 − 11	.530868153 + 01	.529690965 + 03	0
6	0 0 2 0 $\underline{1}$ 0 0 0 0 0 0	.204601961 − 11	.447677127 + 01	.120364607 + 05	0
7	0 0 2 0 0 0 0 0 0 0 0	.925112075 − 11	.444755697 + 01	.125661517 + 05	1
8	0 0 0 0 0 0 0 0 0 0 0	.547547581 − 11	.000000000 + 00	.000000000 + 00	1

Table 14. $c^{-2}A_pv_E^3$

1	0 0 2 0 0 0 0 0 0 0 0	.655677175 − 10	.140895204 + 00	.125661517 + 05	0
2	0 0 0 0 0 0 0 0 0 0 0	.639383896 − 10	.314159265 + 01	.000000000 + 00	0
3	0 0 1 0 0 0 0 0 0 0 0	.565352598 − 11	.203833961 + 01	.628307585 + 04	0
4	0 0 3 0 0 0 0 0 0 0 0	.164339709 − 11	.974112525 − 01	.188492276 + 05	0
5	0 0 0 0 1 0 0 0 0 0 0	.887936804 − 12	.530870026 + 01	.529690965 + 03	0
6	0 0 2 0 $\underline{1}$ 0 0 0 0 0 0	.887068712 − 12	.447675436 + 01	.120364607 + 05	0
7	0 0 2 0 0 0 0 0 0 0 0	.422078332 − 11	.436211843 + 01	.125661517 + 05	1
8	0 0 0 0 0 0 0 0 0 0 0	.276330466 − 11	.000000000 + 00	.000000000 + 00	1

Table 15. $c^{-2}F^1$

1	0 0 1 0 0 0 0 0 1 $\underline{1}$ 0	.631723381 − 11	.410074393 + 01	.337814272 + 03	0
2	0 0 0 0 0 0 0 0 0 0 0	.322941894 − 11	.000000000 + 00	.000000000 + 00	0
3	0 0 0 0 2 $\underline{5}$ 0 0 0 0 0	.957304140 − 12	.524343393 + 01	.711354700 + 01	0
4	0 8 $\underline{13}$ 0 0 0 0 0 0 0 0	.162778812 − 12	.214693239 + 01	.262983048 + 02	0
5	0 3 $\underline{5}$ 0 0 0 0 0 0 0 0	.132992985 − 12	.418857071 + 01	.775522617 + 03	0
6	0 0 1 0 0 0 0 0 0 1 0	.103535306 − 12	.338135431 + 01	.906177374 + 05	0
7	0 0 0 0 0 0 0 0 0 0 0	.368756167 − 10	.000000000 + 00	.000000000 + 00	1
8	0 0 1 0 0 0 0 0 0 0 0	.151143366 − 12	.466915164 + 01	.628307585 + 04	1
9	0 0 0 0 0 0 0 0 0 0 0	.947534621 − 08	.000000000 + 00	.000000000 + 00	2
10	0 0 0 0 0 0 0 0 0 0 0	.291759138 − 08	.000000000 + 00	.000000000 + 00	3
11	0 0 0 0 0 0 0 0 0 0 0	.252474520 − 10	.314159265 + 01	.000000000 + 00	4
12	0 0 0 0 0 0 0 0 0 0 0	.934626985 − 12	.314159265 + 01	.000000000 + 00	5

Table 16. $c^{-2}F^2$

1	0 0	1 0 0 0 0 0 0 0 0	.295269994 − 09	.152766821 + 01	.628307585 + 04	0
2	0 0	0 0 0 0 0 0 0 0 0	.158211797 − 10	.314159265 + 01	.000000000 + 00	0
3	0 0	1 0 0 0 0 0 1 $\underline{1}$ 0	.579617945 − 11	.567155194 + 01	.337814272 + 03	0
4	0 0	2 0 0 0 0 0 0 0 0	.370142256 − 11	.148404865 + 01	.125661517 + 05	0
5	0 0	0 0 2 $\underline{5}$ 0 0 0 0 0	.810981192 − 12	.385154152 + 01	.711354700 + 01	0
6	0 0	0 0 0 0 0 0 1 0 0	.716789259 − 12	.362768051 + 01	.777137714 + 05	0
7	0 0	1 0 $\underline{1}$ 0 0 0 0 0 0	.397469768 − 12	.275372606 + 01	.575338489 + 04	0
8	0 0	4 $\underline{8}$ 3 0 0 0 0 0 0	.336491428 − 12	.597192282 + 01	.352311373 + 01	0
9	0 2	$\underline{2}$ 0 0 0 0 0 0 0 0	.245586218 − 12	.125316176 + 01	.786041939 + 04	0
10	0 1	$\underline{1}$ 0 0 0 0 0 0 0 0	.177341329 − 12	.307423037 + 01	.393020970 + 04	0
11	0 8	$\underline{13}$ 0 0 0 0 0 0 0 0	.176379964 − 12	.420271554 + 01	.262983048 + 02	0
12	0 0	1 0 0 $\underline{1}$ 0 0 0 0 0	.148834847 − 12	.250125409 + 01	.606977676 + 04	0
13	0 3	$\underline{5}$ 0 0 0 0 0 0 0 0	.131063179 − 12	.225580422 + 01	.775522617 + 03	0
14	0 0	1 $\underline{2}$ 0 0 0 0 0 0 0	.107692890 − 12	.572588758 + 00	.398149002 + 03	0
15	0 2	$\underline{3}$ 0 0 0 0 0 0 0 0	.100166058 − 12	.429087451 + 01	.157734354 + 04	0
16	0 0	0 0 0 0 0 0 0 0 0	.370245011 − 04	.314159265 + 01	.000000000 + 00	1
17	0 0	1 0 0 0 0 0 0 0 0	.188921703 − 10	.574491837 + 01	.628307585 + 04	1
18	0 0	2 0 0 0 0 0 0 0 0	.464367536 − 12	.573880012 + 01	.125661517 + 05	1
19	0 0	0 0 0 0 0 0 0 0 0	.971584742 − 07	.000000000 + 00	.000000000 + 00	2
20	0 0	1 0 0 0 0 0 0 0 0	.766331284 − 12	.407953626 + 01	.628307585 + 04	2
21	0 0	0 0 0 0 0 0 0 0 0	.670247397 − 09	.314159265 + 01	.000000000 + 00	3
22	0 0	0 0 0 0 0 0 0 0 0	.545536461 − 10	.314159265 + 01	.000000000 + 00	4
23	0 0	0 0 0 0 0 0 0 0 0	.460220905 − 12	.000000000 + 00	.000000000 + 00	5

Table 17. $c^{-2}F^3$

1	0 0	1 0 0 0 0 0 0 0 0	.681047929 − 09	.466926134 + 01	.628307585 + 04	0
2	0 0	0 0 0 0 0 0 0 0 0	.261172477 − 10	.000000000 + 00	.000000000 + 00	0
3	0 0	2 0 0 0 0 0 0 0 0	.853292926 − 11	.462618401 + 01	.125661517 + 05	0
4	0 0	1 0 0 0 0 0 1 $\underline{1}$ 0	.251265812 − 11	.567151302 + 01	.337814272 + 03	0
5	0 0	0 0 0 0 0 0 1 0 0	.165339994 − 11	.486075462 + 00	.777137714 + 05	0
6	0 0	1 0 $\underline{1}$ 0 0 0 0 0 0	.926911809 − 12	.584308441 + 01	.575338489 + 04	0
7	0 0	4 $\underline{8}$ 3 0 0 0 0 0 0	.776162979 − 12	.283235818 + 01	.352311373 + 01	0
8	0 2	$\underline{2}$ 0 0 0 0 0 0 0 0	.565789276 − 12	.442261203 + 01	.786041939 + 04	0
9	0 1	$\underline{1}$ 0 0 0 0 0 0 0 0	.405731668 − 12	.612004936 + 01	.393020970 + 04	0
10	0 0	1 0 0 $\underline{1}$ 0 0 0 0 0	.352380753 − 12	.558146571 + 01	.606977676 + 04	0
11	0 0	0 0 2 $\underline{5}$ 0 0 0 0 0	.317523425 − 12	.320222812 + 01	.711354700 + 01	0
12	0 0	1 $\underline{2}$ 0 0 0 0 0 0 0	.234591707 − 12	.355646595 + 01	.398149002 + 03	0
13	0 2	$\underline{3}$ 0 0 0 0 0 0 0 0	.229336954 − 12	.107011138 + 01	.157734354 + 04	0
14	0 0	2 $\underline{2}$ 0 0 0 0 0 0 0	.203286078 − 12	.523337253 + 01	.588492685 + 04	0
15	0 8	$\underline{13}$ 0 0 0 0 0 0 0 0	.188535618 − 12	.234742427 + 01	.262983048 + 02	0
16	0 0	2 0 $\underline{2}$ 0 0 0 0 0 0	.185432080 − 12	.739785935 + 00	.115067698 + 05	0
17	0 0	0 0 1 0 0 0 0 0 0	.178233805 − 12	.546602530 + 01	.529690965 + 03	0
18	0 0	1 0 $\underline{2}$ 0 0 0 0 0 0	.159992576 − 12	.719100579 + 00	.522369392 + 04	0
19	0 3	$\underline{4}$ 0 0 0 0 0 0 0 0	.151390853 − 12	.252876275 + 01	.550755323 + 04	0
20	0 0	3 0 0 0 0 0 0 0 0	.139900610 − 12	.458313575 + 01	.188492276 + 05	0
21	0 0	0 0 0 0 0 0 1 0 $\underline{1}$	.115552944 − 12	.186947067 + 01	.557314280 + 04	0
22	0 3	$\underline{5}$ 0 0 0 0 0 0 0 0	.113123168 − 12	.379080051 + 01	.775522617 + 03	0
23	0 0	0 0 0 0 0 0 0 0 0	.853979154 − 04	.000000000 + 00	.000000000 + 00	1
24	0 0	1 0 0 0 0 0 0 0 0	.417297520 − 10	.269273986 + 01	.628307585 + 04	1
25	0 0	2 0 0 0 0 0 0 0 0	.104916266 − 11	.264234995 + 01	.125661517 + 05	1
26	0 0	0 0 0 0 0 0 0 0 0	.414106554 − 07	.000000000 + 00	.000000000 + 00	2
27	0 0	1 0 0 0 0 0 0 0 0	.178452866 − 11	.109726309 + 01	.628307585 + 04	2
28	0 0	0 0 0 0 0 0 0 0 0	.386704263 − 09	.314159265 + 01	.000000000 + 00	3
29	0 0	0 0 0 0 0 0 0 0 0	.223195616 − 10	.314159265 + 01	.000000000 + 00	4
30	0 0	0 0 0 0 0 0 0 0 0	.226641280 − 12	.000000000 + 00	.000000000 + 00	5

Table 18. $c^{-2}\dot{F}^{1}$

1	0 0 1 0 0 0 0 0 0 1 0	.938213522 − 08	.495215064 + 01	.906177374 + 05	0
2	$0\,0\,1\,0\,0\,0\,0\,0\,0\,\underline{1}\,0$	.589176757 − 08	.144525345 + 01	.780515857 + 05	0
3	$0\,0\,1\,0\,0\,0\,0\,0\,1\,\underline{1}\,0$	.213405174 − 08	.567154021 + 01	.337814272 + 03	0
4	0 0 0 0 0 0 0 0 0 0 0	.189506924 − 07	.000000000 + 00	.000000000 + 00	1
5	0 0 0 0 0 0 0 0 0 0 0	.875277413 − 08	.000000000 + 00	.000000000 + 00	2

Table 19. $c^{-2}\dot{F}^{2}$

1	0 0 0 0 0 0 0 0 0 0 0	.370245011 − 04	.314159265 + 01	.000000000 + 00	0
2	0 0 1 0 0 0 0 0 0 0 0	.185518715 − 05	.309846938 + 01	.628307585 + 04	0
3	0 0 0 0 0 0 0 0 1 0 0	.557043967 − 07	.519847690 + 01	.777137714 + 05	0
4	0 0 2 0 0 0 0 0 0 0 0	.465122208 − 07	.305484939 + 01	.125661517 + 05	0
5	0 0 1 0 0 0 0 0 0 1 0	.860730719 − 08	.338139405 + 01	.906177374 + 05	0
6	0 0 0 0 0 0 0 0 1 0 1	.674496965 − 08	.127084601 + 01	.161000686 + 06	0
7	$0\,0\,1\,0\,0\,0\,0\,0\,0\,\underline{1}\,0$	.540399862 − 08	.301599871 + 01	.780515857 + 05	0
8	$0\,0\,1\,0\,\underline{1}\,0\,0\,0\,0\,0\,0$	.228679604 − 08	.432452275 + 01	.575338489 + 04	0
9	$0\,0\,1\,0\,0\,0\,0\,0\,1\,\underline{1}\,0$	.195803161 − 08	.959159994 + 00	.337814272 + 03	0
10	$0\,2\,\underline{2}\,0\,0\,0\,0\,0\,0\,0\,0$	.193041068 − 08	.282395867 + 01	.786041939 + 04	0
11	0 0 0 0 0 0 0 0 0 0 1	.153982399 − 08	.549714754 + 01	.832869142 + 05	0
12	$0\,0\,0\,0\,0\,0\,0\,0\,3\,0\,\underline{1}$	.128254344 − 08	.673416743 + 00	.149854400 + 06	0
13	0 0 3 0 0 0 0 0 0 0 0	.114491893 − 08	.301073075 + 01	.188492276 + 05	0
14	0 0 0 0 0 0 0 0 3 0 0	.102259236 − 08	.302903181 + 01	.233141314 + 06	0
15	0 0 0 0 0 0 0 0 0 0 0	.194316948 − 06	.000000000 + 00	.000000000 + 00	1
16	0 0 1 0 0 0 0 0 0 0 0	.118699413 − 06	.103253061 + 01	.628307585 + 04	1
17	0 0 2 0 0 0 0 0 0 0 0	.583524834 − 08	.102641377 + 01	.125661517 + 05	1
18	0 0 1 0 0 0 0 0 0 0 0	.481484075 − 08	.565033303 + 01	.628307585 + 04	2
19	0 0 0 0 0 0 0 0 0 0 0	.201074219 − 08	.314159265 + 01	.000000000 + 00	2

Table 20. $c^{-2}\dot{F}^3$

1	0 0 0 0 0 0 0 0 0 0 0	$.853979154-04$	$.000000000+00$	$.000000000+00$	0
2	0 0 1 0 0 0 0 0 0 0 0	$.427903745-05$	$.624006151+01$	$.628307585+04$	0
3	0 0 0 0 0 0 0 0 1 0 0	$.128491945-06$	$.205687178+01$	$.777137714+05$	0
4	0 0 2 0 0 0 0 0 0 0 0	$.107225123-06$	$.619698427+01$	$.125661517+05$	0
5	0 0 0 0 0 0 0 0 1 0 1	$.155582988-07$	$.441242789+01$	$.161000686+06$	0
6	0 0 1 0 $\underline{1}$ 0 0 0 0 0 0	$.533288050-08$	$.113069545+01$	$.575338489+04$	0
7	0 2 $\underline{2}$ 0 0 0 0 0 0 0 0	$.444734078-08$	$.599340820+01$	$.786041939+04$	0
8	0 0 1 0 0 0 0 0 0 1 0	$.373140172-08$	$.338140966+01$	$.906177374+05$	0
9	0 0 0 0 0 0 0 0 0 0 1	$.355204498-08$	$.235555555+01$	$.832869142+05$	0
10	0 0 0 0 0 0 0 0 3 0 $\underline{1}$	$.295821714-08$	$.381500935+01$	$.149854400+06$	0
11	0 0 3 0 0 0 0 0 0 0 0	$.263699480-08$	$.615393603+01$	$.188492276+05$	0
12	0 0 0 0 0 0 0 0 3 0 0	$.235863377-08$	$.617062443+01$	$.233141314+06$	0
13	0 0 1 0 0 0 0 0 0 $\underline{1}$ 0	$.234591373-08$	$.301604017+01$	$.780515857+05$	0
14	0 0 1 0 0 $\underline{1}$ 0 0 0 0 0	$.213887252-08$	$.869076685+00$	$.606977676+04$	0
15	0 0 2 0 $\underline{2}$ 0 0 0 0 0 0	$.213372423-08$	$.231058229+01$	$.115067698+05$	0
16	0 0 1 0 0 0 0 0 $\underline{1}$ 0 0	$.194614513-08$	$.210000307+01$	$.714306956+05$	0
17	0 1 $\underline{1}$ 0 0 0 0 0 0 0 0	$.159461069-08$	$.140766028+01$	$.393020970+04$	0
18	0 0 0 0 0 0 0 0 1 0 2	$.148218731-08$	$.484802610+00$	$.244287600+06$	0
19	0 0 2 $\underline{2}$ 0 0 0 0 0 0 0	$.119632368-08$	$.520983306+00$	$.588492685+04$	0
20	0 0 1 0 0 0 0 0 0 0 0	$.262187629-06$	$.426353652+01$	$.628307585+04$	1
21	0 0 0 0 0 0 0 0 0 0 0	$.828213108-07$	$.000000000+00$	$.000000000+00$	1
22	0 0 2 0 0 0 0 0 0 0 0	$.131837889-07$	$.421314844+01$	$.125661517+05$	1
23	0 0 1 0 0 0 0 0 0 0 0	$.112121512-07$	$.266805824+01$	$.628307585+04$	2

Table 21. z_E^1(TT) (ICRS)

1	001000000000	$.999826263+00$	$.175348528+01$	$.628307585+04$	0
2	002000000000	$.835254790-02$	$.171034482+01$	$.125661517+05$	0
3	000000000000	$.590519388-02$	$.000000000+00$	$.000000000+00$	0
4	000010000000	$.493123615-02$	$.374115795+01$	$.529690965+03$	0
5	000001000000	$.271697274-02$	$.401601399+01$	$.213299095+03$	0
6	000000010000	$.154647692-02$	$.217052025+01$	$.381330356+02$	0
7	000000100000	$.836647708-03$	$.233967709+01$	$.747815986+02$	0
8	000000000000	$.122106596-02$	$.000000000+00$	$.000000000+00$	1
9	002000000000	$.515000016-03$	$.600266210+01$	$.125661517+05$	1

Table 22. z_E^2(TT) (ICRS)

1	001000000000	$.917380368+00$	$.182658535+00$	$.628307585+04$	0
2	000000000000	$.220960396-01$	$.314159265+01$	$.000000000+00$	0
3	002000000000	$.766363633-02$	$.139528008+00$	$.125661517+05$	0
4	000010000000	$.453127036-02$	$.218025318+01$	$.529690965+03$	0
5	000001000000	$.251550288-02$	$.246154598+01$	$.213299095+03$	0
6	000000010000	$.143167360-02$	$.609170505+00$	$.381330356+02$	0
7	000000100000	$.766049695-03$	$.774439757+00$	$.747815986+02$	0
8	001000000000	$.909408593-03$	$.269013703+00$	$.628307585+04$	1
9	000000000000	$.831610152-03$	$.000000000+00$	$.000000000+00$	1

Table 23. z_E^3(TT) (ICRS)

1	001000000000	$.397732879+00$	$.182658320+00$	$.628307585+04$	0
2	000000000000	$.959237348-02$	$.314159265+01$	$.000000000+00$	0
3	002000000000	$.332258690-02$	$.139529338+00$	$.125661517+05$	0
4	000010000000	$.194477282-02$	$.211821614+01$	$.529690965+03$	0
5	000001000000	$.104351978-02$	$.234916313+01$	$.213299095+03$	0
6	000000010000	$.586898458-03$	$.543535503+00$	$.381330356+02$	0
7	001000000000	$.208884483-02$	$.341431350+01$	$.628307585+04$	1

Table 24. $x^1_{(C)S}$(TDB) (BRSC)

1	000010000000	$.495675760-02$	$.374107340+01$	$.529690965+03$	0
2	000001000000	$.271848958-02$	$.401601132+01$	$.213299095+03$	0
3	000000010000	$.154649144-02$	$.217052033+01$	$.381330356+02$	0
4	000000100000	$.836686623-03$	$.233967992+01$	$.747815986+02$	0
5	000000000000	$.293757368-03$	$.000000000+00$	$.000000000+00$	0
6	000020000000	$.120131527-03$	$.409073192+01$	$.105938193+04$	0
7	000000000000	$.129648364-04$	$.314159265+01$	$.000000000+00$	1

Table 25. $x^2_{(C)S}$(TDB) (BRSC)

1	000010000000	$.495539239-02$	$.217046696+01$	$.529690965+03$	0
2	000001000000	$.272232470-02$	$.244443347+01$	$.213299095+03$	0
3	000000010000	$.154657739-02$	$.599269940+00$	$.381330356+02$	0
4	000000100000	$.836315130-03$	$.768801711+00$	$.747815986+02$	0
5	000000000000	$.338645302-03$	$.000000000+00$	$.000000000+00$	0
6	000020000000	$.120119004-03$	$.252003115+01$	$.105938193+04$	0

Table 26. $x^3_{(C)S}$(TDB) (BRSC)

1	000001000000	$.118126725-03$	$.460786726+00$	$.213299095+03$	0
2	000010000000	$.112777688-03$	$.416899276+00$	$.529690965+03$	0

Table 27. $z^1_{(C)S}$(TT) (BRSC)

1	000010000000	$.495675768-02$	$.374107340+01$	$.529690965+03$	0
2	000001000000	$.271848962-02$	$.401601132+01$	$.213299095+03$	0
3	000000010000	$.154649147-02$	$.217052033+01$	$.381330356+02$	0
4	000000100000	$.836686635-03$	$.233967992+01$	$.747815986+02$	0
5	000000000000	$.293757372-03$	$.000000000+00$	$.000000000+00$	0
6	000020000000	$.120131529-03$	$.409073192+01$	$.105938193+04$	0
7	000000000000	$.129648366-04$	$.314159265+01$	$.000000000+00$	1

Table 28. $z^2_{(C)S}$(TT) (BRSC)

1	000010000000	$.495539247-02$	$.217046696+01$	$.529690965+03$	0
2	000001000000	$.272232474-02$	$.244443347+01$	$.213299095+03$	0
3	000000010000	$.154657741-02$	$.599269940+00$	$.381330356+02$	0
4	000000100000	$.836315143-03$	$.768801711+00$	$.747815986+02$	0
5	000000000000	$.338645307-03$	$.000000000+00$	$.000000000+00$	0
6	000020000000	$.120119006-03$	$.252003115+01$	$.105938193+04$	0

Table 29. $z^3_{(C)S}$(TT) (BRSC)

1	000001000000	$.118126726-03$	$.460786726+00$	$.213299095+03$	0
2	000010000000	$.112777690-03$	$.416899276+00$	$.529690965+03$	0

Table 30. $x^1_{(C)S}(\mathrm{TDB}) - x^1_{(C)E}(\mathrm{TDB})$ (BRSC)

1	001000000000	.999829288 + 00	.489507817 + 01	.628307585 + 04	0
2	002000000000	.835257300 − 02	.485193771 + 01	.125661517 + 05	0
3	000000000000	.561144206 − 02	.314159265 + 01	.000000000 + 00	0
4	003000000000	.104666280 − 03	.480881860 + 01	.188492276 + 05	0
5	001000001000	.311083836 − 04	.381034400 + 01	.839968473 + 05	0
6	000010000000	.255249796 − 04	.372469456 + 01	.529690965 + 03	0
7	023000000000	.213725626 − 04	.423394472 + 01	.157734354 + 04	0
8	005830000000	.170910281 − 04	.363699489 + 01	.627955274 + 04	0
9	003830000000	.170788154 − 04	.301156249 + 01	.628659896 + 04	0
10	012000000000	.144524153 − 04	.331135016 + 00	.235286616 + 04	0
11	000000000000	.123403056 − 02	.314159265 + 01	.000000000 + 00	1
12	002000000000	.515001561 − 03	.286106968 + 01	.125661517 + 05	1
13	003000000000	.129072646 − 04	.281783809 + 01	.188492276 + 05	1
14	001000000000	.106862704 − 04	.515713425 + 01	.628307585 + 04	1
15	000000000000	.414321698 − 04	.000000000 + 00	.000000000 + 00	2
16	002000000000	.217569548 − 04	.125840551 + 01	.125661517 + 05	2

Table 31. $x^2_{(C)S}(\mathrm{TDB}) - x^2_{(C)E}(\mathrm{TDB})$ (BRSC)

1	001000000000	.999892110 + 00	.332425139 + 01	.628307585 + 04	0
2	000000000000	.244269904 − 01	.000000000 + 00	.000000000 + 00	0
3	002000000000	.835292314 − 02	.328112111 + 01	.125661517 + 05	0
4	003000000000	.104669649 − 03	.323800906 + 01	.188492276 + 05	0
5	001000001000	.311083782 − 04	.223954775 + 01	.839968473 + 05	0
6	000010000000	.257033846 − 04	.215944691 + 01	.529690965 + 03	0
7	023000000000	.214747320 − 04	.580412821 + 01	.157734354 + 04	0
8	005830000000	.170921895 − 04	.206621136 + 01	.627955274 + 04	0
9	003830000000	.170798696 − 04	.144073560 + 01	.628659896 + 04	0
10	012000000000	.144026477 − 04	.504227413 + 01	.235286616 + 04	0
11	000000000000	.930463241 − 03	.314159265 + 01	.000000000 + 00	1
12	002000000000	.515066093 − 03	.129021200 + 01	.125661517 + 05	1
13	003000000000	.129079956 − 04	.124701233 + 01	.188492276 + 05	1
14	001000000000	.464550057 − 05	.268570631 + 01	.628307585 + 04	1
15	000000000000	.508020784 − 04	.314159265 + 01	.000000000 + 00	2
16	002000000000	.217801602 − 04	.597116776 + 01	.125661517 + 05	2
17	001000000000	.102048661 − 04	.149587436 + 01	.628307585 + 04	2

Table 32. $x^3_{(C)S}(\mathrm{TDB}) - x^3_{(C)E}(\mathrm{TDB})$ (BRSC)

1	000000000100	.279620336 − 05	.571088092 − 01	.843346616 + 05	0
2	034000000000	.101625375 − 05	.228088862 + 01	.550755323 + 04	0
3	001000000000	.227822442 − 02	.272132221 + 00	.628307585 + 04	1
4	000000000000	.542928153 − 04	.314159265 + 01	.000000000 + 00	1
5	002000000000	.190318304 − 04	.229019720 + 00	.125661517 + 05	1
6	001000000000	.972198866 − 04	.201074444 + 01	.628307585 + 04	2

Table 33. $w^1_{C(S)}$(TT) (DGRSC)

1	001000000000	.999829318 + 00	.489507817 + 01	.628307585 + 04	0
2	002000000000	.835257312 − 02	.485193771 + 01	.125661517 + 05	0
3	000000000000	.561144214 − 02	.314159265 + 01	.000000000 + 00	0
4	003000000000	.104666281 − 03	.480881860 + 01	.188492276 + 05	0
5	001000001000	.311083840 − 04	.381034400 + 01	.839968473 + 05	0
6	000010000000	.255249800 − 04	.372469456 + 01	.529690965 + 03	0
7	$02\underline{3}000000000$	.213725629 − 04	.423394472 + 01	.157734354 + 04	0
8	$005\underline{8}30000000$	.170910283 − 04	.363699489 + 01	.627955274 + 04	0
9	$003\underline{8}30000000$	.170788156 − 04	.301156249 + 01	.628659896 + 04	0
10	$01\underline{2}000000000$	.144524155 − 04	.331135016 + 00	.235286616 + 04	0
11	000000000000	.123175694 − 02	.314159265 + 01	.000000000 + 00	1
12	002000000000	.515711556 − 03	.286168449 + 01	.125661517 + 05	1
13	001000000000	.908876963 − 04	.343805847 + 01	.628307585 + 04	1
14	003000000000	.129161600 − 04	.281814577 + 01	.188492276 + 05	1
15	000000000000	.413455480 − 04	.000000000 + 00	.000000000 + 00	2
16	002000000000	.218048745 − 04	.125847520 + 01	.125661517 + 05	2

Table 34. $w^2_{C(S)}$(TT) (DGRSC)

1	001000000000	.999892140 + 00	.332425139 + 01	.628307585 + 04	0
2	000000000000	.244269907 − 01	.000000000 + 00	.000000000 + 00	0
3	002000000000	.835292326 − 02	.328112111 + 01	.125661517 + 05	0
4	003000000000	.104669650 − 03	.323800906 + 01	.188492276 + 05	0
5	001000001000	.311083787 − 04	.223954775 + 01	.839968473 + 05	0
6	000010000000	.257033850 − 04	.215944691 + 01	.529690965 + 03	0
7	$02\underline{3}000000000$	.214747323 − 04	.580412821 + 01	.157734354 + 04	0
8	$005\underline{8}30000000$	.170921898 − 04	.206621136 + 01	.627955274 + 04	0
9	$003\underline{8}30000000$	.170798699 − 04	.144073560 + 01	.628659896 + 04	0
10	$01\underline{2}000000000$	.144026479 − 04	.504227413 + 01	.235286616 + 04	0
11	000000000000	.929940950 − 03	.314159265 + 01	.000000000 + 00	1
12	002000000000	.515776033 − 03	.129082682 + 01	.125661517 + 05	1
13	001000000000	.959041937 − 04	.179238919 + 01	.628307585 + 04	1
14	003000000000	.129168905 − 04	.124732002 + 01	.188492276 + 05	1
15	000000000000	.506872207 − 04	.314159265 + 01	.000000000 + 00	2
16	002000000000	.218280744 − 04	.597123669 + 01	.125661517 + 05	2
17	001000000000	.102050976 − 04	.149590644 + 01	.628307585 + 04	2

Table 35. $w^3_{C(S)}$(TT) (DGRSC)

1	000000000100	.279620340 − 05	.571088092 − 01	.843346616 + 05	0
2	$03\underline{4}000000000$	.101625379 − 05	.228088870 + 01	.550755323 + 04	0
3	001000000000	.227822453 − 02	.272132216 + 00	.628307585 + 04	1
4	000000000000	.542928164 − 04	.314159265 + 01	.000000000 + 00	1
5	002000000000	.190318311 − 04	.229019720 + 00	.125661517 + 05	1
6	001000000000	.971153599 − 04	.201092674 + 01	.628307585 + 04	2

Table 36. $x^1_{(C)L}(\mathrm{TDB}) - x^1_{(C)E}(\mathrm{TDB})$ (BRSC)

1	001000001000	$.256024110 - 02$	$.381034400 + 01$	$.839968473 + 05$	0
2	0010000010$\underline{1}$0	$.210839483 - 03$	$.459633356 + 01$	$.709933052 + 03$	0
3	001000001010	$.706718204 - 04$	$.616589962 + 01$	$.167283762 + 06$	0
4	00100000$\underline{1}$010	$.416838748 - 04$	$.519365786 + 01$	$.118562187 + 05$	0
5	00100000$\underline{1}$000	$.214904049 - 04$	$.344606385 + 01$	$.714306956 + 05$	0
6	0010000030$\underline{1}$0	$.155419779 - 04$	$.556853615 + 01$	$.156137476 + 06$	0
7	001000001000	$.841152656 - 05$	$.223954055 + 01$	$.839968473 + 05$	2
8	0010000010$\underline{1}$0	$.391101997 - 05$	$.302550739 + 01$	$.709933052 + 03$	2

Table 37. $x^2_{(C)L}(\mathrm{TDB}) - x^2_{(C)E}(\mathrm{TDB})$ (BRSC)

1	001000001000	$.256024056 - 02$	$.223954775 + 01$	$.839968473 + 05$	0
2	0010000010$\underline{1}$0	$.210798143 - 03$	$.302563223 + 01$	$.709933052 + 03$	0
3	001000001010	$.706718003 - 04$	$.459510340 + 01$	$.167283762 + 06$	0
4	00100000$\underline{1}$010	$.416689655 - 04$	$.362303498 + 01$	$.118562187 + 05$	0
5	00100000$\underline{1}$000	$.213957440 - 04$	$.501472051 + 01$	$.714306956 + 05$	0
6	0010000030$\underline{1}$0	$.155419757 - 04$	$.399773986 + 01$	$.156137476 + 06$	0
7	001000001000	$.841278338 - 05$	$.667969677 + 00$	$.839968473 + 05$	2
8	0010000010$\underline{1}$0	$.391072633 - 05$	$.145472425 + 01$	$.709933052 + 03$	2

Table 38. $x^3_{(C)L}(\mathrm{TDB}) - x^3_{(C)E}(\mathrm{TDB})$ (BRSC)

1	000000000100	$.230129534 - 03$	$.571088093 - 01$	$.843346616 + 05$	0
2	0000000001$\underline{1}$0	$.187020868 - 04$	$.843145825 + 00$	$.104774732 + 04$	0
3	001000001000	$.583690198 - 05$	$.547483962 + 01$	$.839968473 + 05$	1
4	000000000100	$.146565996 - 05$	$.476909062 + 01$	$.843346616 + 05$	2

Table 39. $z^1_{(C)L}(\mathrm{TT}) - z^1_{(C)E}(\mathrm{TT})$ (BRSC)

1	001000001000	$.256024114-02$	$.381034400+01$	$.839968473+05$	0
2	$00100000101\underline{1}0$	$.210839486-03$	$.459633356+01$	$.709933052+03$	0
3	001000001010	$.706718214-04$	$.616589962+01$	$.167283762+06$	0
4	$00100000\underline{1}010$	$.416838755-04$	$.519365786+01$	$.118562187+05$	0
5	$00100000\underline{1}000$	$.214904052-04$	$.344606385+01$	$.714306956+05$	0
6	$0010000030\underline{1}0$	$.155419781-04$	$.556853615+01$	$.156137476+06$	0
7	001000001000	$.841152668-05$	$.223954055+01$	$.839968473+05$	2
8	$0010000010\underline{1}0$	$.391102003-05$	$.302550739+01$	$.709933052+03$	2

Table 40. $z^2_{(C)L}(\mathrm{TT}) - z^2_{(C)E}(\mathrm{TT})$ (BRSC)

1	001000001000	$.256024060-02$	$.223954775+01$	$.839968473+05$	0
2	$0010000010\underline{1}0$	$.210798146-03$	$.302563223+01$	$.709933052+03$	0
3	001000001010	$.706718014-04$	$.459510340+01$	$.167283762+06$	0
4	$00100000\underline{1}010$	$.416689661-04$	$.362303498+01$	$.118562187+05$	0
5	$00100000\underline{1}000$	$.213957444-04$	$.501472051+01$	$.714306956+05$	0
6	$0010000030\underline{1}0$	$.155419760-04$	$.399773986+01$	$.156137476+06$	0
7	001000001000	$.841278351-05$	$.667969677+00$	$.839968473+05$	2
8	$0010000010\underline{1}0$	$.391072638-05$	$.145472425+01$	$.709933052+03$	2

Table 41. $z^3_{(C)L}(\mathrm{TT}) - z^3_{(C)E}(\mathrm{TT})$ (BRSC)

1	000000000100	$.230129537-03$	$.571088096-01$	$.843346616+05$	0
2	$0000000001\underline{1}0$	$.187020871-04$	$.843145825+00$	$.104774732+04$	0
3	001000001000	$.583690207-05$	$.547483962+01$	$.839968473+05$	1
4	000000000100	$.146565998-05$	$.476909062+01$	$.843346616+05$	2

Table 42. $w^1_{C(L)}$(TT) (DGRSC)

1	001000001000	$.256024117-02$	$.381034400+01$	$.839968473+05$	0
2	0010000010$\underline{1}$0	$.210839488-03$	$.459633356+01$	$.709933052+03$	0
3	001000001010	$.706718223-04$	$.616589962+01$	$.167283762+06$	0
4	00100000$\underline{1}$010	$.416838755-04$	$.519365786+01$	$.118562187+05$	0
5	00100000$\underline{1}$000	$.214904118-04$	$.344606385+01$	$.714306956+05$	0
6	0010000030$\underline{1}$0	$.155419783-04$	$.556853615+01$	$.156137476+06$	0
7	001000001000	$.841152515-05$	$.223954055+01$	$.839968473+05$	2
8	0010000010$\underline{1}$0	$.391102014-05$	$.302550744+01$	$.709933052+03$	2

Table 43. $w^2_{C(L)}$(TT) (DGRSC)

1	001000001000	$.256024063-02$	$.223954775+01$	$.839968473+05$	0
2	0010000010$\underline{1}$0	$.210798148-03$	$.302563223+01$	$.709933052+03$	0
3	001000001010	$.706718023-04$	$.459510340+01$	$.167283762+06$	0
4	00100000$\underline{1}$010	$.416689661-04$	$.362303498+01$	$.118562187+05$	0
5	00100000$\underline{1}$000	$.213957509-04$	$.501472051+01$	$.714306956+05$	0
6	0010000030$\underline{1}$0	$.155419762-04$	$.399773986+01$	$.156137476+06$	0
7	001000001000	$.841278198-05$	$.667969677+00$	$.839968473+05$	2
8	0010000010$\underline{1}$0	$.391072609-05$	$.145472420+01$	$.709933052+03$	2

Table 44. $w^3_{C(L)}$(TT) (DGRSC)

1	000000000100	$.230129539-03$	$.571088096-01$	$.843346616+05$	0
2	0000000001$\underline{1}$0	$.187020873-04$	$.843145825+00$	$.104774732+04$	0
3	001000001000	$.583690226-05$	$.547483961+01$	$.839968473+05$	1
4	000000000100	$.146566000-05$	$.476909062+01$	$.843346616+05$	2

Appendix C

Final Expansions

Newtonian (N) and relativistic SMART solutions in terms of the dynamical and kinematical Euler angles are given in the tables below in the same form as in Appendix B.

Table 45. Differences $\underset{1}{\psi}_N - \underset{1}{\psi}$ (TT) (DGRSC)
Table 46. Differences $\underset{1}{\theta}_N - \underset{1}{\theta}$ (TT) (DGRSC)
Table 47. Differences $\underset{1}{\varphi} - \underset{1}{\varphi}_N$ (TT) (DGRSC)

Table 48. Differences $\underset{1}{\psi} - \underset{0}{\psi}$ (TT) (GRSC)
Table 49. Differences $\underset{1}{\theta} - \underset{0}{\theta}$ (TT) (GRSC)
Table 50. Differences $\underset{0}{\varphi} - \underset{1}{\varphi}$ (TT) (GRSC)

Table 51. Differences $\underset{0}{\psi}_N - \underset{0}{\psi}$ (TT) (KGRSC)
Table 52. Differences $\underset{0}{\theta}_N - \underset{0}{\theta}$ (TT) (KGRSC)
Table 53. Differences $\underset{0}{\varphi} - \underset{0}{\varphi}_N$ (TT) (KGRSC)

Table 45. Differences $\underset{1}{\psi}_N - \underset{1}{\psi}$ (TT) (DGRSC)

1	0 0 1 $000001\underline{1}00$	$.231266586-10$	$.252922703+01$	$.337814272+03$	0
2	0 $010\underline{19}03000000$	$.759550646-11$	$.284970036+01$	$.980309527+00$	0
3	0 0 2 $000002\underline{2}00$	$.285365891-12$	$.348721854+01$	$.675628545+03$	0
4	0 0 0 $0001\underline{2}0000$	$.177282778-12$	$.363336072+01$	$.148447271+01$	0
5	010 $\underline{3}$ $0000000\underline{1}0$	$.141872493-12$	$.607912246+01$	$.328633904+01$	0
6	0 0 2 000000000	$.754544995-13$	$.193506521+01$	$.125661517+05$	0
7	0 0 1 $000001\underline{1}00$	$.780322332-08$	$.959162770+00$	$.337814272+03$	1
8	0 0 2 000000000	$.115291472-08$	$.365346519+00$	$.125661517+05$	1
9	0 0 2 000002000	$.199920367-09$	$.447910095+01$	$.167993695+06$	1
10	0 0 2 $000002\underline{2}00$	$.188795275-09$	$.191827564+01$	$.675628545+03$	1
11	0 0 1 $000001\underline{1}00$	$.190446025-08$	$.568655793+01$	$.337814272+03$	2
12	0 0 2 000000000	$.562966495-09$	$.194262028+01$	$.125661517+05$	2
13	0 0 0 000000000	$.117110967-09$	$.000000000+00$	$.000000000+00$	2
14	0 0 0 000000000	$.382641250-09$	$.314159265+01$	$.000000000+00$	3
15	0 0 1 $000001\underline{1}00$	$.234166411-09$	$.420125496+01$	$.337814272+03$	3
16	0 0 2 000000000	$.137527601-09$	$.352339891+01$	$.125661517+05$	3
17	0 0 0 000000000	$.250403613-08$	$.000000000+00$	$.000000000+00$	4

Table 46. Differences $\underset{1}{\theta}_N - \underset{1}{\theta}$ (TT) (DGRSC)

1	00 1 $000001\underline{1}00$	$.123517565-10$	$.410001583+01$	$.337814272+03$	0
2	00 2 $000002\underline{2}00$	$.122942182-12$	$.505796083+01$	$.675628545+03$	0
3	$0010\underline{19}03000000$	$.527537894-13$	$.715313761+00$	$.980309527+00$	0
4	00 1 $000001\underline{1}00$	$.416734462-08$	$.252995484+01$	$.337814272+03$	1
5	00 2 000000000	$.499381069-09$	$.507773508+01$	$.125661517+05$	1
6	00 1 $000001\underline{1}00$	$.101645481-08$	$.965285399+00$	$.337814272+03$	2
7	00 0 000000000	$.323583375-09$	$.314159265+01$	$.000000000+00$	2
8	00 2 000000000	$.243819609-09$	$.373329495+00$	$.125661517+05$	2
9	00 0 000000000	$.859605900-08$	$.314159265+01$	$.000000000+00$	3
10	00 1 $000001\underline{1}00$	$.124723317-09$	$.577800493+01$	$.337814272+03$	3

Table 47. Differences $\underset{1}{\varphi} - \underset{1}{\varphi}_N$ (TT) (DGRSC)

1	0	0	1	000001$\underline{1}$00	$.212112060 - 10$	$.567081957 + 01$	$.337814272 + 03$	0
2	0	0101$\underline{19}$03000000			$.265955171 - 11$	$.625381360 + 00$	$.980309527 + 00$	0
3	0	0	2	000002$\underline{2}$00	$.261989022 - 12$	$.345626689 + 00$	$.675628545 + 03$	0
4	010		$\underline{3}$	0000000$\underline{1}$0	$.123819694 - 12$	$.329819546 + 01$	$.328633904 + 01$	0
5	0	0	0	0001$\underline{2}$0000	$.162652877 - 12$	$.491774874 + 00$	$.148447271 + 01$	0
6	0	0	2	000000000	$.692284581 - 13$	$.507665785 + 01$	$.125661517 + 05$	0
7	0	0	1	000001$\underline{1}$00	$.715812097 - 08$	$.410075543 + 01$	$.337814272 + 03$	1
8	0	0	2	000000000	$.105778263 - 08$	$.350693917 + 01$	$.125661517 + 05$	1
9	0	0	2	000002000	$.183423115 - 09$	$.133750830 + 01$	$.167993695 + 06$	1
10	0	0	2	000002$\underline{2}$00	$.173274092 - 09$	$.505986832 + 01$	$.675628545 + 03$	1
11	0	0	1	000001$\underline{1}$00	$.174700634 - 08$	$.254497057 + 01$	$.337814272 + 03$	2
12	0	0	2	000000000	$.516512839 - 09$	$.508421183 + 01$	$.125661517 + 05$	2
13	0	0	0	000000000	$.107461963 - 09$	$.314159265 + 01$	$.000000000 + 00$	2
14	0	0	0	000000000	$.340588092 - 09$	$.000000000 + 00$	$.000000000 + 00$	3
15	0	0	1	000001$\underline{1}$00	$.214853467 - 09$	$.105830183 + 01$	$.337814272 + 03$	3
16	0	0	2	000000000	$.126190964 - 09$	$.381970499 + 00$	$.125661517 + 05$	3
17	0	0	0	000000000	$.250626787 - 08$	$.314159265 + 01$	$.000000000 + 00$	4

Table 48. Differences $\psi_1 - \psi_0$ (TT) (GRSC)

1	001000000000	.742300349 − 09	.466926087 + 01	.628307585 + 04	0
2	000000000000	.397739929 − 10	.000000000 + 00	.000000000 + 00	0
3	001000001$\underline{1}$00	.145708061 − 10	.252987892 + 01	.337814272 + 03	0
4	002000000000	.930524508 − 11	.462564700 + 01	.125661517 + 05	0
5	00002$\underline{5}$000000	.203878362 − 11	.709948865 + 00	.711354700 + 01	0
6	000000000000	.930785387 − 04	.000000000 + 00	.000000000 + 00	1
7	001000000000	.474945452 − 10	.260332598 + 01	.628307585 + 04	1
8	001000001$\underline{1}$00	.355965693 − 11	.960586737 + 00	.337814272 + 03	1
9	002000000000	.116661765 − 11	.259734787 + 01	.125661517 + 05	1
10	000000000000	.244280065 − 06	.314159265 + 01	.000000000 + 00	2
11	001000001$\underline{1}$00	.299152471 − 10	.417657693 + 01	.337814272 + 03	2
12	001000000000	.203316536 − 11	.956674751 + 00	.628307585 + 04	2
13	002000000000	.172295028 − 11	.340748453 + 01	.125661517 + 05	2
14	000000000000	.365594907 − 08	.314159265 + 01	.000000000 + 00	3
15	001000001$\underline{1}$00	.117386023 − 10	.261099272 + 01	.337814272 + 03	3
16	002000000000	.121017860 − 11	.500979962 + 01	.125661517 + 05	3
17	000000000000	.579411556 − 08	.000000000 + 00	.000000000 + 00	4
18	001000001$\underline{1}$00	.269183168 − 11	.106628445 + 01	.337814272 + 03	4
19	000000000000	.388062532 − 10	.314159265 + 01	.000000000 + 00	5
20	000000000000	.247036977 − 10	.314159265 + 01	.000000000 + 00	6

Table 49. Differences $\theta_1 - \theta_0$ (TT) (GRSC)

1	001000001$\underline{1}$00	.631723382 − 11	.410068897 + 01	.337814272 + 03	0
2	000000000000	.322968763 − 11	.000000000 + 00	.000000000 + 00	0
3	00002$\underline{5}$000000	.957304138 − 12	.524343393 + 01	.711354700 + 01	0
4	000000000000	.473867892 − 10	.000000000 + 00	.000000000 + 00	1
5	001000001$\underline{1}$00	.154310985 − 11	.252997314 + 01	.337814272 + 03	1
6	000000000000	.947355383 − 08	.000000000 + 00	.000000000 + 00	2
7	001000001$\underline{1}$00	.883496788 − 11	.569290077 + 01	.337814272 + 03	2
8	000000000000	.228920555 − 07	.314159265 + 01	.000000000 + 00	3
9	001000001$\underline{1}$00	.216768803 − 11	.418750886 + 01	.337814272 + 03	3
10	000000000000	.653596505 − 10	.314159265 + 01	.000000000 + 00	4
11	000000000000	.184605964 − 09	.000000000 + 00	.000000000 + 00	5
12	000000000000	.179171613 − 11	.314159265 + 01	.000000000 + 00	6

Table 50. Differences $\underset{0}{\varphi} - \underset{1}{\varphi}$ (TT) (GRSC)

1	001000001$\underline{1}$00	$.158810048 - 10$	$.567147813 + 01$	$.337814272 + 03$	0
2	000000000000	$.103746803 - 10$	$.314159265 + 01$	$.000000000 + 00$	0
3	00002$\underline{5}$000000	$.213211607 - 11$	$.376137398 + 01$	$.711354700 + 01$	0
4	000000000000	$.191527491 - 10$	$.000000000 + 00$	$.000000000 + 00$	1
5	001000000000	$.423492232 - 11$	$.466929724 + 01$	$.628307585 + 04$	1
6	001000001$\underline{1}$00	$.387872333 - 11$	$.409816916 + 01$	$.337814272 + 03$	1
7	000000000000	$.265537808 - 06$	$.000000000 + 00$	$.000000000 + 00$	2
8	001000001$\underline{1}$00	$.274735254 - 10$	$.104917664 + 01$	$.337814272 + 03$	2
9	002000000000	$.164479145 - 11$	$.278147098 + 00$	$.125661517 + 05$	2
10	000000000000	$.388849875 - 08$	$.000000000 + 00$	$.000000000 + 00$	3
11	001000001$\underline{1}$00	$.115340467 - 10$	$.575444761 + 01$	$.337814272 + 03$	3
12	002000000000	$.116412840 - 11$	$.185998408 + 01$	$.125661517 + 05$	3
13	000000000000	$.631104235 - 08$	$.314159265 + 01$	$.000000000 + 00$	4
14	001000001$\underline{1}$00	$.265593077 - 11$	$.420628407 + 01$	$.337814272 + 03$	4
15	000000000000	$.380544861 - 10$	$.000000000 + 00$	$.000000000 + 00$	5
16	000000000000	$.267916561 - 10$	$.000000000 + 00$	$.000000000 + 00$	6

Table 51. Differences $\underset{0}{\psi}_N - \underset{0}{\psi}$ (TT) (KGRSC)

1	0 0 1 $000001\underline{1}00$	$.231266586 - 10$	$.252922703 + 01$	$.337814272 + 03$	0
2	0 $010\underline{19}03000000$	$.759550646 - 11$	$.284970036 + 01$	$.980309527 + 00$	0
3	0 0 2 $000002\underline{2}00$	$.285365890 - 12$	$.348721854 + 01$	$.675628545 + 03$	0
4	0 0 0 $0001\underline{2}0000$	$.177282778 - 12$	$.363336072 + 01$	$.148447271 + 01$	0
5	010 $\underline{3}$ $0000000\underline{1}0$	$.141872493 - 12$	$.607912246 + 01$	$.328633904 + 01$	0
6	0 0 2 000000000	$.754544995 - 13$	$.193506521 + 01$	$.125661517 + 05$	0
7	0 0 1 $000001\underline{1}00$	$.780322332 - 08$	$.959162770 + 00$	$.337814272 + 03$	1
8	0 0 2 000000000	$.115291472 - 08$	$.365346519 + 00$	$.125661517 + 05$	1
9	0 0 2 000002000	$.199920367 - 09$	$.447910095 + 01$	$.167993695 + 06$	1
10	0 0 2 $000002\underline{2}00$	$.188795275 - 09$	$.191827564 + 01$	$.675628545 + 03$	1
11	0 0 1 $000001\underline{1}00$	$.190446025 - 08$	$.568655793 + 01$	$.337814272 + 03$	2
12	0 0 2 000000000	$.562966495 - 09$	$.194262028 + 01$	$.125661517 + 05$	2
13	0 0 0 000000000	$.117110967 - 09$	$.000000000 + 00$	$.000000000 + 00$	2
14	0 0 0 000000000	$.382641250 - 09$	$.314159265 + 01$	$.000000000 + 00$	3
15	0 0 1 $000001\underline{1}00$	$.234166521 - 09$	$.420126686 + 01$	$.337814272 + 03$	3
16	0 0 2 000000000	$.137527571 - 09$	$.352339649 + 01$	$.125661517 + 05$	3
17	0 0 0 000000000	$.250403635 - 08$	$.000000000 + 00$	$.000000000 + 00$	4

Table 52. Differences $\underset{0}{\theta}_N - \underset{0}{\theta}$ (TT) (KGRSC)

1	00 1 $000001\underline{1}00$	$.123517565 - 10$	$.410001583 + 01$	$.337814272 + 03$	0
2	00 2 $000002\underline{2}00$	$.122942182 - 12$	$.505796083 + 01$	$.675628545 + 03$	0
3	$0010\underline{19}03000000$	$.527537894 - 13$	$.715313761 + 00$	$.980309527 + 00$	0
4	00 1 $000001\underline{1}00$	$.416734462 - 08$	$.252995484 + 01$	$.337814272 + 03$	1
5	00 2 000000000	$.499381069 - 09$	$.507773508 + 01$	$.125661517 + 05$	1
6	00 1 $000001\underline{1}00$	$.101645481 - 08$	$.965285401 + 00$	$.337814272 + 03$	2
7	00 0 000000000	$.323583375 - 09$	$.314159265 + 01$	$.000000000 + 00$	2
8	00 2 000000000	$.243819609 - 09$	$.373329495 + 00$	$.125661517 + 05$	2
9	00 0 000000000	$.859605900 - 08$	$.314159265 + 01$	$.000000000 + 00$	3
10	00 1 $000001\underline{1}00$	$.124723404 - 09$	$.577801150 + 01$	$.337814272 + 03$	3

Table 53. Differences $\underset{0}{\varphi} - \underset{0}{\varphi}_N$ (TT) (KGRSC)

1	0	0	1	00000$\underline{1}$100	.212112060 − 10	.567081957 + 01	.337814272 + 03	0
2	0	010$\underline{19}$03000000			.265955171 − 11	.625381360 + 00	.980309527 + 00	0
3	0	0	2	00000$\underline{2}$200	.261989021 − 12	.345626689 + 00	.675628545 + 03	0
4	010		$\underline{3}$	0000000$\underline{1}$0	.123819694 − 12	.329819546 + 01	.328633904 + 01	0
5	0	0	0	0001$\underline{2}$0000	.162652877 − 12	.491774874 + 00	.148447271 + 01	0
6	0	0	2	000000000	.692284581 − 13	.507665785 + 01	.125661517 + 05	0
7	0	0	1	00000$\underline{1}$100	.715812097 − 08	.410075543 + 01	.337814272 + 03	1
8	0	0	2	000000000	.105778263 − 08	.350693917 + 01	.125661517 + 05	1
9	0	0	2	000002000	.183423115 − 09	.133750830 + 01	.167993695 + 06	1
10	0	0	2	00000$\underline{2}$200	.173274092 − 09	.505986832 + 01	.675628545 + 03	1
11	0	0	1	00000$\underline{1}$100	.174700634 − 08	.254497058 + 01	.337814272 + 03	2
12	0	0	2	000000000	.516512839 − 09	.508421183 + 01	.125661517 + 05	2
13	0	0	0	000000000	.107461963 − 09	.314159265 + 01	.000000000 + 00	2
14	0	0	0	000000000	.340588092 − 09	.000000000 + 00	.000000000 + 00	3
15	0	0	1	00000$\underline{1}$100	.214853535 − 09	.105831374 + 01	.337814272 + 03	3
16	0	0	2	000000000	.126190931 − 09	.381968078 + 00	.125661517 + 05	3
17	0	0	0	000000000	.250626807 − 08	.314159265 + 01	.000000000 + 00	4

Lecture 10. Relativistic reduction of optical observations

The information given by astronomical observations characterizes not only the object of observation but the observer as well. This information depends on the position of the observer, its velocity and the value of the gravitational potential at the point of observation. In order to use information obtained by different observers or even by one and the same observer but at different moments of time it is necessary to perform a reduction of measurements (observations), i.e. to refer them to some conventional point of the space–time events. This reduction has been performed even in classical astronomy (e.g., aberration, parallax, etc.) but GRT introduces its own corrections to classical reduction and in addition makes its specific contribution due to the influence of the gravitational field on the results of measurement. The second type of reduction specific to GRT is related to the reduction of coordinates, i.e. the transformation of coordinate–dependent quantities into measurable quantities).

Just as in Newtonian astrometry there might be various options to treat the problem of GRT reduction. The exposition below is based on algorithms of (Brumberg, 1991; Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992; Klioner, 2003; Brumberg, 2000). The main tool herewith is the three–level (deep) RS hierarchy of Lecture 7 (7.52 to be more specific).

Measurable vs. coordinate light direction

Any relativistic RS may be described by metric of the type

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \qquad x^0 = ct \,, \tag{10.1}$$

greek indices running values from 0 to 3 with summation over repeating indices. Even in case of Newtonian rotation of the spatial axes coefficients $g_{\mu\nu}$ are supposed to differ from their Minkowski values $\eta_{\mu\nu}$ (special relativity flat space–time) by small corrections $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{10.2}$$

$$\eta_{00} = 1 \,, \qquad \eta_{0i} = 0 \,, \qquad \eta_{ij} = -\delta_{ij} \,, \tag{10.3}$$

$$h_{00} \,, h_{ij} \sim \mathrm{O}(c^{-2}), \qquad h_{0i} \sim \begin{cases} \mathrm{O}(c^{-1}) & \text{for} \quad \text{rotating} \quad \mathrm{RS} \,, \\ \mathrm{O}(c^{-3}) & \text{for} \quad \text{nonrotating} \quad \mathrm{RS} \,, \end{cases}$$

latin indices running values from 1 to 3. Four–dimensional quadratic form (3.1) may be reduced locally to the algebraic sum of squares. At first, one has

$$ds^2 = c^2 d\tau^2 - dl^2 \tag{10.4}$$

with

$$d\tau = \frac{1}{\sqrt{g_{00}}} g_{0\alpha} dx^\alpha \tag{10.5}$$

and

$$dl^2 = \gamma_{ik} dx^i dx^k \tag{10.6}$$

where

$$\gamma_{ik} = \frac{1}{g_{00}} g_{0i} g_{0k} - g_{ik} \,. \tag{10.7}$$

By substituting (10.2) into (10.5) and (10.7) one yields

$$d\tau = (1 + h_{00})^{1/2} c dt + (1 + h_{00})^{-1/2} h_{0i} dx^i \tag{10.8}$$

and

$$\gamma_{ik} = \delta_{ik} - h_{ik} + (1 + h_{00})^{-1} h_{0i} h_{0k} \,. \tag{10.9}$$

The three–dimensional quadratic form (10.6) may be easily reduced to the sum of squares

$$dl^2 = \delta_{ik} dx^{(i)} dx^{(k)} \tag{10.10}$$

by linear transformation

$$dx^{(i)} = dx^i + \lambda_{ij} dx^j \tag{10.11}$$

with symmetrical coefficients λ_{ij} to be determined from the equations

$$2\lambda_{ik} + \lambda_{mi}\lambda_{mk} = -h_{ik} + (1 + h_{00})^{-1} h_{0i} h_{0k} \,. \tag{10.12}$$

Metric (10.4) with (10.10) presented locally as Minkowski metric enables one to find the measurable light direction in the form

$$p^{(i)} = c^{-1} \frac{dx^{(i)}}{d\tau} \,. \tag{10.13}$$

These components are to be compared with the components $\dot{x}^i$ of the coordinate light velocity in the field (10.1). Using (10.8) and (10.13) one obtains

$$p^{(i)} = \frac{c^{-1}(\dot{x}^i + \lambda_{ij}\dot{x}^j)}{(1 + h_{00})^{1/2} + c^{-1}(1 + h_{00})^{-1/2} h_{0k}\dot{x}^k} \tag{10.14}$$

and

$$c^{-1}\dot{x}^i = \frac{p^{(i)} - \lambda_{ij} p^{(j)}}{(1 + h_{00})^{-1/2} - (1 + h_{00})^{-1} h_{0k} p^{(k)}} \,. \tag{10.15}$$

Until now all the above formulas are rigorous. The approximate solution of (10.12) results in

$$\begin{aligned} \lambda_{ik} = & -\tfrac{1}{2} h_{ik} + \tfrac{1}{2} h_{0i} h_{0k} - \tfrac{1}{8} h_{im} h_{km} - \tfrac{1}{2} h_{00} h_{0i} h_{0k} + \\ & + \tfrac{1}{8}(h_{0i} h_{km} + h_{0k} h_{im}) h_{0m} - \tfrac{1}{8} h_{0i} h_{0k} h_{0m} h_{0m} + \mathrm{O}(c^{-6}). \end{aligned} \tag{10.16}$$

In case of non–rotating RS all terms containing h_{0i} should be omitted. For non–rotating systems in harmonic coordinates within the post-Newtonian approximation one has

$$h_{ij} = \delta_{ij} h_{00} \tag{10.17}$$

and

$$\lambda_{ik} = -\tfrac{1}{2}\delta_{ik}h_{00} + \mathrm{O}(c^{-4}), \tag{10.18}$$

$$p^{(i)} = (1 - h_{00})c^{-1}\dot{x}^i + \mathrm{O}(c^{-4}), \tag{10.19}$$

$$c^{-1}\dot{x}^i = (1 + h_{00})p^{(i)} + \mathrm{O}(c^{-4}). \tag{10.20}$$

General Reduction for BRS, GRS, SRS

First of all, to distinguish between BRS, GRS and SRS quantities let us mark GRS and SRS quantities by hat and tilde, respectively. The starting point in the general reduction technique developed in (Brumberg, 1991) is the solution of the BRS equations of the light propagation. This solution may be presented in the form

$$\mathbf{x}(t) = \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\mathbf{x}\,, \tag{10.21}$$

$$\dot{\mathbf{x}}(t) = c\boldsymbol{\sigma} + \Delta\dot{\mathbf{x}}\,, \tag{10.22}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0\,, \qquad \dot{\mathbf{x}}(-\infty) = c\boldsymbol{\sigma}\,, \qquad \boldsymbol{\sigma}^2 = 1\,, \tag{10.23}$$

$\mathbf{x}_0$ and $\boldsymbol{\sigma}$ being two arbitrary vectorial constants. Here $\Delta\mathbf{x}$ and $\Delta\dot{\mathbf{x}}$ stand for general relativity terms. Their expressions may be found in (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992) but one may use any other suitable expressions for these quantities as well. The technique exposed below does not demand their explicit expressions. One may relate BRS coordinate light velocity $\dot{x}^i$ and BRS measurable light direction $p^{(i)}$ to result in virtue of (10.22) in the BRS $(t\,,x^i)$ reduction formula

$$v^i = \frac{dx^i}{dt}\,, \qquad p^{(i)} = \sigma^i + \delta p^{(i)}\,. \tag{10.24}$$

In the same manner by applying (10.14), (10.15) one finds the relationship between the GRS coordinate light velocity $\hat{v}^i$ and GRS measurable light direction $\hat{p}^{(i)}$. On the other hand, BRS→GRS transformation involves the relationship between v^i and $\hat{v}^i$. In result, one gets the GRS $(\hat{t}\,,\hat{x}^i\,,q)$ reduction formula

$$\hat{v}^i = \frac{d\hat{x}^i}{d\hat{t}} = v^i + \delta\hat{v}^i\,, \qquad \hat{p}^{(i)} = p^{(i)} + \delta\hat{p}^{(i)}\,. \tag{10.25}$$

Repeating this procedure once more one relates the SRS coordinate light velocity $\tilde{v}^i$ and SRS measurable light direction $\tilde{p}^{(i)}$. On the other hand, GRS→SRS transformation involves the relationship between $\hat{v}^i$ and $\tilde{v}^i$. In result, one gets the SRS $(\tilde{t}\,,\tilde{x}^i\,,\hat{q}\,,\tilde{q})$ reduction formula

$$\tilde{v}^i = \frac{d\tilde{x}^i}{d\tilde{t}} = \hat{v}^i + \delta\tilde{v}^i\,, \qquad \tilde{p}^{(i)} = \hat{p}^{(i)} + \delta\tilde{p}^{(i)}\,. \tag{10.26}$$

Combination of (10.24)–(10.26) leads to the final reduction formula

$$\tilde{p}^{(i)} = \sigma^i + \delta p^{(i)} + \delta\hat{p}^{(i)} + \delta\tilde{p}^{(i)} \tag{10.27}$$

relating the actually observed light direction $\tilde{p}^{(i)}$ and the coordinate light direction σ^i. In what follows we will give correction terms $\delta p^{(i)}$, $\delta\hat{p}^{(i)}$, $\delta\tilde{p}^{(i)}$ as well as auxiliary corrections $\delta\hat{v}^i$, $\delta\tilde{v}^i$. The rigorous expressions (10.12), (10.14), (10.15) enable one to compute these quantities within the accuracy of BRS→GRS and GRS→SRS transformations. To have simple analytical formulas for fast evaluation of the relativistic effects we give below the post–Newtonian expressions based on (10.18)–(10.20). An equivalent reduction formula slightly different from (10.27) by its form is presented in (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992). Formula (10.27) is directly valid for observations from an Earth's artificial satellite. It may be easily modified for other cases as, for instance, observations from an interplanetary probe. In the latter case it is sufficient to do only one transformation from BRS to the system related to such probe.

BRS Reduction (gravitational deflection of light, annual parallax, proper motion, planetary aberration)
Rewriting (10.22) in the form

$$c^{-1}\dot{x}^i = \sigma^i + c^{-1}\Delta\dot{x}^i \tag{10.28}$$

one finds the relativistic term of the BRS reduction formula (10.24)

$$\delta p^{(i)} = -h_{00}\sigma^i + c^{-1}\Delta\dot{x}^i\,. \tag{10.29}$$

$\Delta\dot{x}^i$ is to be taken as a solution of the BRS equations of light propagation, σ^i is to be transformed, if necessary, to take into account parallax and proper motion corrections (see below).

General reduction formulas (10.28), (10.29) may be specified using more explicit approximate expressions. First of all, the conversion to spherical coordinates is often needed

$$\mathbf{p} = -(\cos\alpha\cos\delta\,,\,\sin\alpha\cos\delta\,,\,\sin\delta)$$

$$\cos\delta\Delta\alpha = \sin\alpha\Delta p^1 - \cos\alpha\Delta p^2 + \sin\delta\Delta\alpha\Delta\delta$$

$$\Delta\delta = \sin\delta(\cos\alpha\Delta p^1 + \sin\alpha\Delta p^2) - \cos\delta\Delta p^3 - \tfrac{1}{2}\sin\delta\cos\delta(\Delta\alpha)^2\,.$$

Then, the 1–PNA solution foe the Cauchy problem of light propagation in BRS reads

$$\Delta\mathbf{x} = 2\sum_A m_a\left(\frac{\boldsymbol{\sigma}\times(\mathbf{r}_{0A}\times\boldsymbol{\sigma})}{r_{0A}-\boldsymbol{\sigma}\mathbf{r}_{0A}} - \frac{\boldsymbol{\sigma}\times(\mathbf{r}_A\times\boldsymbol{\sigma})}{r_A-\boldsymbol{\sigma}\mathbf{r}_A} - \boldsymbol{\sigma}\ln\frac{r_A+\boldsymbol{\sigma}\mathbf{r}_A}{r_{0A}+\boldsymbol{\sigma}\mathbf{r}_{0A}}\right), \tag{10.30}$$

$$\Delta\dot{\mathbf{x}} = -2c\sum_A\frac{m_a}{r_A}\left(\boldsymbol{\sigma} + \frac{\boldsymbol{\sigma}\times(\mathbf{r}_A\times\boldsymbol{\sigma})}{r_A-\boldsymbol{\sigma}\mathbf{r}_A}\right), \tag{10.31}$$

$$\Delta\mathbf{p} = -2\sum_A\frac{m_a}{r_A}\frac{\boldsymbol{\sigma}\times(\mathbf{r}_A\times\boldsymbol{\sigma})}{r_A-\boldsymbol{\sigma}\mathbf{r}_A} \tag{10.32}$$

with

$$\mathbf{r}_A = \mathbf{x}-\mathbf{x}_A\,,\qquad \mathbf{r}_{0A} = \mathbf{x}_0-\mathbf{x}_A\,,\qquad m_A = GM_A/c^2\,.$$

As usually, summation is performed here over all solar system bodies marked by capital Latin letters. Relativistic terms due to the non–sphericity of the bodies and their rotation as well as the second order monopole terms (post–post–Newtonian terms) may be added, by example, from (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992) or elsewhere.

The formulas given above involve the expression for the angular distance between two sources

$$\cos\psi = \frac{(\mathbf{PQ})_{rel}}{P_{rel}Q_{rel}}\,, \qquad (\mathbf{PQ})_{rel} = \gamma_{ik}P^iQ^k\,, \qquad P_{rel} = (\gamma_{ik}P^iP^k)^{1/2},$$

$$\mathbf{P} = c^{-1}\dot{\mathbf{r}}_1(t), \qquad \mathbf{Q} = c^{-1}\dot{\mathbf{r}}_2(t),$$

$$\cos\psi = \mathbf{p}_1\mathbf{p}_2 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + 2\sum_A \frac{m_A}{r_A}\left(\frac{\mathbf{r}_A\times\boldsymbol{\sigma}_1}{r_A - \boldsymbol{\sigma}_1\mathbf{r}_A} - \frac{\mathbf{r}_A\times\boldsymbol{\sigma}_2}{r_A - \boldsymbol{\sigma}_2\mathbf{r}_A}\right)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2). \tag{10.33}$$

Corrections for parallax and proper motion may be taken just as in Newtonian astrometry (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992). These corrections are related to the boundary problem of light propagation. Denoting

$$\mathbf{x}(t_0) = \mathbf{x}_0\,, \qquad \mathbf{x}(t) = \mathbf{x}\,, \qquad \mathbf{R}(t\,,\,t_0) = \mathbf{x} - \mathbf{x}_0 \qquad (t > t_0) \tag{10.34}$$

one may rewrite (10.21) in the form

$$c(t-t_0)\boldsymbol{\sigma} = \mathbf{R} - \Delta\mathbf{x}\,, \qquad \Delta\mathbf{x}(t_0) = 0\,. \tag{10.35}$$

Hence,

$$c(t-t_0) = R\left[1 - \frac{2}{R^2}\mathbf{R}\Delta\mathbf{x} + \frac{1}{R^2}(\Delta\mathbf{x})^2\right]^{1/2}. \tag{10.36}$$

Introducing the unit vector directed to the light source

$$\mathbf{k} = -\frac{\mathbf{R}}{R} \tag{10.37}$$

one gets

$$c(t-t_0) = R\left[1 + \frac{1}{R}\mathbf{k}\Delta\mathbf{x} + \frac{1}{2R^2}(\Delta\mathbf{x}\times\mathbf{k})^2 + \ldots\right] \tag{10.38}$$

and

$$\begin{aligned}\boldsymbol{\sigma} = &-\mathbf{k} - \frac{1}{R}[\mathbf{k}\times(\Delta\mathbf{x}\times\mathbf{k})] + \frac{1}{2R^2}(\Delta\mathbf{x}\times\mathbf{k})^2\mathbf{k} + \\ &+ \frac{1}{R^2}(\mathbf{k}\Delta\mathbf{x})[\mathbf{k}\times(\Delta\mathbf{x}\times\mathbf{k})] + \ldots\,.\end{aligned} \tag{10.39}$$

For monopole structure masses in 1–PNA it follows

$$\boldsymbol{\sigma} = \frac{\mathbf{R}}{R} + 2\sum_A \frac{m_A}{R}\frac{r_A - r_{0A} + R}{|\mathbf{r}_{0A}\times\mathbf{r}_A|^2}[\mathbf{R}\times(\mathbf{r}_{0A}\times\mathbf{r}_A)] \tag{10.40}$$

resulting to the coordinate direction

$$c^{-1}\frac{d\mathbf{x}(t)}{dt} = \frac{\mathbf{R}}{R} - \frac{2}{R}\sum_A \frac{m_A}{r_A}\left(\mathbf{R} + \frac{\mathbf{R}\times(\mathbf{r}_{0A}\times\mathbf{r}_A)}{r_{0A}r_A + \mathbf{r}_{0A}\mathbf{r}_A}\right), \tag{10.41}$$

the observed direction

$$\mathbf{p} = \frac{\mathbf{R}}{R} - \frac{2}{R}\sum_A \frac{m_A}{r_A}\frac{\mathbf{R}\times(\mathbf{r}_{0A}\times\mathbf{r}_A)}{r_{0A}r_A + \mathbf{r}_{0A}\mathbf{r}_A} \tag{10.42}$$

and the time of flight

$$c(t-t_0) = R + 2\sum_A m_A \ln\frac{r_A + r_{0A} + R}{r_A + r_{0A} - R}\,. \tag{10.43}$$

Parallactic expansions ($\mathbf{x} \neq 0$) in powers ρ/ρ_0 for the source outside the solar system

$$\rho \equiv |\mathbf{x}| \ll |\mathbf{x}_0| \equiv \rho_0 \tag{10.44}$$

take the form

$$\boldsymbol{\sigma} = -\frac{\mathbf{x}_0}{\rho_0} + \frac{1}{\rho_0^3}[\mathbf{x}_0\times(\mathbf{x}\times\mathbf{x}_0)] + \ldots + \frac{2}{\rho_0^3}\sum_A \frac{m_A}{r_A}\frac{\mathbf{x}_0\times(\mathbf{r}_A\times\mathbf{x}_0)}{1 + (\mathbf{x}_0\mathbf{r}_A/\rho_0 r_A)}\,, \tag{10.45}$$

$$\mathbf{p} = -\frac{\mathbf{x}_0}{\rho_0} + \frac{1}{\rho_0^3}[\mathbf{x}_0\times(\mathbf{x}\times\mathbf{x}_0)] + \ldots - \frac{2}{\rho_0^2}\sum_A \frac{m_A}{r_A^2}\frac{\mathbf{x}_0\times(\mathbf{r}_A\times\mathbf{x}_0)}{1 + (\mathbf{x}_0\mathbf{r}_A/\rho_0 r_A)}\,, \tag{10.46}$$

$$c(t-t_0) = \rho_0\left(1 - \frac{\mathbf{x}\mathbf{x}_0}{\rho_0^2} + \ldots\right) + 2\sum_A m_A \ln\frac{2\rho_0^2}{\rho_0 r_A + \mathbf{x}_0\mathbf{r}_A}\,. \tag{10.47}$$

Comparing (10.45) and (10.46) it is seen that for $\rho_0 \to \infty$ the GRT correction (light deflection) disappears in $\boldsymbol{\sigma}$ and remains in $\mathbf{p}$. The unit vector $\mathbf{k}$ admits the expansion

$$\begin{aligned}\mathbf{k} =& \frac{\mathbf{x}_0}{\rho} - \frac{1}{\rho^3}[\mathbf{x}_0\times(\mathbf{x}\times\mathbf{x}_0)] - \frac{1}{2\rho^5}(\mathbf{x}\times\mathbf{x}_0)^2\mathbf{x}_0 - \\ &- \frac{1}{\rho^5}(\mathbf{x}\mathbf{x}_0)[\mathbf{x}_0\times(\mathbf{x}\times\mathbf{x}_0)] + \ldots\,.\end{aligned} \tag{10.48}$$

Proper motion correction is introduced to take into account the time interval between the initial epoch of emission t_0^* and the moment t_0 of the light emission. One has therewith

$$\mathbf{x}_0(t_0) = \mathbf{x}_0^* + \dot{\mathbf{x}}_0^*\Delta t_0 + \tfrac{1}{2}\ddot{\mathbf{x}}_0^*(\Delta t_0)^2 + \ldots \tag{10.49}$$

with

$$\Delta t_0 = t_0 - t_0^*\,, \qquad \mathbf{x}_0^* = \mathbf{x}_0(t_0^*), \qquad \dot{\mathbf{x}}_0^* = \dot{\mathbf{x}}_0(t_0^*), \qquad \ddot{\mathbf{x}}_0^* = \ddot{\mathbf{x}}_0(t_0^*). \tag{10.50}$$

Putting

$$\rho^* = |\mathbf{x}_0^*|, \qquad \mathbf{k}_0 = \frac{\mathbf{x}_0^*}{\rho^*} \tag{10.51}$$

and using auxiliary expansions

$$\frac{1}{\rho} = \frac{1}{\rho^*}\Big\{1 - \frac{1}{\rho^*}\mathbf{k}_0\dot{\mathbf{x}}_0^*\Delta t_0 + \frac{1}{2\rho^*}\Big[\frac{3}{\rho^*}(\mathbf{k}_0\dot{\mathbf{x}}_0^*)^2 - \frac{1}{\rho^*}(\dot{\mathbf{x}}_0^*)^2 - \mathbf{k}_0\ddot{\mathbf{x}}_0^*\Big](\Delta t_0)^2 + \ldots\Big\},$$

and

$$\begin{aligned}\frac{\mathbf{x}_0}{\rho} =& \mathbf{k}_0 + \frac{1}{\rho^*}[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)]\Delta t_0 + \frac{1}{\rho^*}\Big\{\tfrac{1}{2}[\mathbf{k}_0 \times (\ddot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \\ &- \frac{1}{\rho^*}(\mathbf{k}_0\dot{\mathbf{x}}_0^*)[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \frac{1}{2\rho^*}(\mathbf{k}_0 \times \dot{\mathbf{x}}_0^*)^2\mathbf{k}_0\Big\}(\Delta t_0)^2 + \ldots\end{aligned}$$

one gets

$$\begin{aligned}\mathbf{k} =& \mathbf{k}_0 - \frac{1}{\rho^*}[\mathbf{k}_0 \times (\mathbf{x} \times \mathbf{k}_0)] - \frac{1}{2\rho^{*2}}(\mathbf{x} \times \mathbf{k}_0)^2\mathbf{k}_0 - \frac{1}{\rho^{*2}}(\mathbf{x}\mathbf{k}_0)[\mathbf{k}_0 \times (\mathbf{x} \times \mathbf{k}_0)] + \\ &+ \frac{1}{\rho^*}\Big\{[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] + \frac{1}{\rho^*}(\mathbf{x}\mathbf{k}_0)[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] + \\ &+ \frac{1}{\rho^*}(\dot{\mathbf{x}}_0^*\mathbf{k}_0)[\mathbf{k}_0 \times (\mathbf{x} \times \mathbf{k}_0)] + \frac{1}{\rho^*}\mathbf{k}_0\big(\mathbf{x}[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)]\big)\Big\}\Delta t_0 + \\ &+ \frac{1}{\rho^*}\Big\{\tfrac{1}{2}[\mathbf{k}_0 \times (\ddot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \frac{1}{\rho^*}(\dot{\mathbf{x}}_0^*\mathbf{k}_0)[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \\ &- \frac{1}{2\rho^*}(\mathbf{x} \times \mathbf{k}_0^*)^2\mathbf{k}_0\Big\}(\Delta t_0)^2 + \ldots .\end{aligned} \tag{10.52}$$

Introducing now the vector of parallax

$$\boldsymbol{\pi} = \frac{1}{\rho^*}[\mathbf{k}_0 \times (\mathbf{x} \times \mathbf{k}_0)] \tag{10.53}$$

and vector of proper motion

$$\boldsymbol{\mu} = \mathbf{k}_0 \times (\dot{\mathbf{k}}_0 \times \mathbf{k}_0) \tag{10.54}$$

and using the evident relations

$$\mathbf{k}_0\boldsymbol{\pi} = 0\,, \qquad \mathbf{k}_0\boldsymbol{\mu} = 0$$

as well as the derivatives

$$\dot{\mathbf{k}}_0 = \frac{1}{\rho^*}[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] \tag{10.55}$$

and

$$\dot{\boldsymbol{\mu}} = \frac{1}{\rho^*}\Big\{[\mathbf{k}_0 \times (\ddot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \frac{1}{\rho^*}(\mathbf{k}_0 \times \dot{\mathbf{x}}_0^*)^2\mathbf{k}_0 - \frac{2}{\rho^*}(\mathbf{k}_0\dot{\mathbf{x}}_0^*)[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)]\Big\} \tag{10.56}$$

one may present (10.52) in the more compressed form

$$\mathbf{k} = \big(1 + \boldsymbol{\pi}\boldsymbol{\mu}\Delta t_0 - \tfrac{1}{2}\boldsymbol{\pi}^2\big)\mathbf{k}_0 + \big(1 + \frac{1}{\rho^*}\mathbf{x}\mathbf{k}_0\big)\boldsymbol{\mu}\Delta t_0 - \\ - \big(1 + \frac{1}{\rho^*}\mathbf{x}\mathbf{k}_0 - \frac{1}{\rho^*}\dot{\mathbf{x}}_0^*\mathbf{k}_0\Delta t_0\big)\boldsymbol{\pi} + \tfrac{1}{2}\dot{\boldsymbol{\mu}}(\Delta t_0)^2 + \ldots . \tag{10.57}$$

It remains to eliminate the unmeasurable time interval Δt_0 at the point of the light emission by means of the transformation $\Delta t_0 \to \Delta t = t - t^*$, t^* being the BRS moment corresponding to t_0^* at the point of the light reception. The expression for the measurable time interval Δt at the point of observation may be found from the simple relations of light propagation. One has

$$t - t_0 = c^{-1}R + \ldots , \qquad \boldsymbol{R} = \mathbf{x} - \mathbf{x}_0(t_0) = \mathbf{x} - \mathbf{x}_0^* - \dot{\mathbf{x}}_0^*\Delta t_0 + \ldots \tag{10.58}$$

and

$$t^* - t_0^* = c^{-1}R^* + \ldots , \qquad \mathbf{R}^* = \mathbf{x}^* - \mathbf{x}_0^* , \qquad \mathbf{x}^* = \mathbf{x}(t^*) \tag{10.59}$$

with

$$R = \rho^* + \mathbf{k}_0\dot{\mathbf{x}}_0^*\Delta t_0 - \mathbf{x}\mathbf{k}_0 + \ldots , \qquad R^* = \rho^* - \mathbf{x}^*\mathbf{k}_0 + \ldots . \tag{10.60}$$

Taking the difference of (10.58) and (10.59) one gets

$$\Delta t - \Delta t_0 = c^{-1}(R - R^*) + \ldots = c^{-1}\mathbf{k}_0(\dot{\mathbf{x}}_0^*\Delta t_0 - \mathbf{x} + \mathbf{x}^*) + \ldots \tag{10.61}$$

and finally

$$\Delta t_0 = (1 + c^{-1}\mathbf{k}_0\dot{\mathbf{x}}_0^*)^{-1}(\Delta t + c^{-1}\mathbf{k}_0\mathbf{x} - c^{-1}\mathbf{k}_0\mathbf{x}^*). \tag{10.62}$$

Combination of (10.39), (10.57) and (10.62) enables one to include completely the parallax and proper motion corrections.

GRS reduction (annual aberration, geodesic rotation, relativistic acceleration terms) Applied to GRS quantities the equation (10.19) involves

$$\hat{p}^{(i)} = (1 - \hat{h}_{00})c^{-1}\hat{v}^i \tag{10.63}$$

and then

$$\hat{p}^{(i)} = (1 + h_{00} - \hat{h}_{00})p^{(i)} + c^{-1}\delta\hat{v}^i . \tag{10.64}$$

BRS→GRS transformation involves

$$\delta\hat{v}^i = - v_E^i + (c^{-1}v^k v_E^k)c^{-1}v^i + c^{-1}[(c^{-1}v^k v_E^k)^2 c^{-1}v^i - \tfrac{1}{2}c^{-1}v^i v_E^2 - \\ - \tfrac{1}{2}v_E^i(c^{-1}v^k v_E^k) + (qF^{ik} + 2D^{ik} + 2D^{ikm}\hat{x}^m)c^{-1}v^k + a_E^k\hat{x}^k c^{-1}v^i]. \tag{10.65}$$

By substituting this expression into (10.64) one gets

$$\hat{p}^{(i)} = (1 + h_{00} - \hat{h}_{00})p^{(i)} + c^{-1}[\mathbf{p} \times (\mathbf{p} \times \mathbf{v}_E)]^{(i)} + c^{-2}(\mathbf{p}\mathbf{v}_E)[\mathbf{p} \times (\mathbf{p} \times \mathbf{v}_E)]^{(i)} - \\ - \tfrac{1}{2}c^{-2}[\mathbf{v}_E \times (\mathbf{p} \times \mathbf{v}_E)]^{(i)} + c^{-2}(qF^{ik} + 2D^{ik} + 2D^{ikm}\hat{x}^m)p^{(k)} + \\ + c^{-2}a_E^k\hat{x}^k p^{(i)} . \tag{10.66}$$

In virtue of the relation
$$h_{00} - \hat{h}_{00} = -2\bar{U}_E(\mathbf{x}_E) - 2a^k_E \hat{x}^k \tag{10.67}$$
one obtains the GRS reduction formula
$$\begin{aligned}\delta\hat{p}^{(i)} =& c^{-1}[\mathbf{p}\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} + c^{-2}(\mathbf{p}\mathbf{v}_E)[\mathbf{p}\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} - \\ &- \tfrac{1}{2}c^{-2}[\mathbf{v}_E\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} + c^{-2}qF^{ik}p^{(k)} + c^{-2}(a^k_E\hat{x}^i - a^i_E\hat{x}^k)p^{(k)}\,. \end{aligned}\tag{10.68}$$
In (10.66) we have used
$$D^{ik}(t) = \delta_{ik}\bar{U}_E(\mathbf{x}_E) \tag{10.69}$$
and
$$D^{ikm}(t) = \tfrac{1}{2}(\delta_{ik}a^m_E + \delta_{im}a^k_E - \delta_{km}a^i_E) \tag{10.70}$$
where $\bar{U}_E(\mathbf{x})$ stands for the Newtonian potential of all solar system bodies excepting the Earth, x^i_E, v^i_E and a^i_E being Earth's BRS position, velocity and acceleration, respectively, with
$$a^i_E = \bar{U}_{E,i}(\mathbf{x}_E) - Q_i\,, \tag{10.70}$$
Q_i being non–geodesic acceleration in the Earth's BRS motion. Besides,
$$\dot{F}^{ij} = \tfrac{3}{2}(v^i_E a^j_E - v^j_E a^i_E) - 2\big[\bar{U}^i_{E,j}(\mathbf{x}_E) - \bar{U}^j_{E,i}(\mathbf{x}_E)\big] + 2(v^i_E Q_j - v^j_E Q_i), \tag{10.71}$$
$\bar{U}^i_E(\mathbf{x})$ denoting the Newtonian vector–potential of all solar system bodies excepting the Earth. As usually, comma denotes the partial derivative with respect to the variable separated by comma.

From the GRS reduction formula it follows that the angular distance between two sources considered in GRS will be
$$\begin{aligned}\cos\hat{\psi} =& \hat{\mathbf{p}}_1\hat{\mathbf{p}}_2 = \cos\psi + c^{-1}(\mathbf{p}_1\mathbf{p}_2 - 1)(\mathbf{p}_1\mathbf{v}_E + \mathbf{p}_2\mathbf{v}_E) + \\ &+ c^{-2}(\mathbf{p}_1\mathbf{p}_2 - 1)[(\mathbf{p}_1\mathbf{v}_E)^2 + (\mathbf{p}_2\mathbf{v}_E)^2 + (\mathbf{p}_1\mathbf{v}_E)(\mathbf{p}_2\mathbf{v}_E) - \mathbf{v}^2_E].\end{aligned}\tag{10.72}$$
One should not the disappearance of the geodesic rotation and GRT acceleration terms as compared with the corresponding expression in BRS.

SRS reduction (diurnal aberration, GRT rotation and acceleration terms)
By applying now (10.19) to SRS one has
$$\tilde{p}^{(i)} = (1 - \tilde{h}_{00})c^{-1}\tilde{v}^i \tag{10.73}$$
and then
$$\tilde{p}^{(i)} = (1 + \hat{h}_{00} - \tilde{h}_{00})\hat{p}^{(i)} + c^{-1}\delta\tilde{v}^i\,. \tag{10.74}$$
GRS→SRS transformation involves
$$\begin{aligned}\delta\tilde{v}^i =& -\hat{v}^i_S + (c^{-1}\hat{v}^k\hat{v}^k_S)c^{-1}\hat{v}^i + c^{-1}\big\{(c^{-1}\hat{v}^k\hat{v}^k_S)^2c^{-1}\hat{v}^i - \tfrac{1}{2}c^{-1}\hat{v}^i\hat{v}^2_S - \\ &- \tfrac{1}{2}\hat{v}^i_S(c^{-1}\hat{v}^k\hat{v}^k_S) + \big[\hat{q}R^{ik} + (\tilde{q}-1)K^{ik} + 2\mathcal{D}^{ik} + 2\mathcal{D}^{ikm}\tilde{x}^m\big]c^{-1}\hat{v}^k + \\ &+ \hat{a}^k_S\tilde{x}^k c^{-1}\hat{v}^i\big\}.\end{aligned}\tag{10.75}$$

Combining (10.74) and (10.75) one gets

$$\begin{aligned}\tilde{p}^{(i)} =&(1+\hat{h}_{00}-\tilde{h}_{00})\hat{p}^{(i)} + c^{-1}[\hat{\mathbf{p}}\times(\hat{\mathbf{p}}\times\hat{\mathbf{v}}_S)]^{(i)} + c^{-2}(\hat{\mathbf{p}}\hat{\mathbf{v}}_S)[\hat{\mathbf{p}}\times(\hat{\mathbf{p}}\times\hat{\mathbf{v}}_S)]^{(i)}-\\ &-\tfrac{1}{2}c^{-2}\big[\hat{\mathbf{v}}_S\times(\hat{\mathbf{p}}\times\hat{\mathbf{v}}_S)\big]^{(i)} + c^{-2}\big[\hat{q}R^{ik}+(\tilde{q}-1)K^{ik}+2\mathcal{D}^{ik}+\\ &+2\mathcal{D}^{ikm}\tilde{x}^m\big]\hat{p}^{(k)} + c^{-2}\hat{a}^k_S\tilde{x}^k\hat{p}^{(i)}. \end{aligned} \tag{10.76}$$

By means of the relation

$$\hat{h}_{00}-\tilde{h}_{00} = -2\big[\hat{U}_E(\hat{\mathbf{x}}_S)+Q_k\hat{x}^k_S+T(\hat{\mathbf{x}}_S)\big]-2\hat{a}^k_S\tilde{x}^k \tag{10.77}$$

one gets the SRS reduction formula

$$\begin{aligned}\delta\tilde{p}^{(i)} =&c^{-1}[\hat{\mathbf{p}}\times(\hat{\mathbf{p}}\times\hat{\mathbf{v}}_S)]^{(i)} + c^{-2}(\hat{\mathbf{p}}\hat{\mathbf{v}}_S)[\hat{\mathbf{p}}\times(\hat{\mathbf{p}}\times\hat{\mathbf{v}}_S)]^{(i)}-\\ &-\tfrac{1}{2}c^{-2}[\hat{\mathbf{v}}_S\times(\hat{\mathbf{p}}\times\hat{\mathbf{v}}_S)]^{(i)} + c^{-2}\big[\hat{q}R^{ik}+(\tilde{q}-1)K^{ik}\big]\hat{p}^{(k)}+\\ &+c^{-2}(\hat{a}^k_S\tilde{x}^i-\hat{a}^i_S\tilde{x}^k)\hat{p}^{(k)}\,. \end{aligned} \tag{10.78}$$

In (10.76) one meets the coefficients

$$\mathcal{D}^{ik}(\hat{t}) = \big[\hat{U}_E(\hat{\mathbf{x}}_S)+Q_m\hat{x}^m_S+T(\hat{\mathbf{x}}_S)\big] \tag{10.79}$$

and

$$\mathcal{D}^{ikm}(\hat{t}) = \tfrac{1}{2}(\delta_{ik}\hat{a}^m_S+\delta_{im}\hat{a}^k_S-\delta_{km}\hat{a}^i_S) \tag{10.80}$$

with the GRS tidal potential

$$T(\hat{\mathbf{x}}) = \bar{U}_E(\mathbf{x}_E+\hat{\mathbf{x}})-\bar{U}_E(\mathbf{x}_E)-\bar{U}_{E,j}(\mathbf{x}_E)\hat{x}^j \tag{10.81}$$

and geocentric satellite acceleration $\hat{a}^i_S$. Introducing the quantity

$$E_i = -\hat{a}^i_S+\hat{U}_{E,i}(\hat{\mathbf{x}}_S)+Q_i+\bar{U}_{E,i}(\mathbf{x}_E+\hat{\mathbf{x}}_S)-\bar{U}_{E,i}(\mathbf{x}_E) \tag{10.82}$$

one may present the $\hat{t}$–derivative of the topocentric–type precession in the form

$$\begin{aligned}\dot{R}_{ij} =&\tfrac{3}{2}(\hat{v}^i_S\hat{a}^j_S-\hat{v}^j_S\hat{a}^i_S)+(\dot{a}^i_E\hat{x}^j_S-\dot{a}^j_E\hat{x}^i_S)-2[\hat{U}^i_{E,j}(\hat{\mathbf{x}}_S)-\hat{U}^j_{E,i}(\hat{\mathbf{x}}_S)]+\\ &+2[v^i_E\bar{U}_{E,jk}(\mathbf{x}_E)-v^j_E\bar{U}_{E,ik}(\mathbf{x}_E)-\bar{U}^i_{E,jk}(\mathbf{x}_E)+\bar{U}^j_{E,ik}(\mathbf{x}_E)]\hat{x}^k_S+\\ &+2(\hat{v}^i_SE_j-\hat{v}^j_SE_i), \end{aligned} \tag{10.83}$$

$\hat{U}_E(\hat{\mathbf{x}})$ and $\hat{U}^i_E(\hat{\mathbf{x}})$ being the GRS geopotential and vector–geopotential. For SRS the quantity E_i vanishes resulting to the Newtonian GRS satellite equations of motion. If S denotes a point on the surface of the Earth then E_i does not vanish and the right–hand of (10.83) relates just to the topocentric precession. The purely kinematical precession caused by two subsequent BRS→GRS and GRS→SRS transformations (Klioner, 1993) is given by

$$K^{ij} = \hat{x}^i_Sa^j_E-\hat{x}^j_Sa^i_E+\tfrac{1}{2}(\hat{v}^i_Sv^j_E-\hat{v}^j_Sv^i_E). \tag{10.84}$$

From the SRS reduction formula it follows that the angular distance between two sources considered in SRS will be

$$\begin{aligned}\cos\tilde{\psi} =& \tilde{\mathbf{p}}_1\tilde{\mathbf{p}}_2 = \cos\hat{\psi} + c^{-1}(\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2 - 1)(\hat{\mathbf{p}}_1\hat{\mathbf{v}}_S + \hat{\mathbf{p}}_2\hat{\mathbf{v}}_S) + \\ &+ c^{-2}(\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2 - 1)[(\hat{\mathbf{p}}_1\hat{\mathbf{v}}_S)^2 + (\hat{\mathbf{p}}_2\hat{\mathbf{v}}_S)^2 + (\hat{\mathbf{p}}_1\hat{\mathbf{v}}_S)(\hat{\mathbf{p}}_2\hat{\mathbf{v}}_S) - \hat{\mathbf{v}}_S^2].\end{aligned} \tag{10.85}$$

Again one should note the disappearance of the GRT rotation and acceleration terms.

Rotating reference systems and Rotation Matrices

To conclude this lecture it may be useful to remind a set of well–known formulas to deal with rotating RSs and rotation matrices. In GRT framework the rotating systems demand special treatment. Very luckily this treatment is not necessary for practical application. In relativistic celestial mechanics and relativistic astrometry one may use (at least for present applications) just classical theory of three–dimensional rotation. It violates the agreement to use only harmonic coordinates but enables one to avoid difficulties related with rigorous GRT treatment (Suen, 1986).

Direct transformation from 'nonrotating' RS x^i to 'rotating' RS y^i reads

$$y^i = P_{ik}x^k \,. \tag{10.86}$$

The inverse transformation from y^i to x^i is given with the transposed matrix P'

$$x^i = P_{ki}y^k \,, \qquad P' = P^{-1} \,. \tag{10.87}$$

Being orthogonal, the rotation matrix P satisfies the relations

$$P_{ik}P_{jk} = \delta_{ij} \,, \qquad \varepsilon_{ijk}P_{jm}P_{kn} = \varepsilon_{lmn}P_{il} \,, \qquad \varepsilon_{ijk}P_{il}P_{jm}P_{kn} = \varepsilon_{lmn} \tag{10.88}$$

(these relations remain valid when interchanging the indices everywhere in P_{ik}). Denoting the components of the angular velocity on y-axes by Ω^i and on x-axes by ω^i one has

$$\Omega^i = P_{ik}\omega^k \,, \qquad \omega^i = P_{ki}\Omega^k \,. \tag{10.89}$$

Elements of the rotation matrix are functions of time with the time derivatives

$$\dot{P}_{ik} = \varepsilon_{imj}\Omega^j P_{mk} = \varepsilon_{kjm}\omega^j P_{im} \,, \tag{10.90}$$

so that

$$\Omega^i = \tfrac{1}{2}\varepsilon_{ijk}P_{km}\dot{P}_{jm} \,, \qquad \omega^i = \tfrac{1}{2}\varepsilon_{ijk}P_{mj}\dot{P}_{mk} \tag{10.91}$$

and

$$\dot{y}^i = P_{ik}\dot{x}^k - \varepsilon_{ijk}\Omega^j y^k \,, \qquad \dot{x}^i = P_{ki}\dot{y}^k + \varepsilon_{ijk}\omega^j x^k \,. \tag{10.92}$$

Kinematical definition of a rigid-body rotation is given by the conditions

$$\dot{y}^i = 0 \tag{10.93}$$

to be satisfied for all points of the body.

Let us note that in correlating Ω^i or ω^i with the actual angular velocity of Earth's rotation one should remember that matrix P is usually split into matrices of precession, nutation and diurnal rotation. The Earth's angular velocity obtained from observations corresponds just to the matrix of diurnal rotation but not to the complete matrix P.

If t, x^i is a harmonic RS then t, y^i will be a nonharmonic RS with metric tensor components

$$h^+_{00} = h_{00} - c^{-2}\big(\varepsilon_{ijk}\Omega^j y^k\big)^2(1-h_{00}) + 2c^{-1}\ \varepsilon_{mjk}\Omega^j y^k P_{mn} h_{0n}\,, \tag{10.94}$$

$$h^+_{0i} = -c^{-1}\varepsilon_{ijk}\Omega^j y^k(1-h_{00}) + h_{0k}P_{ik} \tag{10.95}$$

$$h^+_{ij} = h_{00}\delta_{ij} \tag{10.96}$$

with

$$\varepsilon_{ijk}\Omega^j y^k = P_{il}\varepsilon_{ljk}\omega^j x^k\,. \tag{10.97}$$

Returning to the reduction formulas one may add the TRS$^+$ reduction formula for the topocentric observations

$$\tilde{p}^{(i)+} = \tilde{P}_{ik}\tilde{p}^{(k)} + c^{-1}\varepsilon_{krs}\tilde{\omega}^r\tilde{x}^s P_{in}\tilde{p}^{(k)}\tilde{p}^{(n)} - c^{-1}\varepsilon_{krs}\tilde{\omega}^r\tilde{x}^s P_{ik} + \dots \tag{10.97}$$

resulting in particular to the precession formula

$$\tilde{p}^{(i)+} = \Big[\tilde{P}_{im} + c^{-2}\tilde{P}_{ik}\Big(qF^{km} + \hat{q}R^{km} + (\tilde{q}-1)K^{km}\Big)\Big]\sigma^m\,. \tag{10.97}$$

For angular distance between two sources observed in TRS$^+$ one has

$$\cos\tilde{\psi}^+ = \cos\tilde{\psi} + \mathrm{O}(c^{-2}) \tag{10.98}$$

(with no term of $\mathrm{O}(c^{-1})$).

As an open question to the topic of this lecture one may indicate the problem of reduction using the more sophisticated solution for the light propagation developed in (Kopeikin and Schäfer, 1999).

Lecture 11. Relativistic reduction of radio observations. Time scales

VLBI observations

The relativistic hierarchy of RSs turned out to be the most effective and reliable tool for GRT treating of VLBI observations. The exposition below presents only a slight modification of algorithms exposed in (Klioner, 1991).

Let's remind that a model VLBI observation consists in recording by two ground stations ($i = 1, 2$) with BRS time–space coordinates t_i , $\mathbf{x}_i$ one and the same impulse emitted from a distant radio source $\mathbf{x}_s(t_0)$. As the actually measurable quantity one may consider the time interval between two recordings expressed in the proper time of the first station. Describing the boundary value problem for the light propagation in BRS in general terms

$$c(t_i - t_0) = R_i + \Delta R_i\,, \qquad i = 1, 2 \tag{11.1}$$

$$R_i = |\mathbf{x}_i(t_i) - \mathbf{x}_s(t_0)| \tag{11.2}$$

one has the BRS time delay

$$c\Delta t \equiv c(t_2 - t_1) = R_2 - R_1 + c\Delta t_{gr}\,, \qquad \Delta t_{gr} = c^{-1}(\Delta R_2 - \Delta R_1), \tag{11.3}$$

relativistic contributions ΔR_i being specified below. Using the parallactic expansion:

$$R_i = x_s - \mathbf{k}\mathbf{x}_i(t_i) + \frac{1}{2x_s}\Big[\mathbf{x}_i^2(t_i) - \big(\mathbf{k}\mathbf{x}_i(t_i)\big)^2\Big] + \ldots\,, \qquad x_s = |\mathbf{x}_s|, \qquad \mathbf{k} = \frac{\mathbf{x}_s}{x_s} \tag{11.4}$$

one gets

$$\begin{aligned} c\Delta t = &- \mathbf{k}\big[\mathbf{x}_2(t_2) - \mathbf{x}_1(t_1)\big] + \frac{1}{2x_s}\Big\{\big[\mathbf{k}\times\big(\mathbf{x}_2(t_2) - \mathbf{x}_1(t_1)\big)\big]^2 + \\ &+ 2\mathbf{x}_1(t_1)\Big[\mathbf{k}\times\Big(\big(\mathbf{x}_2(t_2) - \mathbf{x}_1(t_1)\big)\times\mathbf{k}\Big)\Big]\Big\} + c\Delta t_{gr} + \ldots\,. \end{aligned} \tag{11.5}$$

Representing the BRS motion of the station 2 in terms of the coordinate time of the first station

$$\mathbf{x}_2(t_2) = \mathbf{x}_2(t_1) + \dot{\mathbf{x}}_2(t_1)\Delta t + \tfrac{1}{2}\ddot{\mathbf{x}}_2(t_1)(\Delta t)^2 + \ldots \tag{11.6}$$

one obtains

$$\mathbf{x}_2(t_2) - \mathbf{x}_1(t_1) = \mathbf{b} + \dot{\mathbf{x}}_2(t_1)\Delta t + \tfrac{1}{2}\ddot{\mathbf{x}}_2(t_1)(\Delta t)^2 + \ldots\,, \qquad \mathbf{b} = \mathbf{b}(t_1) = \mathbf{x}_2(t_1) - \mathbf{x}_1(t_1) \tag{11.7}$$

resulting to

$$\begin{aligned} c\Delta t = &-\mathbf{k}\big[\mathbf{b} + \dot{\mathbf{x}}_2(t_1)\Delta t + \tfrac{1}{2}\ddot{\mathbf{x}}_2(t_1)(\Delta t)^2\big] + c\Delta t_{gr} + \frac{1}{x_s}\Big\{\mathbf{x}_1(t_1)[\mathbf{k}\times(\mathbf{b}\times\mathbf{k})] + \\ &+ \mathbf{x}_1(t_1)\big[\mathbf{k}\times\big(\dot{\mathbf{x}}_2(t_1)\times\mathbf{k}\big)\big]\Delta t + \tfrac{1}{2}(\mathbf{k}\times\mathbf{b})^2\Big\} + \ldots\,. \end{aligned} \tag{11.8}$$

Hence, the explicit BRS time–delay formula will be

$$\Delta t = c^{-1}\Big\{-\mathbf{kb} + \frac{1}{x_s}\mathbf{x}_1(t_1)[\mathbf{k}\times(\mathbf{b}\times\mathbf{k})] + \frac{1}{2x_s}(\mathbf{k}\times\mathbf{b})^2 - \tfrac{1}{2}c^{-2}\mathbf{k}\ddot{\mathbf{x}}_2(t_1)(\mathbf{kb})^2 + c\Delta t_{gr}\Big\}\times$$
$$\times\Big\{1 + c^{-1}\mathbf{k}\dot{\mathbf{x}}_2(t_1) - c^{-1}\frac{1}{x_s}\mathbf{x}_1(t_1)\big[\mathbf{k}\times\big(\dot{\mathbf{x}}_2(t_1)\times\mathbf{k}\big)\big]\Big\}^{-1} + \ldots\,. \qquad (11.9)$$

The idea is to compute Δt from the measurable proper time interval between recordings of the impulse at two stations. The left–hand member being known, the right–hand member enables one to determine the projection $\mathbf{kb}$ of the base vector.

Vector $\mathbf{k}$ should be corrected for the proper motion (if known)

$$\mathbf{k} = \mathbf{k}^* + \boldsymbol{\mu}\Delta t_0 + \tfrac{1}{2}\dot{\boldsymbol{\mu}}(\Delta t_0)^2 + \ldots\,. \qquad (11.10)$$

Indeed, the BRS motion of the source with the known proper motion reads

$$\mathbf{x}_s(t_0) = \mathbf{x}_s^* + \dot{\mathbf{x}}_s^*\Delta t_0 + \tfrac{1}{2}\ddot{\mathbf{x}}_s^*(\Delta t_0)^2 + \ldots\,, \qquad \Delta t_0 = t_0 - t_0^*\,, \qquad (11.11)$$

$$\boldsymbol{\mu} = \mathbf{k}^*\times(\dot{\mathbf{k}}^*\times\mathbf{k}^*) = \dot{\mathbf{k}}^* = \frac{1}{x_s^*}[\mathbf{k}^*\times(\dot{\mathbf{x}}_s^*\times\mathbf{k}^*)], \qquad (11.12)$$

$$\dot{\boldsymbol{\mu}} = \frac{1}{x_s^*}\Big\{[\mathbf{k}^*\times(\ddot{\mathbf{x}}_s^*\times\mathbf{k}^*)] - 2(\mathbf{k}^*\dot{\mathbf{x}}_s^*)\dot{\mathbf{k}}^* - \frac{1}{x_s^*}(\mathbf{k}^*\times\dot{\mathbf{x}}_s^*)^2\mathbf{k}^*\Big\}, \qquad (11.13)$$

t_0^* the initial epoch of emission. Denoting by t_1^* the TCB moment at the station 1 corresponding to t_0^* the measurable interval $t_1 - t_1^*$ at station 1 is related to the unmeasurable interval Δt_0 at the point of emission by means of

$$\Delta t_0 = \big[t_1 - t_1^* + c^{-1}\mathbf{k}^*\big(\mathbf{x}_1(t_1) - \mathbf{x}_1(t_1^*)\big)\big](1 + c^{-1}\mathbf{k}^*\dot{\mathbf{x}}_s^*)^{-1}\,. \qquad (11.14)$$

This formula follows from the relations

$$t_1 - t_0 = c^{-1}R_1 + \ldots\,, \qquad \mathbf{R}_1 = \mathbf{x}_1(t_1) - \mathbf{x}_s(t_0) = \mathbf{x}_1(t_1) - \mathbf{x}_s^* - \dot{\mathbf{x}}_s^*\Delta t_0 + \ldots\,, \quad (11.15)$$

$$t_1^* - t_0^* = c^{-1}R_1^* + \ldots\,, \qquad \mathbf{R}_1^* = \mathbf{x}_1(t_1^*) - \mathbf{x}_s^*\,, \qquad (11.16)$$

$$R_1 = x_s^* + \mathbf{k}^*\dot{\mathbf{x}}_s^*\Delta t_0 - \mathbf{x}_1(t_1)\mathbf{k}^* + \ldots\,, \qquad R_1^* = x_s^* - \mathbf{x}_1(t_1^*)\mathbf{k}^* + \ldots\,, \qquad (11.17)$$

$$t_1 - t_1^* - \Delta t_0 = c^{-1}(R_1 - R_1^*) + \ldots = c^{-1}\mathbf{k}^*\big(\dot{\mathbf{x}}_s^*\Delta t_0 - \mathbf{x}_1(t_1) + \mathbf{x}_1(t_1^*)\big) + \ldots\,. \quad (11.18)$$

Hence, for the sources with known proper motion $\mathbf{x}_s^*$, x_s^*, and $\mathbf{k}^*$ have the same meaning as $\mathbf{x}_s$, x_s, and $\mathbf{k}$ but referred to the epoch of emission t_0^*. Vector $\mathbf{k}$ may be considered therewith as a function of t_1 with the principal term for the derivative

$$\frac{d\mathbf{k}}{dt_1} = \frac{\boldsymbol{\mu}}{1 + c^{-1}\mathbf{k}^*\dot{\mathbf{x}}_s^*}\,. \qquad (11.19)$$

Until now the GRT corrections are involved only by Δt_{gr}. Except for this term all formulas are known in Newtonian astrometry.

Transformation to the GRS quantities

Let the coordinate time (TCG) and spatial coordinates of GRS be denoted by u, $\mathbf{w}$, respectively. Let u_i (i=1,2) be the TCG moment of the radio signal reception at station i and let $\mathbf{w}_i$ (i=1,2) be the GRS position vector of station i. The treating in GRS means the transformation from BRS time interval Δt to GRS time interval $\Delta u = u_2 - u_1$ and from BRS base vector $\mathbf{b}$ to GRS base vector $\mathbf{B} = \mathbf{B}(u_1) = \mathbf{w}_2(u_1) - \mathbf{w}_1(u_1)$. These substitutions are based on the BRS→GRS transformation of Lecture 7

$$u = t - c^{-2}[A(t) + \mathbf{v}_E\mathbf{r}_E] + c^{-4}[B(t) + B^k(t)r_E^k + B^{km}(t)r_E^k r_E^m - \tfrac{1}{10}\mathbf{r}_E^2 \dot{a}_E^k r_E^k] + \dots\,, \qquad (11.20)$$

$$w^i = r_E^i + c^{-2}[\tfrac{1}{2}\mathbf{v}_E\mathbf{r}_E v_E^i - q\varepsilon_{ijk}F^j r_E^k + \bar{U}_E(t\,,\mathbf{x}_E)r_E^i + \mathbf{a}_E\mathbf{r}_E r_E^i - \tfrac{1}{2}\mathbf{r}_E^2 a_E^i] + \dots\,. \qquad (11.21)$$

The post–post–Newtonian c^{-4} terms in (11.20) omitted in formulas of Lecture 7 may be found now in IAU resolutions (IAU, 2001). One may use them also in the slightly different form given in (Klioner and Voinov, 1993). For the transformation of the base vector it is sufficient to have in (11.20) only c^{-2} terms with

$$\dot{A}(t) = \tfrac{1}{2}\mathbf{v}_E^2 + \bar{U}_E(t\,,\mathbf{x}_E). \qquad (11.22)$$

For time interval transformation with high accuracy one may need in (11.20) the term with

$$B^k(t) = -\tfrac{1}{2}\mathbf{v}_E^2 v_E^k - 3v_E^k\bar{U}_E(t\,,\,\mathbf{x}_E) + 4\bar{U}_E^k(t\,,\,\mathbf{x}_E) \qquad (11.23)$$

but without the last term containing the vector potential. Omitting the quadratic terms with respect to $\mathbf{w}$ one gets from (11.21)

$$\mathbf{r}_E = \mathbf{w} - c^{-2}[\bar{U}_E(t\,,\,\mathbf{x}_E)\mathbf{w} + \tfrac{1}{2}(\mathbf{v}_E\mathbf{w})\mathbf{v}_E - q(\mathbf{F}\times\mathbf{w})] + \dots\,. \qquad (11.24)$$

By substituting (11.23), (11.24) into (11.21) one finds

$$u = t - c^{-2}[A(t) + \mathbf{v}_E(t)\mathbf{w}] - c^{-4}\,[2\bar{U}_E(t\,,\,\mathbf{x}_E)\mathbf{w} + q(\mathbf{F}\times\mathbf{w})]\mathbf{v}_E(t). \qquad (11.25)$$

Relations (11.24), (11.25) replace now the BRS→GRS transformation (11.20), (11.21).

The BRS base vector $b(t_1)$ is determined by two events having in BRS the time–space coordinates t_1 , $\mathbf{x}_1(t_1)$ and t_1 , $\mathbf{x}_2(t_1)$. These two events correspond to GRS events with the space–time coordinates u_1 , $\mathbf{w}_1(u_1)$ and u' , $\mathbf{w}_2(u')$, respectively, with $u' \neq u_1$. Taking the difference of two expressions (11.20) written for u_1 and u' (within c^{-2} accuracy) one gets

$$u' - u_1 = -c^{-2}\mathbf{v}_E(t_1)\mathbf{b}\,. \qquad (11.26)$$

Expanding

$$w_2^i(u') = w_2^i(u_1) - c^{-2}\mathbf{v}_E(t_1)\mathbf{b}\dot{w}_2^i(u_1) + \dots \qquad (11.27)$$

and taking the difference of (11.21) applied to $\mathbf{w}_1(u_1)$ and $\mathbf{w}_2(u')$ one obtains the transformation law $\mathbf{b} \to \mathbf{B}$ for the base vector

$$b^i = B^i - c^{-2}\Big[(\mathbf{B}\mathbf{v}_E)(\tfrac{1}{2}v^i_E + \dot{w}^i_2) - q\varepsilon_{ijk}F^jB^k + \bar{U}_E\big(t_1\,, \mathbf{x}_E(t_1)\big)B^i + \\ + (\mathbf{a}_E\mathbf{w}_2)w^i_2 - (\mathbf{a}_E\mathbf{w}_1)w^i_1 - \tfrac{1}{2}\mathbf{B}(\mathbf{w}_1 + \mathbf{w}_2)a^i_E\Big]. \tag{11.28}$$

It remains to find the transformation $\Delta t \to \Delta u$. Taking the difference of (11.25) applied for moments u_1 and u_2 and using expansions

$$\mathbf{v}_E(t_2) = \mathbf{v}_E(t_1) + \mathbf{a}_E(t_1)\Delta t + \dots\,, \tag{11.29}$$

$$\mathbf{w}_2(u_2) = \mathbf{w}_2(u_1) + \dot{\mathbf{w}}_2(u_1)\Delta u + \tfrac{1}{2}\ddot{\mathbf{w}}_2(u_1)(\Delta u)^2 + \dots\,, \tag{11.30}$$

$$A(t_2) = A(t_1) + \big[\tfrac{1}{2}\mathbf{v}^2_E(t_1) + \bar{U}_E\big(t_1\,,\, \mathbf{x}_E(t_1)\big)\big]\Delta t + \dots \tag{11.31}$$

one derives

$$\Delta u = \Big\{1 - c^{-2}\Big[\tfrac{1}{2}\mathbf{v}^2_E(t_1) + \bar{U}_E\big(t_1\,, \mathbf{x}_E(t_1)\big) + \mathbf{a}_E(t_1)\mathbf{w}_2(u_1) + \\ + \mathbf{v}_E(t_1)\dot{\mathbf{w}}_2(u_1) + \tfrac{1}{2}\mathbf{v}_E(t_1)\ddot{\mathbf{w}}_2(t_1)\Delta t\Big]\Big\}\Delta t - \\ - c^{-2}\mathbf{v}_E(t_1)\Big\{\mathbf{B} + c^{-2}\Big[2\bar{U}_E\big(t_1\,, \mathbf{x}_E(t_1)\big)\mathbf{B} + q\big(\mathbf{F}(t_1) \times \mathbf{B}\big)\Big]\Big\}. \tag{11.32}$$

Expressions (11.28) and (11.32) represent two basic relations for GRS treatment of VLBI observations. Sometimes the transformation of the base vector (11.28) is prolonged for the topocentric (TRS) level. This GRS/TRS dualism may lead to misunderstanding. Working in the TRS framework one can construct a local set of distances (proper distances in the vicinity of the TRS origin). Working in the GRS framework one constructs a global set of coordinate distances for the whole Earth.

BRS gravitational time delay

Before going further let us complete the BRS time–delay formula (11.9) by reproducing the expression of Δ_{gr} (Klioner, 1991). This GRT contribution to BRS time–delay reads

$$\Delta t_{gr} = \Delta t_{pN} + \Delta t_M + \Delta t_Q + \Delta t_R + \Delta t_{ppN} \tag{11.33}$$

with contributions due to fixed monopoles (pN), their motion (M), the quadrupole structure of bodies (Q), their rotation (R) and post–post–Newtonian terms (PPN). The main contributions caused by these factors are as follows

$$c\Delta t_{pN} + c\Delta t_M = \sum_A \frac{2GM_A}{c^2}\Big\{\Big[1 + c^{-1}\mathbf{k}\dot{\mathbf{x}}_A(t_1)\Big]\ln\frac{r_{A1} + \mathbf{k}\mathbf{r}_{A1}}{r_{A2} + \mathbf{k}\mathbf{r}_{A2}} + \\ + \frac{1}{x_s}\left(\frac{\mathbf{x}_2\mathbf{d}_{A2}}{r_{A2} + \mathbf{k}\mathbf{r}_{A2}} - \frac{\mathbf{x}_1\mathbf{d}_{A1}}{r_{A1} + \mathbf{k}\mathbf{r}_{A1}}\right) + \dots\Big\}, \tag{11.34}$$

$$c\Delta t_Q = \sum_A \frac{G}{c^2}(f^{pq}_{A2} - f^{pq}_{A1})(I^{pq}_A - \tfrac{1}{3}\delta_{pq}I^{kk}_A)\,, \tag{11.35}$$

$$c\Delta t_R = \sum_A \frac{2G}{c^3}(\mathbf{k}\times\mathbf{S}_A)\left[\frac{\mathbf{r}_{A2}}{r_{A2}(r_{A2}+\mathbf{k}\mathbf{r}_{A2})} - \frac{\mathbf{r}_{A1}}{r_{A1}(r_{A1}+\mathbf{k}\mathbf{r}_{A1})}\right], \tag{11.36}$$

$$\begin{aligned} c\Delta t_{ppN} = &\sum_A \frac{G^2M_A^2}{c^4}\left\{-\frac{4}{r_{A2}+\mathbf{k}\mathbf{r}_{A2}} + \frac{15}{4|\mathbf{k}\times\mathbf{r}_{A2}|}\arccos\frac{\mathbf{k}\mathbf{r}_{A2}}{r_{A2}} + \frac{\mathbf{k}\mathbf{r}_{A2}}{4r^2_{A2}} + \right. \\ &\left. + \frac{4}{r_{A1}+\mathbf{k}\mathbf{r}_{A1}} - \frac{15}{4|\mathbf{k}\times\mathbf{r}_{A1}|}\arccos\frac{\mathbf{k}\mathbf{r}_{A1}}{r_{A1}} - \frac{\mathbf{k}\mathbf{r}_{A1}}{4r^2_{A1}}\right\}. \end{aligned} \tag{11.37}$$

Here

$$\mathbf{r}_{Ai} = \mathbf{x}_i(t_i) - \mathbf{x}_A(t_{Ai})\,, \tag{11.38}$$

t_{Ai} being the moment of the closest approach between the light photon and body A

$$t_{Ai} = t_i + c^{-1}\mathbf{k}[\mathbf{x}_i(t_i) - \mathbf{x}_A(t_i)] - c^{-2}\big[2\mathbf{k}\big(\mathbf{k}\dot{\mathbf{x}}_A(t_i)\big) - \dot{\mathbf{x}}_A(t_i)\big][\mathbf{x}_i(t_i) - \mathbf{x}_A(t_i)] + \ldots\,, \tag{11.39}$$

$$\mathbf{d}_{Ai} = \mathbf{k}\times(\mathbf{r}_{Ai}\times\mathbf{k})\,, \tag{11.40}$$

$$f^{pq}_{Ai} = \frac{1}{d^2_{Ai}}\left[1 - \frac{(\mathbf{k}\mathbf{r}_{Ai})^3}{r^3_{Ai}}\right]k^pk^q + \frac{2}{r^3_{Ai}}k^pd^q_{Ai} + \frac{1}{d^4_{Ai}}\left[2 - 3\frac{\mathbf{k}\mathbf{r}_{Ai}}{r_{Ai}} + \frac{(\mathbf{k}\mathbf{r}_{Ai})^3}{r^3_{Ai}}\right]d^p_{Ai}d^q_{Ai}\,. \tag{11.41}$$

The spin vector in the rigid–body approximation is determined by

$$S^m = \omega^m I^{ss} - I^{ms}\omega^s \tag{11.42}$$

with angular velocity ω^s and (Fock) inertia moments I^{ms}.

In treating the effect of interference frequency one needs the time derivatives of the gravitational time delay. The main (monopole) contributions for these derivatives are determined by

$$c\frac{\partial}{\partial t_1}\Delta t_{gr} = \sum \frac{2GM_A}{c^2}\frac{\mathbf{r}_{A1} + r_{A1}\mathbf{k}}{r_{A1}(r_{A1}+\mathbf{k}\mathbf{r}_{A1})}[\dot{\mathbf{x}}_1(t_1) - \dot{\mathbf{x}}_A(t_{A1})] \tag{11.43}$$

and

$$c\frac{\partial}{\partial t_2}\Delta t_{gr} = -\sum \frac{2GM_A}{c^2}\frac{\mathbf{r}_{A2} + r_{A2}\mathbf{k}}{r_{A2}(r_{A2}+\mathbf{k}\mathbf{r}_{A2})}[\dot{\mathbf{x}}_2(t_2) - \dot{\mathbf{x}}_A(t_{A2})]\,. \tag{11.44}$$

More precise expressions may be found by direct differentiation of (11.34)–(11.37).

Transformation to the proper time

The differential relationship between the proper (τ) and coordinate (u) time in GRS reads

$$d\tau = \left\{1 - c^{-2}\left[\tfrac{1}{2}\dot{\mathbf{w}}^2 + \Phi(u\,,\mathbf{w})\right]\right\}du \tag{11.45}$$

with

$$\Phi(u\,,\mathbf{w}) = \hat{U}_E + Q_jw^j + T(\mathbf{w})\,, \tag{11.46}$$

where $\hat{U}_E$ stands for the GRS geopotential, Q_j is the value of the non-geodesic acceleration in the BRS motion of the Earth and T is the tidal potential.

Let u^* be a TCG synchronization moment for the clocks at the ground stations 1 and 2 so that

$$\tau_1\big(u^*\,,\mathbf{w}_1(u^*)\big) = \tau_2\big(u^*\,,\mathbf{w}_2(u^*)\big) = \tau^*\,. \tag{11.47}$$

Then in virtue of (11.45)

$$\tau_1 = u_1 - c^{-2}\int_{u^*}^{u_1}(\tfrac{1}{2}\dot{\mathbf{w}}_1^2 + \Phi_1)du \tag{11.48}$$

and

$$\tau_2 = u_2 - c^{-2}\int_{u^*}^{u_1}(\tfrac{1}{2}\dot{\mathbf{w}}_2^2 + \Phi_2)du - c^{-2}\int_{u_1}^{u_2}(\tfrac{1}{2}\dot{\mathbf{w}}_2^2 + \Phi_2)du \tag{11.49}$$

with

$$\Phi_k(u\,,\mathbf{w}) = \Phi(u\,,\mathbf{w}_k), \qquad k = 1,2\,. \tag{11.50}$$

The difference of these relations results in the proper time delay $\Delta\tau = \tau_2 - \tau_1$ for space VLBI observations

$$\Delta\tau = \Delta u - c^{-2}\int_{u_1}^{u_2}(\tfrac{1}{2}\dot{\mathbf{w}}_2^2 + \Phi_2)du - c^{-2}\int_{u^*}^{u_1}(\tfrac{1}{2}\dot{\mathbf{w}}_2^2 + \Phi_2 - \tfrac{1}{2}\dot{\mathbf{w}}_1^2 - \Phi_1)du\,. \tag{11.51}$$

For ground VLBI observations the last integral may be neglected for sufficiently close moments u^* and u_1 whereas the first integral in virtue of the ground relation

$$\tfrac{1}{2}\dot{\mathbf{w}}^2 + \Phi(u\,,\mathbf{w}) = c^2 L_G - g(\varphi)h \tag{11.52}$$

may be transformed to

$$\Delta\tau = \big[1 - L_G + c^{-2}g(\varphi_2)h_2\big]\Delta u \tag{11.53}$$

with the value of the Earth force gravity $g(\varphi)$ at latitude φ

$$g(\varphi) = \frac{GM_E}{A_E^2}(1 + \tfrac{3}{2}J_2) - \omega_E^2 A_E + \big(2\omega_E^2 A_E - \frac{3GM_E}{2A_E^2}J_2\big)\sin^2\varphi\,. \tag{11.54}$$

Here h is the height of the terrestrial station over geoid, J_2 is the coefficient of the second zonal harmonic of the geopotential and L_G is a definition constant (IAU Resolution B1, 2000).

To resume, the VLBI GRT reduction reduces to the sequence of formulas (11.51) or (11.53) for $\Delta\tau$, (11.32) for Δu, (11.28) for GRS bases and (11.9) for Δt.

Interference frequency

The measurable value of the interference frequency in VLBI ground measurements is

$$\Delta\dot{\tau} = \frac{d\Delta\tau}{d\tau_1} = \frac{d\tau_2}{d\tau_1} - 1\,. \tag{11.55}$$

Its calculated value is

$$\Delta\dot{\tau} = \frac{d\tau_2}{du_2}\left(\frac{dt_2}{du_2}\right)^{-1}\frac{dt_2}{dt_1}\frac{dt_1}{du_1}\left(\frac{d\tau_1}{du_1}\right)^{-1} - 1 \tag{11.56}$$

with

$$\frac{dt_i}{du_i} = 1 + c^{-2}\big[\tfrac{1}{2}\mathbf{v}_E^2(t_i) + \bar{U}_E\big(t_i\,,\mathbf{x}_E(t_i)\big) + \mathbf{a}_E(t_i)\mathbf{w}_i(u_i) + \mathbf{v}_E(t_i)\dot{\mathbf{w}}_i(u_i)\big], \tag{11.57}$$

$$\frac{d\tau_i}{du_i} = 1 - c^{-2}[c^2 L_G - g(\varphi_i)h_i] \tag{11.58}$$

and

$$\frac{dt_2}{dt_1} = \frac{1 + c^{-1}\mathbf{k}\dot{\mathbf{x}}_1(t_1)}{1 + c^{-1}\mathbf{k}\dot{\mathbf{x}}_2(t_2)} + c^{-1}\frac{1}{x_s}\Big\{\mathbf{x}_1\big[\mathbf{k}\times\big((\dot{\mathbf{x}}_2 - \dot{\mathbf{x}}_1)\times\mathbf{k}\big)\big] + (\mathbf{x}_2 - \mathbf{x}_1)[\mathbf{k}\times(\dot{\mathbf{x}}_2\times\mathbf{k})]\Big\} - \\ - c^{-1}(\mathbf{x}_2 - \mathbf{x}_1)\frac{d\mathbf{k}}{dt_1} + \frac{\partial}{\partial t_1}\Delta t_{gr} + \frac{\partial}{\partial t_2}\Delta t_{gr} + \dots . \tag{11.59}$$

The involved derivatives were given above.

Radio ranging

The basic formula for the radio ranging is again the time of flight relation

$$c(t - t_0) = R + 2\sum_A m_A \ln\frac{r_A + r_{0A} + R}{r_A + r_{0A} - R} \tag{11.60}$$

with

$$\frac{dt}{dt_0} = \left[1 - c^{-1}\frac{\mathbf{R}\dot{\mathbf{x}}_0}{R} - 4c^{-1}\sum_A \frac{m_A}{(r_A + r_{0A})^2 - R^2}\left(\frac{r_A + r_{0A}}{R}\,\mathbf{R}\dot{\mathbf{x}}_0 + \frac{R}{r_{0A}}\,\mathbf{r}_{0A}\dot{\mathbf{r}}_{0A}\right)\right]\times \\ \times\left[1 - c^{-1}\frac{\mathbf{R}\dot{\mathbf{x}}}{R} - 4c^{-1}\sum_A \frac{m_A}{(r_A + r_{0A})^2 - R^2}\left(\frac{r_A + r_{0A}}{R}\,\mathbf{R}\dot{\mathbf{x}} - \frac{R}{r_A}\,\mathbf{r}_A\dot{\mathbf{r}}_A\right)\right]^{-1} \tag{11.61}$$

and

$$\mathbf{r}_A = \mathbf{x}(t) - \mathbf{x}_A(t)\,, \qquad \mathbf{r}_{0A} = \mathbf{x}_0(t_0) - \mathbf{x}_A(t)\,, \qquad \mathbf{R} = \mathbf{x}(t) - \mathbf{x}_0(t_0).$$

In ranging procedure the light signal emitted by the observer at moment t_0 with frequency ν_0 is received and reflected by the ranging object A at moment t_1 (to be determined by iterations) with frequency ν_1 and is received again by the observer at moment t_2 with frequency ν_2. The measurable quantities are the proper time delay by the observer's clock $\Delta\tau = \tau_2 - \tau_0$ and the frequency ratio ν_0/ν_2 to be determined by the repeated application of the relation:

$$\frac{\nu_0}{\nu} = \frac{\delta\tau}{\delta\tau_0} = \frac{1 + \frac{1}{2}(h_{00})_t - \frac{1}{2}c^{-2}v^2 + c^{-1}\underline{(h_{0k})_t}v^k}{1 + \frac{1}{2}(h_{00})_{t_0} - \frac{1}{2}c^{-2}v_0^2 + c^{-1}\underline{(h_{0k})_{t_0}}v_0^k}\,\frac{dt}{dt_0} \tag{11.62}$$

(for non–rotating systems the underlined terms are of order c^{-4} and may be neglected).

LLR (lunar laser ranging)

In LLR the starting formula is also (11.60) with significant simplifications in its relativistic part. If t_0 is the moment of the signal emission by a ground observer and t is the moment of the reception of the signal at a point on the surface of the Moon then

$$\mathbf{x}_0 = \mathbf{x}_E(t_0) + \boldsymbol{\rho}_E(t_0), \qquad \mathbf{x} = \mathbf{x}_L(t) + \boldsymbol{\rho}_L(t),$$

$\mathbf{x}_E$ and $\mathbf{x}_L$ are the BRS positions of the centers of mass of the Earth and the Moon, respectively, $\boldsymbol{\rho}_E$ is the geocentric position vector of the ground station, and $\boldsymbol{\rho}_L$ is the selenocentric position vector of the reflector on the Moon. In the actual case when LLR is performed for the position of the Moon near the meridian of the ground station one has

$$R \approx r - \rho_E - \rho_L\,, \qquad r_{0E} \approx \rho_E\,, \qquad r_E \approx r - \rho_L$$

and

$$r_S \approx |\mathbf{x}_L| = |\mathbf{x}_E| + \frac{\mathbf{x}_E \mathbf{r}}{|\mathbf{x}_E|} + \frac{1}{2|\mathbf{x}_E|^2}\left(r^2 - \frac{(\mathbf{x}_E\mathbf{r})^2}{|\mathbf{x}_E|^2}\right) + \ldots \tag{11.63}$$

$\mathbf{r} = \mathbf{r}(t)$ being the geocentric BRS position vector of the Moon. It gives the expression for the one–way BRS transit time

$$c(t - t_0) = R + 2m_S \frac{r}{|\mathbf{x}_E|}\left(1 - \frac{\mathbf{x}_E \mathbf{r}}{2|\mathbf{x}_E|^2} + \ldots\right) + 2m_E \ln\frac{r - \rho_L}{\rho_E}\,. \tag{11.64}$$

The quantity R is calculated (by numerical iterations) as the magnitude of the BRS vector

$$\mathbf{R} = \mathbf{x}(t) - \mathbf{x}(t_0) = \mathbf{x}_E(t) + \mathbf{r}(t) + \boldsymbol{\rho}_L(t) - \mathbf{x}_E(t_0) - \boldsymbol{\rho}_E(t_0). \tag{11.65}$$

The relativistic contribution to the measurable quantities may be easily evaluated analytically. Neglecting the sizes of the Earth and the Moon one has

$$\mathbf{R} = \mathbf{r}(t) + \dot{\mathbf{x}}_E(t_0)(t - t_0) + \tfrac{1}{2}\ddot{\mathbf{x}}_E(t_0)(t - t_0)^2 + \ldots \tag{11.66}$$

and

$$R = r\left[1 + \frac{\mathbf{r}\dot{\mathbf{x}}_E}{r^2}(t - t_0) + \frac{1}{2r^2}\left(\dot{\mathbf{x}}_E^2 + \mathbf{r}\ddot{\mathbf{x}}_E - \frac{1}{r^2}(\mathbf{r}\dot{\mathbf{x}}_E)^2\right)(t - t_0)^2 + \ldots\right]. \tag{11.67}$$

It results in the approximated expression for the one–way BRS transit time

$$\begin{aligned} c(t - t_0) = r&\left[1 + c^{-1}\frac{\mathbf{r}\dot{\mathbf{x}}_E}{r} + \tfrac{1}{2}c^{-2}\left(\dot{\mathbf{x}}_E^2 + \frac{(\mathbf{r}\dot{\mathbf{x}}_E)^2}{r^2}\right)\right] + \\ &+ m_S\frac{r}{|\mathbf{x}_E|}\left(2 - \frac{3(\mathbf{r}\mathbf{x}_E)}{2|\mathbf{x}_E|^2}\right) + 2m_E \ln\frac{r - \rho_L}{\rho_E} + \ldots\,. \end{aligned} \tag{11.68}$$

Multiplied by two this expression gives the BRS time interval of light propagation in LLR (two–way). The measurable quantity is $2(\tau-\tau_0)$, τ_0 and τ being the moments of the proper time corresponding to the moments t_0 and t of the BRS time. Retaining only the main terms one has

$$\tau_0 = t_0 - c^{-2}\int_0^{t_0} \left(\tfrac{1}{2}\dot{\mathbf{x}}_E^2 + \bar{U}_E(\mathbf{x}_E)\right)dt\,,$$

$$\tau = t - c^{-2}\left(\int_0^{t} \left(\tfrac{1}{2}\dot{\mathbf{x}}_E^2 + \bar{U}_E(\mathbf{x}_E)\right)dt + \dot{\mathbf{x}}_E\mathbf{r}\right),$$

and

$$c(\tau-\tau_0) = c(t-t_0)\left(1 - \tfrac{1}{2}c^{-2}\dot{\mathbf{x}}_E^2 - \frac{m_s}{|\mathbf{x}_E|}\right) - c^{-1}\dot{\mathbf{x}}_E\mathbf{r}\,. \tag{11.69}$$

From this the expression of the measurable proper–time delay (in terms of BRS quantities) reads

$$2c(\tau-\tau_0) = 2r\left(1 + \frac{c^{-2}}{2r^2}(\mathbf{r}\dot{\mathbf{x}}_E)^2 + \frac{m_s}{|\mathbf{x}_E|} - \frac{3m_S}{2|\mathbf{x}_E|^3}(\mathbf{r}\mathbf{x}_E)\right) + 4m_E \ln\frac{r-\rho_L}{\rho_E} + \ldots. \tag{11.70}$$

The main GRT contribution in BRS solution for the radius–vector of the Moon is

$$r/a_0 = 1 - \tfrac{9}{4}\sigma + \sigma m + \tfrac{1}{4}\sigma\cos 2D + \ldots, \qquad \sigma = N^2A^2/c^2, \qquad m = \frac{N}{n-N}, \tag{11.71}$$

where N and n are the mean motions of the Sun and the Moon, respectively, A and a_0 are their semi–major axes, D stands for the difference of the mean longitudes of the Moon and the Sun, and σ is GRT dimensionless small parameter ($\approx 10^{-8}$). The main GRT reduction term in (11.70) due to the Lorentz transformation is

$$\frac{c^{-2}}{2r^2}(\mathbf{r}\dot{\mathbf{x}}_E)^2 = \tfrac{1}{4}\sigma(1-\cos 2D). \tag{11.72}$$

Thus, the trigonometric term in (11.71) with amplitude $\sigma/4$ disappears in (11.70) and the amplitude of the relativistic terms in the measurable quantity (11.70) does not exceed several centimeters.

In constructing the GRS solution for the Moon one comes directly to this conclusion. Indeed, from BRS→GRS transformation one has for the GRS coordinates of the Moon (retaining only the purely solar terms)

$$w^i = r^i + \tfrac{1}{2}c^{-2}v_E^i v_E^k r^k + c^{-2}qF^{ik}r^k + \frac{m_s}{|\mathbf{x}_E|}r^i + \frac{m_S}{|\mathbf{x}_E|^3}\left(\tfrac{1}{2}r^2x_E^i - r^ir^kx_E^k\right). \tag{11.73}$$

Thus,

$$\rho = |\mathbf{w}| = r\left(1 + \frac{c^{-2}}{2r^2}(v_E^k r^k)^2 + \frac{m_s}{|\mathbf{x}_E|} - \frac{m_s}{2|\mathbf{x}_E|}(r^kx_E^k)\right). \tag{11.74}$$

In terms of the GRS quantities the right–hand side of (11.70) is

$$2c(\tau-\tau_0) = 2\rho\left(1 - \frac{m_S}{|\mathbf{x}_E|^3}(\mathbf{w}\mathbf{x}_E)\right) + 4m_E\ln\frac{\rho-\rho_L}{\rho_E} + \ldots \tag{11.75}$$

(in neglecting by the parallactic term the GRT LLR time–delay is reduced just to the Schwarzschild delay due to the Earth).

Pulsar timing
The timing of millisecond pulsars is at present one of the most accurate observational techniques. A relativistic theory of pulsar timing is based again on (11.60). Indeed, if an impulse n emitted at the moment T_n by a millisecond pulsar in position $\mathbf{X}_n$ is recorded by the ground station at moment t_n in the BRS position $\mathbf{x}_n$ then in accordance with (11.60)

$$c(t_n - T_n) = |\mathbf{x}_n - \mathbf{X}_n| + 2\sum_A m_A \ln \frac{|\mathbf{x}_n - \mathbf{x}_A(t_n)| + |\mathbf{X}_n - \mathbf{x}_A(T_n)| + |\mathbf{x}_n - \mathbf{X}_N)|}{|\mathbf{x}_n - \mathbf{x}_A(t_n)| + |\mathbf{X}_n - \mathbf{x}_A(T_n)| - |\mathbf{x}_n - \mathbf{X}_N)|}. \tag{11.76}$$

Assuming that the pulsar moves with constant velocity $\mathbf{V}$ one has

$$\mathbf{X}_n = \mathbf{X}_0 + \mathbf{V}(T_n - T_0). \tag{11.77}$$

Putting

$$\mathbf{k} = \frac{1}{\rho_0}\mathbf{X}_0\,, \qquad \rho_0 = |\mathbf{X}_0|\,, \qquad \Delta t_n = T_n - T_0 \tag{11.78}$$

and expanding $|\,\mathbf{x}_n - \mathbf{X}_n|$ in powers of Δt_n and $|\mathbf{x}_n|/\rho_0$ one gets

$$\begin{aligned} c(t_n - T_n) =& \rho_0 + [(\mathbf{kV})\Delta t_n - \mathbf{kx}_n] + \frac{1}{2\rho_0}[\mathbf{x}_n^2 - (\mathbf{kx}_n)^2] - \frac{1}{\rho_0}[\mathbf{xV} - (\mathbf{kV})(\mathbf{kx}_n)]\Delta t_n + \\ &+ \frac{1}{2\rho_0}[\mathbf{V}^2 - (\mathbf{kV})^2](\Delta t_n)^2 - 2\sum_A m_A \ln \frac{|\,r_A(t_n) + \mathbf{kr}_A(t_n)|}{2\rho_0}\,. \end{aligned} \tag{11.79}$$

Under designation

$$ct_0 = cT_0 + \rho_0 + 2\sum_A m_A \ln(2\rho_0) \tag{11.80}$$

one has finally the pulsar timing formula

$$\begin{aligned} c(t_n - t_0) =& c(T_n - T_0) + [(\mathbf{kV})\Delta t_n - \mathbf{kx}_n] + \frac{1}{2\rho_0}[\mathbf{x}_n^2 - (\mathbf{kx}_n)^2] - \\ &- \frac{1}{\rho_0}[\mathbf{xV} - (\mathbf{kV})(\mathbf{kx}_n)]\Delta t_n + \frac{1}{2\rho_0}[\mathbf{V}^2 - (\mathbf{kV})^2](\Delta t_n)^2 - \\ &- 2\sum_A m_A \ln|\,r_A(t_n) + \mathbf{kr}_A(t_n)|. \end{aligned} \tag{11.81}$$

Each term in this equation admits a simple physical interpretation. Starting with the second term in the right–hand member these terms describe, respectively, the first–order Doppler effect, the parallax, the pulsar proper motion, the second–order Doppler effect, and the Shapiro effect (the relativistic time delay in light propagation). It remains only to convert in (11.81) from t_n to the corresponding moment of proper time τ_n and to replace $\mathbf{x}_n$ by ITRS spatial coordinates.

In the case of a binary pulsar the timing formula becomes more complicated, due, first of all, to replacing (11.77) by

$$\mathbf{X}_n = \mathbf{X}_0 + \mathbf{V}(T_n - T_0) + \mathbf{X}_{1n} \tag{11.82}$$

where $\mathbf{X}_0$ and $\mathbf{V}$ represent the BRS position and velocity of the center of mass of the binary and $\mathbf{X}_{1n}$ is the pulsar's position relative to the binary barycenter.

Time scales

Rigorous relativistic theory of the time scales became possible only with the development of the relativistic hierarchy of astronomical RSs. Considering that the latter theory was exposed in Lecture 7 we will just remind the basic formulations. The basic hierarchy BRS$(t, \mathbf{x}) \to$ GRS$(u, \mathbf{w}) \to$ TRS$(\tau, \mathbf{z})$ involves

t=TCB (barycentric coordinate time), BRS coordinate time,

u=TCG (geocentric coordinate time), GRS coordinate time,

τ, TRS coordinate time (proper time of an observer in the spatial origin of TRS).

Reproduced from Lecture 7, the TCB→TCG transformation reads

$$u = t - c^{-2}[A(t) + \mathbf{v}_E \mathbf{r}_E] + \ldots, \qquad \mathbf{r}_E = \mathbf{x} - \mathbf{x}_E \tag{11.82}$$

with

$$\dot{A}(t) = \tfrac{1}{2}\mathbf{v}_E^2 + \bar{U}_E(t, \mathbf{x}_E), \qquad \bar{U}_E(t, \mathbf{x}_E) = \sum_{A \neq E} \frac{GM_A}{r_{EA}} \tag{11.83}$$

and the solution

$$A(t) = c^2 L_C t + A_p(t)\,, \qquad L_C t = L_C(J - 2443144.5)86400s \tag{11.84}$$

resulting to

$$u = (1 - L_C)t - c^{-2}[A_p(t) + \mathbf{v}_E \mathbf{r}_E] + \ldots\,. \tag{11.85}$$

With the use of the VSOP planetary theories $A_p(t)$ admits the representation

$$c^{-2}A_p(t) = P = \sum_\alpha t^\alpha \left[\sum_k A_k^\alpha \cos(\psi_k^\alpha + \nu_k^\alpha t)\right]$$

with condition $P = 0$ on Jan. 1, 1977 Oh 0m 0s TAI (J=2443144.5 TAI). Finally, there results

$$\mathrm{TCB} - \mathrm{TCG} = L_C(J - 2443144.5)86400s + P + c^{-2}\mathbf{v}_E \mathbf{w}\,. \tag{11.86}$$

The time part of the GRS→TRS transformation reads

$$\tau = u - c^{-2}[V(u) + \hat{v}_T^k(w^k - w_T^k)] + \ldots \tag{11.87}$$

with

$$\frac{dV}{du} = \tfrac{1}{2}\hat{\mathbf{v}}_T^2 + \hat{U}_E(\mathbf{w}_T) + Q_j w_T^j + T(\mathbf{w}_T) \tag{11.88}$$

or in retaining only the main tidal terms

$$\frac{dV}{du} = \tfrac{1}{2}\hat{\mathbf{v}}_T^2 + \hat{U}_E(\mathbf{w}_T) + Q_j w_T^j + \tfrac{1}{2}\bar{U}_{E,km}(\mathbf{x}_E) w_T^k w_T^m + \dots . \qquad (11.89)$$

Function $\mathbf{w}_T$ satisfies the equation

$$\frac{d\mathbf{w}_T}{du} \equiv \hat{\mathbf{v}}_T = \hat{\boldsymbol{\omega}}_E \times \mathbf{w}_T + \hat{\mathbf{v}}_{TT} , \qquad (11.90)$$

$\mathbf{v}_{TT}$ being the relative velocity of the TRS origin due to geophysical factors (deviation of the ground station rotation from the law of rigid–body rotation).

Introducing GRS$^+$ coordinates $y^i = \hat{P}_{ik} w^k$ and neglecting by Q_j one has

$$\frac{dV}{du} = \varepsilon_{ijk}\hat{\Omega}_E^j y_T^k \dot{y}_T^i + \tfrac{1}{2}\dot{y}_T^k \dot{y}_T^k + W_E(\mathbf{y}_T) + \tfrac{1}{2}\bar{U}_{E,km}(\mathbf{x}_E)\hat{P}_{ik}\hat{P}_{jm} y_T^i y_T^j + \dots , \qquad (11.91)$$

$\hat{\Omega}_E^j$ being GRS$^+$ components of the Earth angular velocity and $\dot{y}_T^i = \hat{P}_{ik}\hat{v}_{TT}^k$ being geophysical factors contributions.

The potential of the force of gravity generated by the Earth alone at any point $\mathbf{y}$ on the surface of the Earth is

$$W_E(\mathbf{y}) = \tfrac{1}{2}(\varepsilon_{ijk}\hat{\Omega}_E^j y^k)^2 + \hat{U}_E(\mathbf{y}) . \qquad (11.92)$$

Choosing the direction of the y^3 axis to be along the axis of rotation of the Earth and introducing the geocentric spherical coordinates one gets

$$y_T^1 = r\cos\psi\cos l\, ,\ y_T^2 = r\cos\psi\sin l\, ,\ y_T^3 = r\sin\psi\, ,\ \hat{\Omega}^i = \Omega\delta_{3i}\, ,$$

$$W_E(r,l,\psi) = \tfrac{1}{2}\Omega^2 r^2\cos^2\psi + \hat{U}_E(\mathbf{y}). \qquad (11.93)$$

The main constant term of the tidal quadrupole potential Q_{20} is

$$\bar{Q}_{20}(r,\psi) = -\tfrac{1}{4}r^2(3\sin^2\psi - 1)\sum_{A\neq E}\frac{GM_A}{\bar{r}_{EA}^3} , \qquad (11.94)$$

$\bar{r}_{EA}$ being some mean value of r_{EA}. It is reasonable to add this value to the potential of the force of gravity due to the Earth and to consider the potential

$$W(r,l,\psi) = W_E(r,l,\psi) + \bar{Q}_{20}(r,\psi) \qquad (11.95)$$

although the term $\bar{Q}_{20}$ is usually treated as a perturbation and is not included in the potential for gravitational force.

The surface $r = r(l,\psi)$ providing a constant value for the gravity potential

$$W = W_0 = c^2 L_G \qquad (11.96)$$

us called the geoid. In terms of the astronomical longitude λ, astronomical latitude φ and the height h of the observer above the geoid the potential of the force of gravity at any point on the surface of the Earth may be presented in the form

$$W(h,\lambda,\varphi) = W_0 - g(\varphi,\lambda)h + \mathrm{O}(h^2), \tag{11.97}$$

$g(\varphi,\lambda)$ being the force of gravity on the geoid at the point with astronomical longitude λ and latitude φ.

Returning to (11.91) and using (11.97) one obtains

$$\frac{dV}{du} = \varepsilon_{ijk}\hat{\Omega}^j_E y^k_T \dot{y}^i_T + \tfrac{1}{2}\dot{y}^k_T\dot{y}^k_T + W_0 + Q_2 - \bar{Q}_{20} - g(\varphi,\lambda)h + \ldots \tag{11.98}$$

where φ, λ, h and y^i_T are the coordinates of the ground station. The solution of this equation is presented in the form

$$V(u) = c^2 L_G u + V_p(u) \tag{11.99}$$

where $V_p(u)$ includes secular term due to $-g(\varphi,\lambda)h$ dependent on the ground station coordinates, periodic tidal luni–solar terms from $Q_2-\bar{Q}_{20}$ and geophysical factors contributions from the terms with $\dot{y}^i_T$. As a result, equation (11.87) gives the following relationship between u and τ:

$$\tau = (1-L_G)u - c^{-2}[V_p(u) + \hat{v}^k_T(w^k - w^k_T)] + \ldots. \tag{11.100}$$

According to the IAU Resolution B1.9 (2000) L_G is a defining constant

$$L_G = 6.969290134 \times 10^{-10} \tag{11.101}$$

involving possibly an additive secular term in $V_p(u)$ if the present measured value of W_0 will be improved.

Time scales TT and TDB (T_{eph})
TCB and TCG are 'theoretical' time scales. In practice (e.g., in numerical or semi–analytical ephemerides of the major planets and the Moon) 'practical' time scales TT (terrestrial time) and TDB (barycentric dynamical time) are used

$$\mathrm{TT} = (1-L_G)\mathrm{TCG}\,, \tag{11.102}$$

$$\mathrm{TDB} = (1-L_B)\mathrm{TCB}\,, \tag{11.103}$$

L_B to be determined in terms of L_C and L_G by using (11.85)

$$(1-L_G)^{-1}\mathrm{TT} = (1-L_C)(1-L_B)^{-1}\mathrm{TDB} - c^{-2}[A_p(t) + \mathbf{v}_E\mathbf{r}_E] + \ldots.$$

L_B is determined from this relation in optimal choice, i.e.

$$1 - L_B = (1-L_C)(1-L_G) \qquad (L_B = L_C + L_G - L_C L_G) \tag{11.104}$$

resulting to

$$\mathrm{TT} = \mathrm{TDB} - c^{-2}[A_p(t) + \mathbf{v}_E\mathbf{r}_E] + \ldots . \tag{11.105}$$

Together with (11.101)

$$L_C = 1.48082686741 \times 10^{-8}\,, \qquad L_B = 1.55051976772 \times 10^{-8}\,. \tag{11.106}$$

L_C (or L_B) and L_G characterize, in general, the smallness of relativistic effects in BRS and GRS, respectively. In retaining in $A_p(t)$ only trigonometric terms (by neglecting small power and mixed terms) TT and TDB taken in the geocenter differ only by periodic terms. Under present day accuracy such approximation is not sufficient anymore liquidating the main advantage of using TT and TDB instead of TCG and TCB.

International atomic time TAI

Along the world line of the TRS origin ($w^k = w^k_T$) τ is proper time of the ground station and may be measured. The functions $V_p(u)$ specific to each TRS are known from theoretical calculations. International atomic time TAI is formed by averaging the clock readings of many ground observatories. This may be interpreted as averaging over many TRS so that

$$\mathrm{TAI} = \mathrm{mean}[\tau + c^{-2}V_p(u)]. \tag{11.107}$$

It means that TAI is the physical realization of TT. To prevent the discontinuity between atomic and ephemeris time scales one uses the constant shift

$$\mathrm{TT} = \mathrm{TAI} + 32.184s\,. \tag{11.108}$$

Therefore, the TRS coordinate time scale relates to TAI by means of

$$\tau = \mathrm{TAI} - c^{-2}[V_p(\mathrm{TAI}) + \hat{v}^k_T(w^k - w^k_T)]. \tag{11.109}$$

To resume the theory of the relativistic time scales let's formulate some points (Brumberg and Kopejkin, 1990).

1. TCB is not the proper time of the clock at rest in the solar system barycenter since

$$d\tau = [1 - c^{-2}U(t, \mathbf{x} = 0)]d(\mathrm{TCB}) \neq d(\mathrm{TCB}).$$

On the other hand,

$$d\tau = [1 - c^{-2}U(t, \mathbf{x} = \infty)]d(\mathrm{TCB}) = d(\mathrm{TCB})$$

but BRS does not exist for $\mathbf{x} = \infty$.

2. TCG is not the proper time of the clock at rest in the geocenter since

$$d\tau = [1 - c^{-2}\hat{U}_E(u, \mathbf{w} = 0)]d(\mathrm{TCG}) \neq d(\mathrm{TCG}).$$

3. Just similarly one can construct Galactic time, planetocentric time (for any major planet), solar time (for heliocentric motion), and so on.

4. As stated above,

$$\mathrm{TAI} = \mathrm{mean}[\tau + c^{-2} V_p(u)]$$

may be regarded as a physical realization of TT. Moreover, $V_p(u) = 0$ on geoid neglecting tidal potential.

5. Just in a similar manner one may introduce a modified version of TAI,

$$\mathrm{TAIM} = \mathrm{mean}[\tau + c^{-2} V(u)]$$

as a physical realization of TCG. In this case one would not deal with the notion of geoid and constant L_g. But this possibility remained unused.

6. The presently used time scales TCB, TCG, TDB, TT may be physically realized (directly or indirectly) by means of TAI involving atomic time and atomic constants. On the other hand, the astronomy–based time scales (dynamical time, ephemeris time) involve gravitational time and gravitational constant G. It is assumed that the scales of atomic time and gravitational time coincide. This coincidence can be violated if fundamental physical constants may vary. In this respect the pulsar time being not equivalent to gravitational or atomic time in a world with changing constants might be of crucial importance.

Scale factors related with TDB and TT

To preserve the form of the equations of motion and light propagation in using TDB and TT one applies coordinates and mass–factors differing from their corresponding counterparts when using TCB and TCG, respectively. It was already mentioned in Lectures 7 and 9 but to avoid any confusion we give below the list of relevant formulas once again. TCG/TDB and TCB/TT quantities are as follows:

$$\mathrm{TDB} = (1 - L_B)\mathrm{TCB}\,, \qquad (\mathbf{x})_{\mathrm{TDB}} = (1 - L_B)\mathbf{x}\,, \qquad (GM)_{\mathrm{TDB}} = (1 - L_B)GM\,, \tag{11.110}$$

$$\mathrm{TT} = (1 - L_G)\mathrm{TCG}\,, \qquad (\mathbf{w})_{\mathrm{TT}} = (1 - L_G)\mathbf{w}\,, \qquad (G\hat{M})_{\mathrm{TT}} = (1 - L_G)(G\hat{M})\,. \tag{11.111}$$

Introducing the scalar parameters

$$\mu = \begin{cases} 1\,, & t = \mathrm{TCB}\,, \\ 0\,, & t = \mathrm{TDB}\,, \end{cases} \qquad \nu = \begin{cases} 1\,, & u = \mathrm{TCG}\,, \\ 0\,, & u = \mathrm{TT} \end{cases} \tag{11.112}$$

one can combine all possible options.
Direct BCRS↔GCRS transformation takes the form

$$u = (1 - \mu L_B + \nu L_G)t - c^{-2}(A_p + \mathbf{v}_E \mathbf{r}_E), \tag{11.113}$$

$$w^i = [1 + (1 - \mu)L_B - (1 - \nu)L_G] r_E^i + c^{-2}\Lambda^i(t,\, \mathbf{r}_E), \tag{11.114}$$

$$\Lambda^i(t,\, \mathbf{r}_E) = \tfrac{1}{2}\mathbf{v}_E \mathbf{r}_E v_E^i - q\varepsilon_{ijk} F^j r_E^k + \bar{U}_E(t, \mathbf{x}_E) r_E^i + \mathbf{a}_E \mathbf{r}_E r_E^i - \tfrac{1}{2}\mathbf{r}_E^2 a_E^i\,. \tag{11.115}$$

The inverse transformation reads

$$t = (1 + \mu L_B - \nu L_G)u + c^{-2}(A_p + \mathbf{v}_E \mathbf{w}), \tag{11.116}$$

$$x^i = [1 - (1-\mu)L_B + (1-\nu)L_G](w^i + z^i_E) + c^{-2}\Gamma^i(u,\, \mathbf{w}), \tag{11.117}$$

$$z^i_E(u) = [1 + (1-\mu)L_B - (1-\nu)L_G]x^i_E(t^*) \qquad (11.118).$$

Instead of (11.117) one can use also

$$x^i = [1 - (1-\mu)L_B + (1-\nu)L_G]w^i + x^i_E(t^*) + c^{-2}\Gamma^i(u,\, \mathbf{w}). \tag{11.119}$$

One has herewith

$$t^* = (1 + \mu L_B - \nu L_G)u + c^{-2}A_p\,, \tag{11.120}$$

$$x^i_E(t^*) = x^i_E[(1 + \mu L_B - \nu L_G)u] + c^{-2}A_p v^i_E\,, \tag{11.121}$$

$$\Gamma^i(u,\, \mathbf{w}) = \tfrac{1}{2}\mathbf{v}_E\mathbf{w}v^i_E + q\varepsilon_{ijk}F^j w^k - \bar{U}_E(t, \mathbf{x}_E)w^i - \mathbf{a}_E\mathbf{w}w^i + \tfrac{1}{2}\mathbf{w}^2 a^i_E\,. \tag{11.122}$$

Geocentric position vectors of the disturbing bodies are expressed as follows:

$$w^i_A(u) = z^i_A(u) - z^i_E(u) + c^{-2}\big[\Lambda^i(t^*,\, \mathbf{r}_{AE}) + \mathbf{v}_E\mathbf{r}_{AE}v^i_{AE}\big], \tag{11.123}$$

$$z^i_A(u) - z^i_E(u) = [1 + (1-\mu)L_B - (1-\nu)L_G][x^i_A(t^*) - x^i_E(t^*)], \tag{11.124}$$

$$\mathbf{r}_{AE} = \mathbf{x}_A - \mathbf{x}_E\,, \qquad \mathbf{v}_{AE} = \mathbf{v}_A - \mathbf{v}_E, .$$

Lecture 12. Motion in the cosmological background

The motion of the solar system bodies is regarded in relativistic celestial mechanics (and so more in Newtonian celestial mechanics) almost always under assumption of the isolated existence of the solar system, i.e. by neglecting the cosmological background. This assumption is quite justified within the present observational results. But each time when there are some hints of discrepancies in the discussion of observations (e.g., time–quadratic terms in the planetary longitudes) the influence of the cosmological background is reminded as one of the possible sources of such discrepancies.

The investigation of this question was started by McVittie (1933) and continued by Järnefelt (1940, 1942) by considering the one–body problem in the expanding Universe (the Schwarzschild problem in the cosmological background). The problem was treated as a mathematical one. The corresponding results were formulated in terms of coordinates rather than in terms of physically measurable quantities (the techniques of relativistic reduction of observations were not elaborated by that time). In monograph (Brumberg, 1991) this problem was outlined in relation with the equations in variations for the spherically symmetrical background metric but the inadequate choice of coordinates involved the unnecessary mathematical difficulties. The satisfactory treatment of this problem based on the discussion of observations and the use of an adequate mathematical techniques seemed to be done for the first time in (Krasinsky and Brumberg, 2004). The conclusion is that within the present precision level the cosmological background (the expansion of the Universe) does not affect the motion of the planets and the reason of the possible observational discrepancies might be anything else. This problem needs to be further investigated. We give below the key formulas for this investigation.

Equations in variations for the spherically symmetrical metric:

The starting point is the GRT (general relativity theory) field equations (Brumberg, 1991)

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = -\kappa(\mathcal{T}^{\mu\nu} + T^{\mu\nu})\,, \tag{12.1}$$

where $\mathcal{T}^{\mu\nu}$ is the background field mass tensor, $T^{\mu\nu}$ is the perturbation field mass tensor, $G^{\mu\nu}$ denotes the Einstein tensor, Λ is the cosmological constant, and $\kappa = 8\pi G/c^2$, G being the gravitational constant. The metric form is represented by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu\,, \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{12.2}$$

with background metric tensor $g_{\mu\nu}$ and perturbations $h_{\mu\nu}$. The background isotropic metric for $\Lambda = 0$, $T^{\mu\nu} = 0$ for the spherically symmetrical field can be represented by

$$\eta_{00} = A\,, \qquad \eta_{0m} = 0\,, \qquad \eta_{mn} = -B\delta_{mn}\,, \tag{12.3}$$

$$\eta^{00} = \frac{1}{A}\,, \qquad \eta^{0m} = 0\,, \qquad \eta^{mn} = -\frac{\delta_{mn}}{B}\,, \tag{12.4}$$

A and B being functions of the radial coordinate distance r and time t to be determined from the background field equations. The field equations (12.1) can be rewritten with Ricci tensor $R_{\mu\nu}$ in form

$$R_{\mu\nu} = -\kappa(\mathcal{T}^*_{\mu\nu} + T^*_{\mu\nu}) + \Lambda g_{\mu\nu} \tag{12.5}$$

with

$$\mathcal{T}^*_{\mu\nu} = \mathcal{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{T} \, , \qquad T^*_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \, , \tag{12.6}$$

$\mathcal{T}$ and T being invariants of the mass tensors $\mathcal{T}^{\mu\nu}$ and $T^{\mu\nu}$, respectively. Therefore, the equations in variations for the field equations (12.5) read (Brumberg, 1991)

$$\delta R_{\mu\nu} = -\kappa T^*_{\mu\nu} - \kappa \delta \mathcal{T}^*_{\mu\nu} + \Lambda g_{\mu\nu} \tag{12.7}$$

with

$$\delta \mathcal{T}^*_{\mu\nu} = \mathcal{T}^*_{\mu\nu}(g_{\alpha\beta}) - \mathcal{T}^*_{\mu\nu}(\eta_{\alpha\beta}). \tag{12.8}$$

Under the coordinate conditions

$$h_{00,0} + h_{ss,0} - 2h_{0s,s} = 0 \, , \qquad h_{00,m} - h_{ss,m} + 2h_{ms,s} = 0 \tag{12.9}$$

the equations in variations for the background metric (12.3) are reduced to

$$h_{00,ss} - h_{00,00} = 2L_{00} \, , \tag{12.10}$$

$$h_{0m,ss} = 2L_{0m} \, , \tag{12.11}$$

$$h_{mn,ss} - \frac{B}{A} h_{mn,00} = 2L_{mn} + \Big(\frac{B}{A} - 1\Big) h_{00,mn} - \frac{B}{A}(h_{0m,0n} + h_{0n,0m}) \tag{12.12}$$

with contraction

$$h_{rr,ss} + h_{00,ss} = 2L_{ss} + 2\frac{B}{A} L_{00} \tag{12.13}$$

and the right–hand member functions

$$L_{\mu\nu} = B(\kappa T^*_{\mu\nu} + \kappa \delta \mathcal{T}^*_{\mu\nu} - \Lambda g_{\mu\nu} + Q_{\mu\nu}) \, , \tag{12.14}$$

$Q_{\mu\nu}$ being non–linear contributions in Ricci tensor components given by (4.3.24)– (4.3.26) of (Brumberg, 1991). Equations in variations (12.10)–(12.12) are to be solved by iterations with respect to $h_{\mu\nu}$. At each step of iteration the right–hand members $2L_{\mu\nu}$ are known. Then, equation (12.10) is the wave equation with constant coefficients. The equation (12.11) is the Poisson equation. Equation (12.12) has the form of the wave equation with variable coefficient B/A (for example, for the background Schwarzschild metric this coefficient may be reduced to a function of r alone). But for $A = B$ this equation simplifies to be also the wave equation with constant coefficients. That's why it is reasonable to use the background cosmological solution (12.3) in the conformally Galilean coordinates ensuring the condition $A = B$ (in contrast to the Robertson–Walker metric in comoving coordinates with $A = 1$ and B being a function of r and t).

The background field equations:

The background solution is constructed here for simplicity with the simplest mass tensor

$$\mathcal{T}^{\mu\nu} = \rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \, , \tag{12.15}$$

ρ being the density of the matter (dust matter without pressure). With the aid of (12.3) and (12.4) it leads to

$$\mathcal{T}^*_{00}(\eta_{\alpha\beta}) = \rho A \left(\frac{1}{2} + B\frac{dx^k}{ds}\frac{dx^k}{ds}\right), \qquad \mathcal{T}^*_{0m}(\eta_{\alpha\beta}) = -AB\rho\frac{dx^0}{ds}\frac{dx^m}{ds},$$

$$\mathcal{T}^*_{mn}(\eta_{\alpha\beta}) = \rho B \left(\frac{1}{2}\delta_{mn} + B\frac{dx^m}{ds}\frac{dx^n}{ds}\right). \tag{12.16}$$

These expressions together with the components of the background field Ricci tensor (3.1.14) of (Brumberg, 1991) enable one to get the background field equations as follows:

$$\begin{aligned} &A_{,ss} - \frac{1}{2A}A_{,s}A_{,s} + \frac{1}{2B}A_{,s}B_{,s} - 3B_{,00} + \frac{3}{2A}A_{,0}B_{,0} + \frac{3}{2B}B_{,0}B_{,0} = \\ &= 2AB\kappa\rho\left(\frac{1}{2} + B\frac{dx^k}{ds}\frac{dx^k}{ds}\right), \end{aligned} \tag{12.17}$$

$$-2B_{,0i} + \frac{1}{A}A_{,i}B_{,0} + \frac{2}{B}B_{,i}B_{,0} = -2AB^2\kappa\rho\frac{dx^0}{ds}\frac{dx^i}{ds}, \tag{12.18}$$

$$\begin{aligned} &- B_{,ik} - \delta_{ik}B_{,ss} + \frac{1}{2A}\left(A_{,i}B_{,k} + A_{,k}B_{,i} - \delta_{ik}A_{,s}B_{,s} + \delta_{ik}B_{,0}B_{,0}\right) + \\ &+ \frac{1}{2B}\left(3B_{,i}B_{,k} + \delta_{ik}B_{,s}B_{,s}\right) - \frac{B}{A}\left(A_{,ik} - \delta_{ik}B_{,00} + \frac{1}{2A}\delta_{ik}A_{,0}B_{,0} - \frac{1}{2A}A_{,i}A_{,k}\right) = \\ &= B^2\kappa\rho\left(\delta_{ik} + 2B\frac{dx^i}{ds}\frac{dx^k}{ds}\right). \end{aligned} \tag{12.19}$$

The cosmological background (the isotropic models) in comoving coordinates ($A = 1$): In comoving coordinates one has

$$A = 1, \qquad \frac{dx^0}{ds} = 1, \qquad \frac{dx^m}{ds} = 0\,. \tag{12.20}$$

Under these conditions the field equations (12.17)–(12.19) admit the solution

$$B = \frac{a^2}{\left(1 + \frac{1}{4}kr^2\right)^2}, \qquad r^2 = x^s x^s, \tag{12.21}$$

k being a real constant. Function a dependent only on time is determined by the equations

$$\frac{\ddot{a}}{a} = -\tfrac{1}{6}c^2\kappa\rho\,, \tag{12.22}$$

$$a\ddot{a} + 2\dot{a}^2 + 2kc^2 = \tfrac{1}{2}a^2c^2\kappa\rho\,. \tag{12.23}$$

From these equations it follows

$$k\frac{c^2}{a^2} = \tfrac{1}{3}c^2\kappa\rho - H^2\,, \qquad H = \frac{\dot{a}}{a}\,, \tag{12.24}$$

H being the Hubble constant. One has also

$$2a\ddot{a} + \dot{a}^2 + kc^2 = 0\,, \tag{12.25}$$

$$\dot{\rho} + 3H\rho = 0\,. \tag{12.26}$$

This Robertson–Walker metric reads:

$$ds^2 = dx^0dx^0 - \frac{a^2}{\left(1+\frac{1}{4}kr^2\right)^2}dx^sdx^s\,. \tag{12.27}$$

With a new "time" argument η

$$dx^0 = ad\eta\,, \qquad \dot{\eta} = \frac{c}{a}\,, \qquad H = \frac{c}{a^2}\frac{da}{d\eta} \tag{12.28}$$

it describes explicitly three well–known cosmological models (with arbitrary linear constant q):

Closed model: $k = 1$

$$a = 2q(1-\cos\eta), \qquad t = \frac{2q}{c}(\eta - \sin\eta), \qquad H = \frac{c}{2q}\frac{\sin\eta}{(1-\cos\eta)^2}, \tag{12.29}$$

Flat model: $k = 0$

$$a = q\eta^2\,, \qquad t = \frac{q}{3c}\eta^3\,, \qquad H = \frac{2c}{q}\eta^{-3}, \tag{12.30}$$

Open model: $k = -1$

$$a = 2q(\cosh\eta - 1), \qquad t = \frac{2q}{c}(\sinh\eta - \eta), \qquad H = \frac{c}{2q}\frac{\sinh\eta}{(\cosh\eta - 1)^2}. \tag{12.31}$$

The cosmological background (the isotropic models) in conformally Galilean coordinates ($A = B$):

In this case the field equations (12.17)–(12.19) result in

$$A_{,ss} - 3A_{,00} + \frac{3}{A}A_{,0}A_{,0} = 2A^2\kappa\rho\left(\frac{1}{2} + A\frac{dx^k}{ds}\frac{dx^k}{ds}\right) \tag{12.32}$$

$$-2A_{,0i} + \frac{3}{A}A_{,i}A_{,0} = -2A^3\kappa\rho\frac{dx^0}{ds}\frac{dx^i}{ds} \tag{12.33}$$

$$-2A_{,ik} - \delta_{ik}A_{,ss} + \frac{3}{A}A_{,i}A_{,k} + \delta_{ik}A_{,00} = A^2\kappa\rho\left(\delta_{ik} + 2A\frac{dx^i}{ds}\frac{dx^k}{ds}\right) \tag{12.34}$$

admitting the general expression for all three values of k

$$A = \left(1 - \frac{q}{d}\right)^4 + \left(\frac{x^0}{q}\right)^4 \delta_{k0}\,, \qquad d = \sqrt{k(x^s x^s - x^0 x^0)} + q\delta_{k0}\,, \qquad q = \mathrm{const}\,. \tag{12.35}$$

The expression of the Hubble constant for this case reads

$$H = \sqrt{-k}\,\frac{c}{d}\frac{1 + \frac{q}{d}}{\left(1 - \frac{q}{d}\right)^3} + \frac{2cq^2}{{x^0}^3}\delta_{k0}\,. \tag{12.36}$$

Indeed, taking the derivatives of A from (12.35) and substituting them into the left–hand members of (12.32)–(12.34) one gets for $k \neq 0$

$$-12\left(1 - \frac{q}{d}\right)^2 k\frac{q}{d^3}\left(1 - 2k\frac{r^2}{d^2}\right) = A^2\kappa\rho\left(1 + 2A\frac{dx^k}{ds}\frac{dx^k}{ds}\right),$$

$$-24\left(1 - \frac{q}{d}\right)^2 k^2\frac{q}{d^5}x^0x^i = -2A^3\kappa\rho\frac{dx^0}{ds}\frac{dx^i}{ds}\,,$$

$$-12\left(1 - \frac{q}{d}\right)^2 k\frac{q}{d^3}\delta_{ik} + 24\left(1 - \frac{q}{d}\right)^2 k^2\frac{q}{d^5}x^ix^k = A^2\kappa\rho\left(\delta_{ik} + 2A\frac{dx^i}{ds}\frac{dx^k}{ds}\right).$$

These equations are satisfied by the solution (12.35) for A together with

$$\kappa\rho = -k\frac{12q}{d^3\left(1 - \frac{q}{d}\right)^6}\,, \qquad \frac{dx^\mu}{ds} = \sqrt{-k}\frac{x^\mu}{d\left(1 - \frac{q}{d}\right)^2} \tag{12.37}$$

determining the density and the velocity of the dust matter (for $k = -1$ this solution coincides with the Fock (1955) solution).

Applying the same procedure for $k = 0$ one gets from (12.32)–(12.34)

$$12 = \frac{{x^0}^6}{q^4}\kappa\rho\left(1 + 2\frac{{x^0}^4}{q^4}\frac{dx^k}{ds}\frac{dx^k}{ds}\right),$$

$$0 = \kappa\rho\frac{dx^0}{ds}\frac{dx^i}{ds}\,,$$

$$12\delta_{ik}\frac{{x^0}^2}{q^4} = \frac{{x^0}^8}{q^8}\kappa\rho\left(\delta_{ik} + 2\frac{{x^0}^4}{q^4}\frac{dx^i}{ds}\frac{dx^k}{ds}\right).$$

One comes again to the solution for A from (12.35) together with

$$\kappa\rho = 12\frac{q^4}{x^{0\,6}}\,. \tag{12.38}$$

This value of ρ corresponds to the critical density

$$\kappa\rho = \frac{3H^2}{c^2}\,, \tag{12.39}$$

as seen from (12.24).

In conformally Galilean coordinates the equation (12.26) takes the form

$$\dot\rho = -3H\rho\,\frac{ds}{dx^0}\,, \tag{12.40}$$

where

$$\left(\frac{ds}{dx^0}\right)^2 = \frac{A}{1 + A\frac{dx^k}{ds}\frac{dx^k}{ds}}\,. \tag{12.41}$$

In dealing with the conformally Galilean metric one may note a useful relation

$$\dot d = c\,\frac{ds}{dx^0}\,\frac{\sqrt{-k}}{\left(1-\frac{q}{d}\right)^2}\,. \tag{12.42}$$

Reduction of the Robertson–Walker metric to the conformally Galilean form:
Two different algorithms of such reduction are given below. The first one similar to the Fock (1955) representation for the open model involves complex coordinates for the closed model. The second algorithm involving only real coordinates is based on problem 19.8 of (Lightman et al,, 1975).

Algorithm 1:
Introducing spherical coordinates by means of

$$x^1 = r\cos\varphi\sin\theta\,,\quad x^2 = r\sin\varphi\sin\theta\,,\quad x^3 = r\cos\theta$$

one can represent the Robertson–Walker metric in form

$$ds^2 = a^2(\eta)\left[d\eta^2 - \frac{1}{\left(1+\frac{1}{4}kr^2\right)^2}(dr^2 + r^2 d\Omega^2)\right] \tag{12.43}$$

with

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2\,.$$

The transformation of the radial coordinate

$$\Sigma = \frac{r}{1+\frac{1}{4}kr^2}\,,\qquad d\Sigma = \frac{1-\frac{1}{4}kr^2}{\left(1+\frac{1}{4}kr^2\right)^2}dr \tag{12.44}$$

implying

$$1 - k\Sigma^2 = \frac{\left(1 - \frac{1}{4}kr^2\right)^2}{\left(1 + \frac{1}{4}kr^2\right)^2}\,, \qquad 1 + \tfrac{1}{4}kr^2 = \frac{2}{1 + \sqrt{1 - k\Sigma^2}}$$

transforms (12.43) into

$$ds^2 = a^2(\eta)\left(d\eta^2 - \frac{d\Sigma^2}{1 - k\Sigma^2} - \Sigma^2 d\Omega^2\right). \tag{12.45}$$

A new variable χ determined by differential relation

$$\frac{d\Sigma^2}{1 - k\Sigma^2} = d\chi^2 \tag{12.46}$$

or in explicit form

$$\Sigma = \begin{cases} \sin\chi\,, & k = +1 \quad \text{(closed model)} \\ \chi\,, & k = 0 \quad \text{(flat model)} \\ \sinh\chi\,, & k = -1 \quad \text{(open model)} \end{cases} \tag{12.47}$$

enables one to rewrite (12.45) as

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\chi^2 - \Sigma^2(\chi)d\Omega^2\right). \tag{12.48}$$

The final reduction to the conformally Galilean metric

$$ds^2 = A(c^2dt^2 - dr^2 - r^2d\Omega^2) \tag{12.49}$$

is provided by the transformation

$$ct = \sqrt{-k}\,d\cos(\sqrt{k}\chi) + \eta\,d\,\delta_{k0}\,, \qquad r = d\Sigma\,, \qquad d = q\exp(\sqrt{-k}\eta\,, \tag{12.50}$$

implying $c^2dt^2 - dr^2 = d^2(d\eta^2 - d\chi^2)$ and

$$A = \frac{a^2}{d^2}, \qquad a = -kd\left(1 - \frac{q}{d}\right)^2 + q\eta^2\delta_{k0}\,, \tag{12.51}$$

q being a real constant. Needless to say, $\cos \mathrm{i}\,\chi = \cosh\chi$. It is easy to see that the form (12.49) with (12.51) is identical to the previous found solution with (12.35).

Algorithm 2:
Starting from (12.48) it is possible instead of (12.50) to transform η and χ by means of

$$u = \frac{1}{2}(\eta + \chi), \qquad v = \frac{1}{2}(\eta - \chi), \tag{12.52}$$

reducing (12.48) to

$$ds^2 = a^2(4du\,dv - \Sigma^2d\Omega^2). \tag{7.53}$$

Now u and v are changed to new variables α and β in a similar manner

$$\alpha = g(u), \qquad \beta = g(v), \qquad u = f(\alpha), \qquad v = f(\beta) \tag{12.54}$$

so that

$$du = f'(\alpha)d\alpha\,, \qquad f'(\alpha) = [g'(u)]^{-1}\,,$$

accent denoting the derivative with respect to the corresponding argument. Hence, the metric (12.53) transforms to

$$ds^2 = a^2 f'(\alpha) f'(\beta)\left[4d\alpha d\beta - \frac{\Sigma^2}{f'(\alpha)f'(\beta)}d\Omega^2\right]. \tag{12.55}$$

The function g is to be chosen to provide the condition for the conformally Galilean metric

$$\frac{\Sigma^2}{f'(\alpha)f'(\beta)} = (\alpha-\beta)^2 \tag{12.56}$$

or more specifically

$$g'(u)g'(v)\Sigma^2 = [g(u)-g(v)]^2\,, \tag{12.57}$$

Σ being considered as dependent only on the difference $u-v$. For any $k=+1,\,0,\,-1$ this equation is satisfied by

$$g(u) = \tan(\sqrt{k}\,u) + u\,\delta_{k0} \tag{12.58}$$

with evident replacing $\tan \mathrm{i}\,u = \tanh u$. The metric (12.55) becomes

$$ds^2 = \frac{a^2}{(1+k\alpha^2)(1+k\beta^2)}[4d\alpha d\beta - (\alpha-\beta)^2 d\Omega^2]. \tag{12.59}$$

Finally, the transformation

$$\alpha = \frac{1}{2q}(c\tilde{t}+\tilde{r}), \qquad \beta = \frac{1}{2q}(c\tilde{t}-\tilde{r}) \tag{12.60}$$

reduces (12.59) to the conformally Galilean metric of the type (12.49)

$$ds^2 = \tilde{A}(c^2d\tilde{t}^2 - d\tilde{r}^2 - \tilde{r}^2 d\Omega^2) \tag{12.61}$$

with

$$\tilde{A} = \frac{a^2}{q^2 D}\,, \tag{12.62}$$

$$D = (1+k\alpha^2)(1+k\beta^2) = 1 + \frac{k}{2q^2}(c^2\tilde{t}^2+\tilde{r}^2) + \frac{k^2}{16q^4}(c^2\tilde{t}^2-\tilde{r}^2)^2\,. \tag{12.63}$$

By using

$$\sigma = g(\eta) \tag{12.64}$$

there results

$$a = 2q \frac{\sigma^2}{\sqrt{1+k\sigma^2}(1+\sqrt{1+k\sigma^2})}\,, \qquad \sigma = \frac{\dfrac{c\tilde{t}}{q}}{1 - k\dfrac{c^2\tilde{t}^2 - \tilde{r}^2}{4q^2}}\,. \tag{12.65}$$

The equivalence of (12.49) and (12.61) may be easily seen from the expressions of $\tilde{t}$, $\tilde{r}$ from the one part, and t, r from the other part, in terms of η and χ. By comparing these expressions one gets

$$\tilde{r} = \sqrt{D}\,\frac{q}{d}\,r \tag{12.66}$$

and

$$c^2 d\tilde{t}^2 - d\tilde{r}^2 = q^2 D(d\eta^2 - d\chi^2) = \frac{q^2 D}{d^2}(c^2dt^2 - dr^2) \tag{12.67}$$

(this formula corrects a misprint in (12.51) of Brumberg, 2005).
Hence,

$$A = \frac{q^2 D}{d^2}\,\tilde{A} \tag{12.68}$$

demonstrating the equivalence of (12.59) and (12.61). In what follows the form (12.61) will be used with no specification for $\tilde{A}$ and with omitting tilde over $\tilde{A}$, $\tilde{t}$ and $\tilde{r}$.

Solar gravitational field at the cosmological background

Let the variations of the background gravitational field of the expanding Universe be caused by a spherical massive body (the Sun) located at the spatial origin $\mathbf{r} = (\mathbf{x^k}) = \mathbf{o}$. Investigating a quasi–circular motion of a test particle in this field (Schwarzschild problem at the cosmological background) one may see if the expansion of the Universe affects the motion of the Solar system bodies (Krasinsky and Brumberg, 2004). For this purpose it is sufficient to have the simplest, just quasi–Newtonian solution of equations (12.10)–(12.12) by restricting in (12.14) only by the first term, i.e. $L_{\mu\nu} = \kappa B T^*_{\mu\nu}$. Then the disturbing mass tensor $T^{\mu\nu}$ may be taken in the form

$$T^{\mu\nu} = \frac{\tilde{\rho}}{\sqrt{-g}}\frac{dx^0}{ds}\frac{dx^\mu}{dx^0}\frac{dx^\nu}{dx^0} \tag{12.69}$$

with the density

$$\tilde{\rho} = M\delta(\mathbf{r})\,, \tag{12.70}$$

$\delta(\mathbf{r})$ being delta–function (Infeld and Plebansky 1960). For the background metric (12.3) one easily finds

$$BT^*_{00} = \frac{1}{2}\sqrt{A}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2}\left(\frac{A}{B} + \frac{v^2}{c^2}\right),$$

$$BT^*_{0i} = -\sqrt{A}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2}\frac{v^i}{c}\,,$$

$$BT^*_{ij} = \frac{B}{\sqrt{A}}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2}\left[\frac{1}{2}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)\delta_{ij} + \frac{v^i}{c}\frac{v^j}{c}\right]. \tag{12.71}$$

In integrating the equations (12.10)–(12.12) for the fixed material point in the conformally Galilean background with $A = B$ one may put

$$L_{00} = 4\pi\sqrt{A}m\delta(\mathbf{r}), \qquad L_{0i} = 0, \qquad L_{ij} = 4\pi\sqrt{A}m\delta(\mathbf{r})\delta_{ij} \tag{12.72}$$

with

$$m = \frac{GM}{c^2}. \tag{12.73}$$

Hence, by neglecting the retardation terms one may present the approximate solution of (12.10)–(12.12) in form

$$h_{00} = -\frac{2m}{r}\sqrt{A}, \qquad h_{0i} = 0, \qquad h_{ij} = -\frac{2m}{r}\sqrt{A}\,\delta_{ij}, \tag{12.74}$$

resulting to

$$ds^2 = \left(A - \frac{2m}{r}\sqrt{A}\right)c^2dt^2 - \left(A + \frac{2m}{r}\sqrt{A}\right)dx^s dx^s. \tag{12.75}$$

Equations of motion of a test particle in the field (12.75) with the coordinate time t as an argument follow from equations (4.3.38) of (Brumberg, 1991) for the general metric (12.3). By retaining only the main terms one has

$$\ddot{x}^i = \frac{1}{2A}(\dot{x}^s\dot{x}^s - c^2)(A_{,i} + c^{-1}A_{,0}\dot{x}^i) - \frac{c^2}{2A}h_{00,i} - \frac{c^2}{2A^2}A_{,s}h_{ms} + \ldots \tag{12.76}$$

or else

$$\ddot{x}^i = -\frac{GM}{\sqrt{A}r^3}x^i - \frac{\dot{A}}{2A}\dot{x}^i. \tag{12.77}$$

In reducing (12.76) to (12.77) the last term in the right–hand member of (12.76) is neglected due to the presence of the derivatives $A_{,s}$ much smaller as compared with A. For the same reason $cA_{,0}$ in (12.77) is replaced just by $\dot{A}$. Considering the motion of the test particle for some limited interval of time $t - t_0$ one may use approximation

$$A = A_0 + \dot{A}_0(t - t_0), \tag{12.78}$$

using as a small parameter the ratio $(t - t_0)/T$ where

$$T = \frac{4A_0}{\dot{A}_0}. \tag{12.79}$$

Then, equation (12.77) reads

$$\ddot{x}^i = -\frac{GM^*}{r^3}\left(1 - 2\frac{t - t_0}{T}\right)x^i - \frac{2}{T}\dot{x}^i, \qquad M^* = \frac{M}{\sqrt{A_0}}. \tag{12.80}$$

It is seen that the main effect of these equations may be interpreted as the variability of the constant of gravitation

$$\frac{\dot{G}}{G} = -\frac{2}{T}\,. \tag{12.81}$$

On the other hand, in polar coordinates in the plane of motion $x^3 = 0$

$$x^1 = r\cos\lambda\,, \qquad x^2 = r\sin\lambda$$

these equations read

$$\ddot{r} - r\dot{\lambda}^2 = -GM^*\left(1 - 2\frac{t-t_0}{T}\right)\frac{1}{r^2} - \frac{2}{T}\,\dot{r}\,, \qquad \frac{1}{r}\frac{d}{dt}(r^2\dot{\lambda}) = -\frac{2}{T}\,r\,\dot{\lambda}\,, \tag{12.82}$$

or else

$$r^2\dot{\lambda} = na^2\exp\left(-2\frac{t-t_0}{T}\right), \qquad \dot{\lambda} = n\left(\frac{a}{r}\right)^2\left(1 - 2\frac{t-t_0}{T}\right),$$

$$\ddot{r} = \frac{n^2a^4}{r^3}\left(1 - 4\frac{t-t_0}{T}\right) - \frac{n^2a^3}{r^2}\left(1 - 2\frac{t-t_0}{T}\right) - \frac{2}{T}\,\dot{r}\,, \tag{12.83}$$

a being an arbitrary constant with $n^2a^3 = GM^*$. These equations admit an approximate solution

$$r = a\left(1 - 2\frac{t-t_0}{T}\right), \qquad \dot{\lambda} = n\left(1 + 2\frac{t-t_0}{T}\right), \tag{12.84}$$

involving the quadratic term in the mean longitude

$$\delta\lambda = \frac{n}{T}(t-t_0)^2\,. \tag{12.85}$$

This is just a coordinate–form solution. To get the physically meaningful relativistic effects it is necessary in general to perform the relativistic reduction of observations using the solution of the equations of the light propagation. These equations follow again from equations (4.3.38) of (Brumberg, 1991) under the substitution

$$B\dot{x}^s\dot{x}^s = c^2A + c^2h_{00} + 2c\,h_{0s}\dot{x}^s + h_{rs}\dot{x}^r\dot{x}^s \tag{12.86}$$

resulted from the condition $ds^2 = 0$ for the light propagation. By restricting only by the main terms one has for $A = B$

$$\ddot{x}^m = \frac{GM}{\sqrt{A}r^3}\left(-2x^m + \frac{4}{c^2}\dot{x}^m\dot{x}^s x^s\right). \tag{12.87}$$

In our case it is sufficient to express the longitude λ of the moving particle in terms of the proper time τ of this particle. From (12.75) it is seen that within the first order with respect to the small parameter of this problem the proper time τ reads

$$\frac{d\tau}{dt} = \sqrt{A_0}\left(1 + 2\frac{t-t_0}{T}\right) \tag{12.88}$$

demonstrating that the quadratic term (the only one significant term) in the mean longitude (12.85) does not exist as a measurable effect in $d\lambda/d\tau$. It follows from this that within the present–day observational precision there is no observable effect in the motion of the solar system bodies due to the cosmological background (Krasinsky and Brumberg, 2004).

However, this treatment is only one of the initial steps toward the global problem of the influence of the cosmological background. In particular, it may be of interest to correlate this approach with the exact solution for the one–body problem in an expanding universe by McVittie (1933) and Järnefelt (1940, 1942) or to apply the equations in variations for the conformally Galilean background field ($A = B$) for more wide class of perturbations.

This lecture reproduces Section 7 of (Brumberg, 2008).

Conclusion

In several years GRT will celebrate its first centenary. The same is true for relativistic celestial mechanics (although very captious people may think that relativistic celestial mechanics is younger than GRT for one year). After rather slow development in the first half of its existence relativistic celestial mechanics made a significant advance in the second half. These lectures are aimed to make students familiar with basic techniques and problems of relativistic celestial mechanics without using too complicated tools unnecessary for most problems of applied astronomy in the post–Newtonian approximation. In this respect these lectures may be used as an introductory treatise into relativistic celestial mechanics with a collection of practical post–Newtonian formulas. The interested readers may find different approaches to solve more advanced problems in the references given in these lectures.

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