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RELATIVISTIC TIME SCALES IN THE SOLAR SYSTEM

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Abstract. This paper deals with a self-consistent relativistic theory of time scales in the Solar system based on the construction of the hierarchy of dynamically non-rotating harmonic reference systems for a four-dimensional space-time of general relativity. In our approach barycentric (TB) and terrestrial (TT) times are regarded as the coordinate times of barycentric (BRS) and geocentric (GRS) reference systems, respectively, with an appropriate choice of the units of measurement. This enables us to avoid some of the inconsistencies and ambiguities of the definitions of these scales as these are currently applied. International atomic time (TAI) is shown to be the physical realization of TT on the surface of the Earth. This realization is achieved by a specific procedure to average the readings of atomic clocks distributed over the terrestrial surface, all of them synchronized with respect to TT. Extending TAI beyond the Earth's surface may be performed along a three-dimensional hypersurface TT = const. The unit of measurement of TAI coincides with TB and TT units and is equal to the SI second on the surface of the geoid in rotation. Due to the specific choice of the units of measurement the TB scale differs from the TT (TAI) scale only by relativistic nonlinear and periodic terms resulting from the planetary and lunar theories of motion. The proper time τ_0 of any terrestrial observer coincides with the coordinate time τ of the corresponding topocentric reference system (TRS) evaluated at its origin. τ_0 is reacted to TT (TAI) by the relativistic transformation involving the GRS velocity of the observer, its height above the geoid and the quadrupole tidal gravitational potential of the external masses. The impact of introducing TB and TT on the units of measurement of length and the basic astronomical constants is discussed.

1. Introduction

Development of the astronomical time scales currently presents one of the most urgent tasks of applied astronomy. The experimental precision of time measurements is increasing steadily. This demands an improvement in the accuracy of the theoretical models including physically meaningful and mathematically rigorous definitions of all of the quantities involved in time measurements. In the last few years the problem of time scales has often been discussed during the General Assemblies of the International Astronomical Union and the International Association of Geodesy. But the resolutions on this subject turn out to be of limited longevity. Subsequent increases in the precision with which time can be measured eventually revealed their logical inconsistency. Some insight into the historical development of basic ideas and concepts in this area can be gained from Mulholland (1972), Wilkins (1974), Winkler and Van Flandern (1977), and Murray (1983).

Nowadays one may note a new stage in the elaboration of the problem of time scales. This stage is marked by publication of the paper by Guinot and Seidelmann (1988) in which the authors outlined the actual status of the question and proposed their

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recommendations and declarations for wide discussion. Our paper aims to contribute into this discussion. We do not think that the present situation as described by Guinot and Seidelmann demands radical changes. The aim of our paper is to make their statements concerning time scales more rigorous in the relativistic sense, regarding them as the part of the general problem of four-dimensional reference systems for astronomy and geodesy. This approach, relating in a quite definite way the time scales TB and TT (TAI) with the coordinate times of the appropriate reference systems enables, as we believe, the removal of some inconsistencies and ambiguities occuring both in the IAU resolutions and some recent papers on time scales, for example those of Murray (1983), Japanese Ephemeris (1985), Guinot (1986), Fukushima (1989), and Huang *et al.* (1989).

The first of these ambiguities is related to the definition of TB. TB is often defined as the coordinate time of the barycentric reference system (BRS) or as the independent argument in the barycentric equations of motion of celestial bodies. But the notions of BRS and BRS equations of motion have no single meaning within the framework of the general relativity theory (GRT). To introduce a unique four-dimensional reference system in GRT it is necessary to indicate explicitly the corresponding metric form or to formulate the conditions to be used as the basis for constructing this form. These conditions involve the following points:

(1) coordinate conditions employed in solving the GRT field equations;
(2) the type of the solutions of the GRT field equations;
(3) the choice of the world line of the origin of the reference system;
(4) the choice of the rotation of the spatial axes.

The uniqueness in formulating a reference system results in the unambiguous equations of motion of celestial bodies. Without such uniqueness of BRS and BRS equations of motion the time scale of TB actually remains undetermined.

Still further ambiguities occur in the current definitions of TT. First of all, TT is often considered only as an argument in calculating the apparent geocentric positions of celestial bodies. Such a treatment leads to the restriction of the TT functions because this time scale serves as the most adequate independent argument in constructing theories of motion of the Moon or artificial satellites of the Earth. TT is sometimes defined as the proper time of the clock at rest at the geocentre, provided that the mass of the Earth is negligible. This definition has its origin in the historical traditions when, in investigating reference systems and time scales, the mass of the Earth and its gravitational field may be ignored. These traditions involve more recent attempts to consider the motion of the geocentre in a fictitious time-space due to the external masses surrounding the Earth. This construction introduces new inconsistencies in the definition of TT. At present, the geocentric reference system (GRS) should, like BRS, be defined quite uniquely; only then does the definition of TT become quite rigorous. Sometimes TT is defined as the time of clocks on the geoid. In doing so, the definition and the practical realization of the time scale are mixed together. Moreover, one introduces the notion of the geoid, which itself needs to be correctly defined.

As far as TAI is concerned there are still controversies about whether TAI is a proper or a coordinate time. This question is answered by the operational procedure of the actual determination of TAI.

Finally, the problem of units of time and length is now also rather controversial. It is evident that the elaboration of the whole system of astronomical constants within the framework of GRT is the task for the future (the not too distant future, let us hope). But more particular problem of units of time and length may be resolved satisfactorily, even now. The consistent relativistic theory of four-dimensional reference systems is adequate for such a solution.

The content of this paper is as follows. Section 2 is devoted to the basic principles underlying our approach to the construction of GRT reference systems. The technique itself is described in Section 3. The reference systems constructed in such a manner are described in Section 4. In Section 5 the asymptotic matching technique is applied to derive the coordinate transformations between the systems. Actual realization of the time scales is developed in Section 6. The units of time and length are discussed in Section 7. Section 8 contains some comments on the questions under discussion. Finally, the recommendations resulting from our approach are given in Section 9.

2. Basic Principles

The space-time of GRT is the curved Riemannian manifold characterized by the metric tensor $g_{\alpha\beta}$ (here and below the Greek subscripts run from 0 to 3; the Latin subscripts run from 1 to 3; every pair of repeated subscripts implies summation over corresponding values; the signature of the metric is equal to $+2$). The metric tensor is found by solving the Einstein field equations with complementary boundary and (or) initial values. Usually, to simplify calculations and to obtain specific results, one introduces into the space-time some reference (co-ordinate) system x^α. The coordinate x^0 refers to time and three other coordinates x^i relate to the spatial dimensions. It should be noted that the reference system may be chosen quite arbitrarily. Therefore, one cannot assume in general that the coordinate time x^0 corresponds to the readings of any actual clock. The interval $\mathrm{d}s$ between any two infinitesimal events is expressed with the aid of the coordinate increments $\mathrm{d}x^\alpha$ and the metric tensor as $\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$. Hence, the GRT metric tensor describes both the gravitational field and the specific features of the reference system.

Most commonly, the whole of space-time cannot be covered with a single reference system. In such cases one introduces a set of mutually superimposing reference systems that altogether cover the whole space-time. In the domain common to both reference systems x^α and w^α the coordinates of one system are functions of the coordinates of another system, i.e. $x^\alpha = x^\alpha(w^\beta)$ and $w^\alpha = w^\alpha(x^\beta)$. The tensor transformation law is therefore valid

$$\hat{g}_{\alpha\beta}(w^\gamma) = g_{\mu\nu}(x^\gamma)\frac{\partial x^\mu}{\partial w^\alpha}\frac{\partial x^\nu}{\partial w^\beta} \tag{2.1}$$

where $\hat{g}_{\alpha\beta}(w^\gamma)$ represents the metric tensor in coordinates w^α and $g_{\alpha\beta}(x^\gamma)$ is the metric tensor in coordinates x^α. The choice of any specific coordinate system is not fundamental since it is possible to use any coordinates. But the adequate choice of a coordinate system may significantly facilitate the treatment of a problem to be investigated and elucidate the meaning of the relevant physical events.

Any test body (an observer) in the space-time describes a time-like world line (an isotropic line in case of a photon). The world line of a freely falling body is called its geodesic. Under acceleration the body moves along the accelerated world line, which differs from the geodesic. The deviation from the geodesic is characterized by the curvature of the world line, which is proportional to the acceleration. The curvature of the geodesic is zero. Similarly, one may consider the motion of extended self-gravitating bodies. In this case one introduces the notion of the centre of mass of the body. Special methods have been developed for deriving the corresponding equations of motion (Brumberg, 1972; Misner *et al.*, 1973; Will, 1981; Damour, 1983, 1987; Kopejkin, 1985, 1988; Schäfer, 1986).

Along any time-like world line of the test body one may introduce a parameter called a proper time. The interval of the proper time $d\tau_0$ is related to the interval ds by the formula $c^2\, d\tau_0^2 = -ds^2$, c being the velocity of light. By definition the proper time is a mathematical idealization. In practice the proper time is measured with some definite error using atomic standards of frequency.

It is known that the rates of the proper time are different for different observers. Yet, in astronomical practice it is necessary to have some universal time scale of high accuracy enabling the production of observations of celestial bodies and monitoring of the Earth's rotation. Several time scales of this kind may exist. Each of them should be established with reference to a specific periodic physical process.

At present, it is customary to consider as the most accurate physical processes three types of universal time scales, each having its own specific purpose. These scales are the Barycentric Time, TB, the Terrestrial Time, TT, and the International Atomic Time, TAI (Guinot and Seidelmann, 1988). Among these three scales the practical realization has been accomplished only for TAI, being based on the existence of high-stability periodic processes occurring in atoms. The remaining two scales, TB and TT, are developed on the basis of TAI by calculation with some definite mathematical formulae and representing the coordinate times of BRS and GRS respectively (with appropriate factors). TAI presents the physical realization of TT, achieved by a specific procedure used to average readings of atomic clocks distributed over the Earth's surface and synchronized with respect to TT. It is of great importance that one deals in this process with the coordinate synchronization, which has to be distinguished from the Einstein synchronization. The Einstein synchronization may be performed by displacing the clocks with direct comparison of their readings or by the transmission of electromagnetic signals between them (Landau and Lifshitz, 1975). It is well known that the Einstein synchronization cannot be attained for the whole Earth due to the Sagnac effect caused by the Earth's rotation. The coordinate synchronization involves reducing the readings of all clocks to the coordinate time. Such a synchronization is possible in various physical situations including the case of the rotating Earth. Various versions of the synchronization of clocks are discussed in Ashby and Allan (1979) and Allan and Ashby (1986).

There are some theoretical approaches to the determination of the above mentioned time scales, including Thomas (1975), Moyer (1981), Murray (1983), Fukushima *et al.* (1986a), Ashby and Bertotti (1986), Fukushima (1989) and Soffel (1989). The approach described here is based on the relativistic theory of astronomical reference systems developed in Kopejkin (1988) and Brumberg and Kopejkin (1989a, b).

3. Method

The essence of the method is the construction of a set of reference systems by solving the Einstein field equations. Subsequent matching of these systems enables one to find the coordinate transformations between them. We use nonrotating harmonic coordinate systems imposing the harmonic (De Donder) conditions on the metric tensor: $(\sqrt{-g}g^{\alpha\beta})_{,\beta}=0$, $g=\det(g_{\alpha\beta})$ where the comma denotes a partial derivative. Harmonic coordinates have no physical advantages but they are very convenient mathematically, enabling one to simplify the solution of many problems of relativistic celestial mechanics.

For the sake of simplicity we confine ourselves to the Newtonian approximation of GRT. This approximation is quite sufficient, within the limits of modern observational precision, to discuss the problem of relativistic time scales in the Solar system. The relativistic corrections of higher order have been discussed in Brumberg and Kopejkin (1989a, b) and Kopejkin (1988, 1989a, b). In the Newtonian approximation the metric is of the form

$$\begin{aligned} ds^2 &= g_{00}(dx^0)^2 + 2g_{0i}\,dx^0\,dx^i + g_{ij}\,dx^i\,dx^j, \\ g_{00} &= -1 + c^{-2}2U + O(c^{-4}), \quad g_{0i} = O(c^{-3}), \quad g_{ij} = \delta_{ij} + O(c^{-2}) \end{aligned} \tag{3.1}$$

where the potential U satisfies the Poisson equation

$$U_{,kk} = -4\pi G\rho, \tag{3.2}$$

where G is the gravitational constant and ρ is the density of matter in the space-time domain under consideration.

Our harmonic reference system is dynamically nonrotating (Kovalevsky and Mueller, 1981). This means that the equations of motion of a test body resulting from (3.1) contain neither Coriolis nor centrifugal terms. Such terms may occur only if components g_{0i} include terms like $c^{-1}\varepsilon_{ijk}\Omega^j x^k$ where ε_{ijk} is the fully antisymmetric

Levi-Civita symbol and Ω^j is the (time-dependent or constant) angular rotational velocity of the spatial axes. Besides this we shall deal further with dynamically rotating reference systems which are not harmonic. These systems results from the appropriate nonrotating systems, using the spatial rotation matrix dependent on time only. The time coordinate is not thereby transformed. Hence, the time scales of the corresponding non-rotating and rotating systems coincide.

Solution of Equation (3.2) may be presented in different forms characteristic of the type of reference system chosen. The specific type of the solution depends on the physical conditions in the domain of validity of the reference system. These physical conditions in turn may be described with the aid of boundary conditions for potential U. To simplify the boundary conditions and the solutions of Equation (3.2) we shall ignore the gravitational field of the Galaxy, regarding the Solar system as an isolated one. In BRS coordinates $x^\alpha = (ct, x^i) = (ct, \mathbf{x})$ the potential $U(t, \mathbf{x})$ at the boundary of the Solar system has the asymptotic form

$$U(t, \mathbf{x}) = GM/R + O(R^{-3}) \tag{3.3}$$

with M being the total mass of the all Solar system gravitating bodies and $R = |\mathbf{x}|$. The right-hand side of (3.2) thus incorporates the density of all bodies.

For the nonrotating geocentric reference system (GRS) with coordinates $w^\alpha = (cu, w^i) = (cu, \mathbf{w})$ the boundary conditions are different. The Earth is almost freely falling in the gravitational field of the external masses. Small deviations from the geodesic motion arising as a result of the interaction of the Earth's multipole moments with the tidal gravitational field of the external masses (Kopejkin, 1988; Brumberg and Kopejkin, 1989a) may here be neglected. For this reason the external gravitational field manifests itself in the vicinity of the Earth only in the form of tidal terms. Hence, potential $U(u, \mathbf{w})$ may be presented in GRS as the sum $U_E(u, \mathbf{w}) + V(u, \mathbf{w})$ of the Earth's potential U_E and the tidal external mass potential V, respectively. Potential U_E is to be found by solving (3.2) taking into account only the Earth's mass density and imposing the asymptotic form far from the Earth

$$U_E(u, \mathbf{w}) = GM_E/r + O(r^{-3}) \tag{3.4}$$

with M_E being the Earth's mass and $r = |\mathbf{w}|$. Potential V is found by solving the Laplace equation with the boundary condition

$$V(u, \mathbf{w} = 0) = 0. \tag{3.5}$$

To relate BRS and GRS coordinate times to the proper time of an observer we have to introduce a reference system the origin of which coincides with the observer. If the observer is situated on the ground the corresponding system is called a topocentric reference system (TRS). For the observer on an Earth satellite the corresponding system is called a satellite reference system (SRS). In both cases we shall use the same designations for the coordinates $\xi^\alpha = (c\tau, \xi^i) = (c\tau, \boldsymbol{\xi})$ since we are not interested in the relationship between TRS and SRS. Both systems may be constructed in a quite similar manner. The observer is regarded as a massless (test) body and in his vicinity the gravitational field of all bodies (including the Earth) is of a tidal nature. Therefore, the gravitational potential $U(\tau, \xi) = U_T(\tau, \xi)$ in the vicinity of the observer satisfies the Laplace equation with the boundary condition

$$U_T(\tau, \xi = 0) = 0. \tag{3.6}$$

By virtue of (3.6) the coordinate time τ at the TRS (SRS) origin coincides with the proper time τ_0 of the observer.

Each reference system described above is restricted by some space domain the size of which will be indicated below. Transformation laws between reference systems generalize the Poincaré transformation of the special theory of relativity (we use here the terminology introduced by Misner *et al.* (1973); Poincaré transformation includes Lorentz transformation, translation of the origin and spatial rotation). Specific transformations and expressions for the potentials $V(u, \mathbf{w})$ and $U_T(\tau, \xi)$ are found by matching the corresponding metric tensors with (2.1). In the next section we describe the reference systems and their relationships in detail.

4. Reference Systems

4.1. BRS (BARYCENTRIC REFERENCE SYSTEM)

Coordinates: $x^\alpha = (ct, x^i) = (ct, \mathbf{x})$.

Metric: $$\mathrm{d}s^2 = (-1 + c^{-2} 2U(t, \mathbf{x}))c^2\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + O(c^{-2}). \tag{4.1}$$

Potential: $$U(t, \mathbf{x}) = \sum_A \frac{GM_A}{R_A} + O(R_A^{-3}). \tag{4.2}$$

Capital letter A enumerates here the gravitating bodies (Sun, Earth, Moon and planets); $R_A = (R_A^i R_A^i)^{1/2}$; $R_A^i = x^i - x_A^i(t)$; $x_A^i(t)$ are the spatial coordinates of the centre of mass of body A; M_A is the constant mass of body A. The BRS origin coincides with the Solar system barycentre, i.e. $\Sigma_A M_A x_A^i(t) = 0 + O(c^{-2})$. By its construction BRS covers the finite spatial domain inside the near zone of the Solar system. The near zone boundary is determined by the length of gravitational waves radiated from the Solar system (Fock, 1959; Damour, 1987).

4.2. GRS (GEOCENTRIC REFERENCE SYSTEM)

Coordinates: $w^\alpha = (cu, w^i) = (cu, \mathbf{w})$.

Metric: $$\mathrm{d}s^2 = (-1 + c^{-2} 2U(u, \mathbf{w}))c^2\,\mathrm{d}u^2 + \mathrm{d}\mathbf{w}^2 + 0(c^{-2}). \tag{4.3}$$

Potential: $U(u, \mathbf{w}) = U_E(u, \mathbf{w}) + V(u, \mathbf{w})$,

$$U_E(u, \mathbf{w}) = \frac{GM_E}{r} + \frac{1}{2r^3} G I_E^{ij} \left(-\delta_{ij} + \frac{3}{r^2} w^i w^j \right) + O(r^{-5}), \tag{4.4}$$

$$V(u, \mathbf{w}) = Q_i^{(E)} w^i + \tfrac{3}{2} Q_{ij}^{(E)} w^i w^j + \tfrac{5}{2} Q_{ijk}^{(E)} w^i w^j w^k + O(r^4). \tag{4.5}$$

Capital letter E here denotes the Earth; $r = (w^i w^i)^{1/2}$; M_E is the mass of the Earth; I_E^{ij} are quadrupole moments of the Earth. Functions $Q_{ij}^{(E)}, Q_{ijk}^{(E)}, \ldots$ characterize the tidal gravitational field of the external messes and are found by matching BRS and GRS in the vicinity of the Earth as will be developed in the following section:

$$Q_{ij}^{(E)} = \tfrac{1}{3} \bar{U}_{,ij}(\mathbf{x}_E) + O(c^{-2}), \qquad Q_{ijk}^{(E)} = \tfrac{1}{15} \bar{U}_{,ijk}(\mathbf{x}_E) + O(c^{-2}), \tag{4.6}$$

$$\bar{U}(\mathbf{x}) = \sum_{A \neq E} \frac{GM_A}{R_A} + O(R_A^{-3}), \tag{4.7}$$

the comma denoting differentiation with respect to BRS spatial coordinates. The quantity $Q_i^{(E)}$ (of the order 3×10^{-9} cm/s^2 for the Earth) characterizes the deviation of the world line of the Earth's centre of mass from the geodesic and is found from the condition of coincidence of the GRS origin and the centre of mass of the Earth for any moment of time (Kopejkin, 1988; Brumberg and Kopejkin, 1989a)

$$Q_i^{(E)} = -\tfrac{1}{2} M_E^{-1} I_E^{jk} \bar{U}_{,ijk}(\mathbf{x}_E) + \ldots \tag{4.8}$$

with dots denoting Newtonian and relativistic terms of the higher order of smallness. The main contribution to $Q_i^{(E)}$ is given by interaction of the Earth's quadrupole moments and the tidal gravitational field of the Moon. Under condition (4.8) the dipole moment of the Earth and its time derivatives vanish in expansion (4.4) of the potential $U_E(u, \mathbf{w})$.

By its construction GRS is valid in the space domain bounded by the orbit of the Moon. Extension of GRS beyond the lunar orbit may be performed using the coordinate transformation between BRS and GRS given in the next section.

4.3. TRS (TOPOCENTRIC REFERENCE SYSTEM)

Coordinates: $\xi^\alpha = (c\tau, \xi^i) = (c\tau, \boldsymbol{\xi})$

Metric: $$ds^2 = (-1 + c^{-2} 2U_T(\tau, \boldsymbol{\xi}))c^2 \, d\tau^2 + d\boldsymbol{\xi}^2 + O(c^{-2}) \tag{4.9}$$

Potential: $$U_T(\tau, \boldsymbol{\xi}) = E_i \xi^i + \tfrac{3}{2} E_{ij} \xi^i \xi^j + \tfrac{5}{2} E_{ijk} \xi^i \xi^j \xi^k + O(\xi^4). \tag{4.10}$$

Here the quantity E_i characterizes the curvature of the world line of a terrestrial observer and is equal numerically to the force of gravity which may be measured by the gravimeter. The expression for E_i is found by matching metrices (4.3) and (4.9) as revealed in the next section:

$$E_i = U_{E,i}(\mathbf{w}_T) - a_T^i + Q_i^{(E)} + 3Q_{ij}^{(E)} w_T^j + \tfrac{15}{2} Q_{ijk}^{(E)} w_T^j w_T^k + O(w_T^3) + O(c^{-2}), \tag{4.11}$$

$$a_T^i = (\dot{\boldsymbol{\omega}}_E \times \mathbf{w}_T)^i + (\boldsymbol{\omega}_E \times \mathbf{v}_T)^i + (\boldsymbol{\omega}_E \times \mathbf{v}_{TT})^i + a_{TT}^i \tag{4.12}$$

$$v_T^i = (\boldsymbol{\omega}_E \times \mathbf{w}_T)^i + v_{TT}^i. \tag{4.13}$$

A dot denotes differentiation with respect to time u. A comma denotes differentiation with respect to GRS coordinates w^i. w_T^i, $v_T^i = \dot{w}_T^i$, and $a_T^i = \dot{v}_T^i$ are GRS coordinates, velocity and acceleration of the observer. $\boldsymbol{\omega}_E$ is the GRS instant angular rotation velocity of the Earth. v^i_{TT} and $a^i_{TT} = \dot{v}^i_{TT}$ are the velocity and acceleration of the observer with respect to the reference system GRS$^+$ having its origin at the geocentre and rigidly rotating with angular velocity $\boldsymbol{\omega}_E$ with respect to GRS. The coordinates of GRS$^+$ are not harmonic. The quantities $E_{ij}, E_{ijk}, \ldots$ characterize the tidal gravitational field of the Earth and the external masses and are found by matching metrices (4.3) and (4.9)

$$E_{ij} = \tfrac{1}{3} U_{E,ij}(\mathbf{w}_T) + Q^{(E)}_{ij} + 5Q^{(E)}_{ijk} w^k_T + O(w^2_T) + O(c^{-2}), \tag{4.14}$$

$$E_{ijk} = \tfrac{1}{15} U_{E,ijk}(\mathbf{w}_T) + Q^{(E)}_{ijk} + O(\mathbf{w}_T) + O(c^{-2}). \tag{4.15}$$

By its construction TRS covers only rather a small space domain in the vicinity of the observer's world line, where expansion (4.10) is valid. But this system may be prolonged over greater distances using the transformation laws between GRS and TRS given in the next section.

4.4. SRS (SATELLITE REFERENCE SYSTEM)

Coordinates: $\xi^\alpha = (c\tau, \xi^i) = (c\tau, \boldsymbol{\xi})$

Metric: $\mathrm{d}s^2 = (-1 + c^{-2} 2U_T(\tau, \boldsymbol{\xi}))c^2\,\mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{\xi}^2 + O(c^{-2})$ (4.16)

Potential: $U_T(\tau, \boldsymbol{\xi}) = \tfrac{3}{2} E_{ij}\xi^i\xi^j + \tfrac{5}{2} E_{ijk}\xi^i\xi^j\xi^k + O(\xi^4)$. (4.17)

SRS differs from TRS in the absence of the acceleration E_i in the potential (4.17). This is a consequence of our assumption that the satellite is in free fall and moves along the geodesic. The quantities $E_{ij}, E_{ijk}, \ldots$ are expressed by (4.14), (4.15) where w^i_T now denote the GRS coordinates of the satellite.

5. Coordinate Transformations

5.1. TRANSFORMATION BETWEEN BRS AND GRS

This transformation is found in the form

$$u = t - c^{-2}(A(t) + v^k_E R^k_E) + O(c^{-4}), \tag{5.1}$$

$$w^i = R^i_E + O(c^{-2}) \tag{5.2}$$

where $R^i_E = x^i - x^i_E(t)$, with $x^i_E(t), v^i_E(t) = \mathrm{d}x^i_E/\mathrm{d}t$ being BRS coordinates and the velocity of the centre of mass of the Earth, respectively. Substituting (5.1) and (5.2) with metric tensors (4.1) and (4.3) into (2.1) and expanding the external potential in the vicinity of x^i_E yields

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \tfrac{1}{2} v^2_E + \bar{U}(\mathbf{x}_E), \tag{5.3}$$

$$a^i_E = \frac{\mathrm{d}v^i_E}{\mathrm{d}t} = \bar{U}_{,i}(\mathbf{x}_E) - Q^{(E)}_i + O(c^{-2}). \tag{5.4}$$

Besides, one obtains relations (4.6) for $Q^{(E)}_{ij}, Q^{(E)}_{ijk}, \ldots$. Equation (5.3) gives the differential relationship between the BRS and GRS coordinate times. BRS equations of the centre of mass of the Earth result from (5.4). Therefore, the matching technique enables us to derive both the coordinate transformations and the equations of motion of celestial bodies. In more detail these questions are considered in Kopejkin (1988) and Brumberg and Kopejkin (1989a, b).

5.2. TRANSFORMATION BETWEEN GRS AND TRS (SRS)

This transformation is found in a form similar to (5.1), (5.2)

$$\tau = u - c^{-2}(S(u) + v_T^k r_T^k) + O(c^{-4}), \tag{5.5}$$

$$\xi^i = r_T^i + O(c^{-2}) \tag{5.6}$$

with $r_T^i = w^i - w_T^i$. Substituting (5.5) and (5.6) with metric tensors (4.3) and (4.9) into (2.1) and expanding the potential of (4.3) in the vicinity of w_T^i yields

$$\frac{\mathrm{d}S}{\mathrm{d}u} = \tfrac{1}{2}v_T^2 + U_E(\mathbf{w}_T) + Q_i^{(E)} w_T^i + \tfrac{3}{2}Q_{ij}^{(E)} w_T^i w_T^j + \tfrac{5}{2}Q_{ijk}^{(E)} w_T^i w_T^j w_T^k + O(w_T^4), \tag{5.7}$$

$$a_T^i = \frac{\mathrm{d}v_T^i}{\mathrm{d}u} = U_{E,i}(\mathbf{w}_T) - E_i + Q_i^{(E)} + 3Q_{ij}^{(E)} w_T^j + \tfrac{15}{2}Q_{ijk}^{(E)} w_T^j w_T^k + O(w_T^3) + O(c^{-2}). \tag{5.8}$$

Besides, one obtains expressions (4.14), (4.15) for $E_{ij}, E_{ijk}, \ldots$. Equation (5.7) gives the differential relationship between the GRS and TRS (SRS) coordinate times. On the basis of (5.8) one obtains either relation (4.11) for the force gravity acceleration E_i of the ground observer (TRS) or the GRS equations of satellite motion ($E_i = 0$, SRS). Matching of GRS and TRS (SRS) taking account of post-Newtonian corrections is presented in Kopejkin (1989a, b). Relativistic corrections for the GRS equations of satellite motion have been derived in Brumberg and Kopejkin (1989b, c).

6. Realization of Time Scales

The scales of TB and TT are related with the corresponding BRS and GRS coordinate times by formulae

$$\mathrm{TB} = k_B t, \qquad \mathrm{TT} = k_E u, \tag{6.1}$$

k_B and k_E are constant factors to be chosen so as to minimize the differences between TB, TT and TAI.

For the geocentre where $R_E^i = 0$ the rate of the time u is determined only by Equation (5.3). This equation has been investigated by many authors. The first detailed solution was given by Moyer (1981). This solution was based on the substitution of the Keplerian motion for the Solar system bodies into (5.3) and provided an accuracy of about 20 μs. At present, the most accurate analytical theories of motion of the major planets and the Moon are the VSOP82/ELP2000 theories. Substituting these theories into (5.3) with subsequent integration was performed independently in Fairhead *et al.* (1987) and Hirayma *et al.* (1988). It should be noted that the BRS velocity of the Earth v_E^k corresponding to these theories may now be found in Soma *et al.* (1987). Therefore there is no longer any necessity to perform transformations of the barycentric motion of the Earth to the heliocentric motion of the Earth, the geocentric motion of the Moon and the heliocentric motion of the Earth – Moon centre of mass as was used by Moyer. In the VSOP82 theory the rectangular coordinates and the velocity components of the planets are represented by trigonometric series in the mean longitudes of the planets, the coefficients of the series being slowly changing polynomial functions of time. Hence, the solution of (5.3) is represented in the same form, as follows

$$A = A_0 + B_0 t + C_0 t^2 + D_0 t^3 + \cdots + \sum_i (A_i + B_i t + C_i t^2 + D_i t^3 + \cdots) \times \sin(\omega_i t + \varphi_i). \quad (6.2)$$

Numerical values of the coefficients obtained by the authors cited above confer an accuracy of several nanoseconds. Coefficients $C_0, D_0, \ldots, B_i, C_i, D_i, \ldots$ are caused by the secular terms of the planetary theories and vanish for pure Keplerian motion. These coefficients would also vanish when using a purely trigonometric planetary theory. But such a theory leads to a drastic extension of the number of periodic terms with almost equal periods so that its actual use may result in a loss of accuracy. For astronomical purposes it is convenient to have the time scales differing from one another only by periodic terms. The linear function $A_0 + B_0 t$ in (6.2) does not interfere with this since the constant A_0 may be removed due to the choice of initial moment (which will be assumed further) and the constant B_0 is removed by adopting different units of measurement of different time scales. This is allowed for by the IAU resolution of 1976 adopted in Grenoble (IAU, 1977).

Equation (5.3) may also be integrated numerically by using the numerical theories DE 200/LE 200 in the right-hand side. Such integration is described in Hellings (1986) and Backer and Hellings (1986). When numerically integrating (5.3) the analytical structure of (6.2) is latent but the numerically determined secular trend of function $A(t)$ depends on the interval of integration.

In any case the function A is represented in the form

$$A(t) = A^* t + A_p(t) \quad (6.3)$$

where A^* is constant and $A_p(t)$ includes both the periodic terms and the non-periodic terms caused by the secular evolution of the planetary orbits. Substitution of (6.3) into (5.1) results in

$$u = (1 - c^{-2} A^*)t - c^{-2}(A_p(t) + v_E^k R_E^k) + O(c^{-4}). \quad (6.4)$$

Using (6.1) one finds

$$k_E^{-1}\mathrm{TT} = k_B^{-1}(1 - c^{-2}A^*)\mathrm{TB} - c^{-2}(A_p(t) + v_E^k R_E^k) + O(c^{-4}). \tag{6.5}$$

The choice of constants k_B and k_E is at our disposal. Setting their ratio to

$$k_B/k_E = 1 - c^{-2}A^* \tag{6.6}$$

and considering that both k_B and k_E differ from 1 by the terms $O(c^{-2})$ one gets

$$\mathrm{TB} = \mathrm{TT} + c^{-2}(A_p(t) + v_E^k R_E^k) + O(c^{-4}). \tag{6.7}$$

The constant A^* is not necessarily equal to the coefficient B_0 in (6.2). Representation (6.2) itself is rather conventional. In using (6.2) over the restricted interval of time the terms of very long period actually manifest themselves as secular terms and in numerical averaging are included in the secular part. Therefore, the choice of A^* should be the subject of agreement, intending to provide a minimal influence of $A_p(t)$ on the secular rate of the difference TB − TT over some definite time interval. After completing this interval the value of A^* may be changed so as to compensate the accumulated secular rate in TB − TT. The order of magnitude is $c^{-2}A^* = 1.48 \times 10^{-8}$.

Consider now the relationship (5.5) between the GRS and TRS coordinate times. Let y_T^i be the GRS$^+$ spatial coordinates of the observer related to his GRS coordinates w_T^i by formulae

$$y_T^i(u) = P_{ik}(u)\, w_T^k(u), \qquad \mathrm{d}P_{ik}/\mathrm{d}u = \varepsilon_{imj}\Omega_E^j P_{mk} \tag{6.8}$$

with P_{ik} being an orthogonal rotation matrix resulting from multiplication of the matrices of precession, nutation and diurnal rotation of the Earth. Ω_E^i are GRS$^+$ components of the instant angular rotational velocity of the Earth. It is important that our GRS is dynamically non-rotating whereas most authors use a kinematically non-rotating geocentric reference system. The angular rotational velocities of the Earth referred to these systems differ from one another by the magnitude of geodesic precession. Therefore, in our approach, matrix P_{ik} does not include the geodesic precession which occurs explicitly in the transformation between BRS and GRS (Brumberg and Kopejkin, 1989a, b; Kopejkin, 1989a, b). Taking this fact into account one may find the expression for matrix $P_{ik}(u)$, for example, in Moritz and Mueller (1987). Using (4.13) one may transform (5.7) in coordinates y_T^i as follows:

$$\frac{\mathrm{d}S}{\mathrm{d}u} = \varepsilon_{ijk}\Omega_E^j y_T^k \dot{y}_T^i + \tfrac{1}{2}\dot{y}_T^k \dot{y}_T^k + W_E(\mathbf{y}_T) + V_2(\mathrm{u}, \mathbf{y}_T) + O(y_T^3) \tag{6.9}$$

where

$$W_E(\mathbf{y}_T) = \tfrac{1}{2}(\mathbf{\Omega}_E \times \mathbf{y}_T)^2 + U_E(\mathbf{y}_T) \tag{6.10}$$

represents the potential of the force of gravity due to the Earth at the point $\mathbf{y}_T(u)$ on the surface of the Earth. The last term in (6.9) describes the tidal quadrupole external disturbing potential

$$V_2(u, \mathbf{y}_T) = \tfrac{1}{2}\bar{U}_{,km}(\mathbf{x}_E) P_{ik} P_{jm} y_T^i y_T^j. \tag{6.11}$$

This tidal perturbation from the Sun and the Moon gives contribution into $(d\tau/du)$ of the order of 10^{-17}. The terms with the acceleration $Q_i^{(E)}$ and the higher multipoles occuring in (5.7) have been ignored in (6.9). Our aim is to reveal in the right-hand side of (6.9) the constant part independent of time and coordinates $\mathbf{y}_T$ of a ground station. Introducing the geocentric spherical coordinates such as radius vector r, longitude l, latitude ψ and choosing the direction of y^3 along the instant axis of rotation of the Earth one has $\Omega_E^1 = \Omega_E^2 = 0, \Omega_E^3 = \Omega$ and

$$W_E(r, l, \psi) = \tfrac{1}{2}\Omega^2 r^2 \cos^2\psi + U_E(\mathbf{y}_T). \tag{6.12}$$

The tidal quadrupole external potential may be easily expanded in spherical harmonics (Moritz and Mueller, 1987). The zonal part of this expansion will contain the time independent terms

$$\bar{V}_{20}(r, \psi) = -\tfrac{1}{4}r^2(3\sin^2\psi - 1)\sum_{A\neq E}\frac{GM_A}{\bar{R}_{EA}} \tag{6.13}$$

with $\bar{R}_{EA}$ denoting the mean constant distance between the Earth and disturbing body A (the Sun or the Moon). It is suitable to add the value (6.13) to the potential of the terrestrial force of gravity and to consider the potential

$$W(r, l, \psi) = W_E(r, l, \psi) + \bar{V}_{20}(r, \psi) \tag{6.14}$$

although the term $\bar{V}_{20}$ is usually treated as perturbation (Moritz, 1980) and is not included in the potential of the force of gravity.

The surface $r = r(l, \psi)$, closely approximating the mean sea level and providing a constant value for the force gravity potential

$$W(r, l, \psi) = W_0 = \text{const}, \tag{6.15}$$

is called the geoid. Nowadays, there are attempts to define the geoid within the framework of GRT (Soffel *et al.*, 1987) but for our purposes the Newtonian definition (6.15) is quite satisfactory. Solution of (6.15) for $r = r(l, \psi)$ with the left-hand side (6.14) may be performed using the technique given by Zhongolovich (1957). It is not difficult to replace for astronomical applications the geocentric spherical coordinates by the astromical longitude λ, the astronomical latitude φ and the height h of the observer above the geoid. There results

$$W(h, \lambda, \varphi) = W_0 - g(\varphi, \lambda)h + O(h^2). \tag{6.16}$$

The value of the force of gravity $g(\varphi, \lambda)$ on the geoid may be found by the known formulae (Zhongolovich, 1957).

Hence, the right-hand side of (6.9) is transformed to the form

$$\frac{dS}{du} = \varepsilon_{ijk}\Omega_E^j y_T^k \dot{y}_T^i + \tfrac{1}{2}\dot{y}_T^k \dot{y}_T^k + W_0 + V_2(u, \mathbf{y}_T) - \bar{V}_{20} - g(\varphi, \lambda)h + {}$$
$$+ O(y_T^3) + O(h^2) \tag{6.17}$$

h, λ, φ as well as $\mathbf{y}_T$ being the coordinates of the ground station. The solution of this equation is found in the form

$$S(u) = S^* u + S_p(u) \tag{6.18}$$

where S^* takes the same constant value for all possible TRS. Being independent of the position of a ground station this constant should be

$$S^* = W_0. \tag{6.19}$$

The function $S_p(u)$ contains all the remaining terms resulting from integrating (6.17). The term $-g(\varphi, \lambda)h$ leads to the linear function of u dependent on the ground station coordinates. The contribution from $V_2 - \bar{V}_{20}$ contains the periodic tidal terms caused by the Moon and the Sun. Finally, the terms with $\dot{y}^i_T = P_{ik} v^k_{TT}$ give the contribution due to the geophysical factors. As a result, Equation (5.5) leads to the following relationship between τ and u

$$\tau = (1 - c^{-2} S^*)u - c^{-2}(S_p(u) + v^k_T r^k_T) + O(c^{-4}). \tag{6.20}$$

Putting

$$k_E = 1 - c^{-2} S^* = 1 - 0.7 \times 10^{-9} \tag{6.21}$$

one obtains

$$\tau^{(n)} = \mathrm{TT} - c^{-2}(S_p(u) + v^k_T r^k_T)_n + O(c^{-4}) \tag{6.22}$$

where subscript n means that the corresponding quantity refers to the observer n. As mentioned earlier the TRS coordinate time $\tau^{(n)}$ coincides under $r^i_T = 0$ with the proper time $\tau^{(n)}_0$ of the observer n. This proper time may be measured with the atomic standard of frequency located at the TRS (SRS) origin. Actual reading T_n of the standard is related to the proper time $\tau^{(n)}_0$ by equation

$$\Delta_n(\tau^{(n)}_0) = \tau^{(n)}_0 - T_n \tag{6.23}$$

$\Delta_n(\tau^{(n)}_0)$ is the clock correction for the given standard n. To determine correction $\Delta_n(\tau^{(n)}_0)$ one applies the TAI time scale.

TAI is formed (Guinot, 1986, 1988) by a specific procedure of averaging of readings of all available standards which are supposed to be subjected to coordinate synchronization. Readings of each standard are reduced to coordinate time TT by taking account of the terms depending on geophysical velocity of a frequency standard, its height above the geoid and tidal perturbations

$$\mathrm{TAI} = \mathrm{mean}(T_n + c^{-2}(S_p(u))_n). \tag{6.24}$$

Readings of the atomic standard n are related to TAI by correction

$$E_n(\mathrm{TAI}) = \mathrm{TAI} - T_n - c^{-2}(S_p(\mathrm{TAI}))_n. \tag{6.25}$$

In principle, corrections $\Delta_n(\tau^{(n)}_0)$ and E_n are of different physical origin. But within the best presently attainable relative accuracy of 10^{-14} these corrections are coincident

(Guinot, 1986; Guinot and Seidelmann, 1988). Then from (6.23) and (6.25) it follows

$$\tau_0^{(n)} = \mathrm{TAI} - c^{-2}(S_p(\mathrm{TAI}))_n. \tag{6.26}$$

Substitution of (6.26) into (6.22) yields the relationship between TAI and TT

$$\mathrm{TT} = \mathrm{TAI} + 32.184\ \mathrm{s} + O(c^{-4}). \tag{6.27}$$

The constant shift 32.184 s is due to the historical reasons to prevent the discontinuity between atomic and ephemeris time scales.

Discussion of the procedure for obtaining TAI makes it evident that TAI presents the physical realization of coordinate time scale TT resulting from the application of the rigorous mathematical theory to actual measurements. Therefore, from the theoretical point of view it is reasonable to define TAI as coordinate time TT kept by ideal clocks situated at the points of observation on the physical surface of the Earth and synchronized with respect to TT.

The relationship between TAI and TT in the form of (6.27) was obtained first only for the points of the actual Earth's surface. But it may be extended beyond this surface along a three-dimensional hypersurface of TT = const. This may be performed due to the fact that the visualized ideal clocks lying on the Earth's surface and keeping TAI belong to this hypersurface being synchronized with respect to TT. Hence, TAI is defined in the same region of space as TT but its practical realization is performed only at the points of distribution of the actual atomic standards on the Earth's surface. It may be possible in future to include clocks on Earth satellites within the procedure of forming TAI. The domain of practical realization of TAI will then be significantly extended.

Thus, the final relations for the relativistic time scales have the form (6.7), (6.27) and

$$\begin{aligned}\tau &= \mathrm{TAI} - c^{-2}(S_p(\mathrm{TAI}) + v_T^k r_T^k) + O(c^{-4}) \\ &= \mathrm{TT} - 32.184\ \mathrm{s} - c^{-2}(S_p\mathrm{TT}) + v_T^k r_T^k) + O(c^{-4}) \\ &= \mathrm{TB} - 32.184\ \mathrm{s} - c^{-2}(A_p(\mathrm{TB}) + v_E^k R_E^k) - c^{-2}(S_p(\mathrm{TT}) + \\ &\quad + v_T^k r_T^k) + O(c^{-4}).\end{aligned} \tag{6.28}$$

In using (6.7), (6.27) and (6.28) it should be remembered that the actual clock is located at the point with GRS coordinates w_T^i.

From the procedure for constructing reference systems one has as a result that TB is determined in the near zone of the Solar system. TT (TAI) and τ are valid in the space domains determined by the vanishing of the determinants of coordinate transformations (5.1), (5.2) and (5.5), (5.6), respectively.

7. Astronomical Units of Measurement of Time and Length

So far we have discussed the time scales mainly from the theoretical point of view and we have not gone into the practical aspects. We now turn briefly to one of the most important practical questions related to the astronomical units of measurement of time and length.

At present, it is acknowledged that the standard unit of time is the SI second determined in the TAI scale on the surface of the geoid in rotation. The theory developed here is consistent with this definition. The units of time for the scales TB, TT and τ are assumed to be equal to the TAI second.

The problem of definition of the BRS and GRS units of length is much more complicated. Indeed, in contrast to time intervals, angular distances and frequency rations, linear distances in BRS and GRS are not directly measurable quantities in astronomical observations. Therefore, the definition of the units of length is dictated mainly by considerations of conveniences in reducing calculational effort. From this point of view it seems to be reasonable to seek the fulfilment of two conditions commonly used (generally, in implicit form) in astronomical practice:

(1) the fundamental astronomical constants such as the velocity of light and the universal gravitational constant should have the same numerical values in any reference systems;
(2) the basic dynamical equations of motion described in BRS and GRS and referred to t or u as an independent argument should, where possible, retain an invariant form in exchanging t and u for TB and TT respectively.

Both these conditions may be satisfied with great accuracy provided that changing from t to TB in the BRS equations is accompanied by the transformation

$$x^{*i} = k_B x^i, \qquad (GM_A)^* = k_B GM_A. \tag{7.1}$$

Similarly, changing from t to TT in the GRS equations should be accompanied by the transformation

$$w'^i = k_E w^i, \qquad (GM_A)' = k_E GM_A. \tag{7.2}$$

From (6.1), (7.1) and (7.2) it follows that the units of length in BRS and GRS are equal. As the standard unit of length one may adopt the SI metre, defined as the product of the velocity of light and the SI second.

This approach to the astronomical units of time and length is based on Fukushima *et al.* (1986b) and Hellings (1986). It should be noted that the first of these papers also suggests an alternative approach without transforming the spatial BRS and GRS coordinates. This latter approach actually leads to the same results but, for astronomical problems in which units of length play a purely auxiliary role, it seems to be logically more complicated, implying different values of the fundamental constants (velocity of light, gravitational constant) in various reference systems.

In any case relations (7.1), (7.2) involve definite inconveniences. For instance, the BRS planetary equations involve the mass of the Moon and its coordinates. These quantities should be expressed in BRS units and will differ from the GRS quantities that are natural for the Moon. Quite similarly the GRS equations of motion of the Moon or an Earth satellite involve the mass and the coordinates of the Sun (with masses and coordinates of the planets for the lunar equations). These quantities should be expressed in GRS units and will differ from the BRS quantities that are adequate for the Sun and planets. These considerations are confirmed to some extent in Ries *et al.* (1988).

From the theoretical point of view the most reasonable solution of the problems under discussion might be the direct use as time scales and spatial coordinates of t, u and x^i, w^i, respectively. The units of time and length as well as the astronomical constants are thereby independent of the reference system employed. But in doing so the secular trend between all time scales will remain. For a long time it seemed 'natural' to remove the secular trend. But within the limits of modern accuracy it is impossible to satisfy this condition rigorously due to the presence of nonlinear and mixed terms in (6.2). Besides, as stated above, this involves inconveniences related to the necessity to redetermine the length-dependent quantities (7.1) and (7.2). It may be possible that nowadays a more reasonable solution is to retain the secular trend between the time scales, putting $k_B = k_E = 1$.

8. Comments

8.1 TB is not the proper time of the clock at rest in the Solar system's barycentre. Actually, according to (4.1) the interval $d\tau_0$ of the proper time of the clock at rest in the barycentre is related to the TB interval by means of

$$d\tau_0 = k_B^{-1}(1 - c^{-2}U(t, \mathbf{x} = 0))\,d(\mathrm{TB}) \neq k_B^{-1}\,d(\mathrm{TB}), \tag{8.1}$$

$U(t, \mathbf{x} = 0)$ being the time-dependent value of the gravitational potential of all gravitating masses evaluated in the solar system barycentre. This value cannot be ignored. It might be more correct to consider TB as coinciding with the proper time of an infinitely distant observer at rest in BRS since in virtue of (3.3) such as observer has $d\tau_0 = k_B^{-1}\,d(\mathrm{TB})$. But this definition is not quite adequate since BRS is mathematically determined not for the entire infinite space but only within the boundary of the near zone of the Solar system (Damour, 1987). The most consistent and mathematically rigorous definition of TB is its interpretation as the coordinate time of BRS characterized by metric (4.1). Such interpretation is free of any inconsistencies or ambiguities.

8.2 TT is not the proper time of the clock at rest in the geocentre. In fact, according to (4.3)–(4.5) the interval $d\tau_0$ of such a clock will be related to the TT interval by means of

$$d\tau_0 = k_E^{-1}(1 - c^{-2}U_E(u, \mathbf{w} = 0))\,d(\mathrm{TT}) \neq k_E^{-1}\,d(\mathrm{TT}), \tag{8.2}$$

$U_E(u, \mathbf{w} = 0)$ being the value of the Earth's gravitational potential at the geocentre. This quantity depends on time due to the distortion of the shape of the Earth's surface caused by the solar and lunar tidal gravitational field. It should be noted also that the value of U_E at the geocentre is not a directly measurable quantity, being strongly dependent on the choice of model of the Earth's internal structure. For these reasons it is not suitable to attempt to satisfy the relation $d\tau_0 = d(TT)$ by choosing a time-dependent factor $k_E = 1 - c^{-2} U_E(u, \mathbf{w} = 0)$. Thus the value of the Earth potential in (8.2) cannot be ignored. The most consistent and mathematically rigorous definition of TT is its interpretation as the coordinate time of GRS with metric (4.3).

8.3 TAI cannot be regarded as the proper time of any clocks. Such treatment of TAI would be in flat contradiction to the specific procedure for constructing this time scale. The main reason behind this conclusion is due to the consideration of the tidal potential V_2. Disregarding this potential one might treat TAI as the proper time of a fixed clock on the surface of the geoid as seen from (6.17) and (6.26). Such a treatment is admissible within the limits of present accuracy. But our aim is to develop a self-consistent theory of time scales independent of the accuracy achieved at any given time. Therefore, in accordance with the procedure of forming TAI, the most consistent and mathematically rigorous procedure is to define TAI as a time scale that is the physical realization of TT. This definition turns out to be in agreement with the definition given by CCIR (1982).

8.4 It may be suitable to introduce in the vicinity of each gravitating body P of the Solar system (the Sun, the Moon, any planet, etc.) its own time scale H_P similar to TT. For this purpose a reference system of the type (4.3) is constructed for the body P and its coordinate time H_P is related to the BRS coordinate time t by means of

$$H_P = t - c^{-2}(C_P(t) + v_P^k R_P^k) + O(c^{-4}), \tag{8.3}$$

$$\frac{dC_P}{dt} = \tfrac{1}{2} v_P^2 + \bar{U}_P(\mathbf{x}_P), \qquad \bar{U}_P(\mathbf{x}) = \sum_{A \neq P} \frac{GM_A}{R_A} + O(R_A^{-3}). \tag{8.4}$$

$x_P^k(t)$ and $v_P^k(t) = dx_P^k/dt$ are the BRS coordinates and velocity of the centre of mass of body P, $R_P^k = \mathrm{x}^k - \mathrm{x}_P^k(\mathrm{t})$. The time scales τ, TAI and TT may be related to H_P by means of BRS coordinate time t using (6.7), (6.27), (6.28) and (8.3).

8.5 In solving some astronomical problems it may be necessary to use time scales associated with the gravitating systems of a higher level such as the Galaxy, systems of galaxies, etc. It is possible, for example, to introduce galactic time scale T related to the BRS time t by formula

$$t = T - c^{-2}(D(T) + V_B^k B_B^k) + O(c^{-4}), \tag{8.5}$$

$$\frac{dD}{dT} = \tfrac{1}{2} V_B^2 + U_G(\mathbf{X}_B) \tag{8.6}$$

where $R_B^k = X^k - X_B^k(T)$, X^k are spatial coordinates of the galactic reference system, $X_B^k(T)$ and $V_B^k(T) = dX_B^k/dT$ are coordinates and velocity components of the Solar system's barycentre with respect to the centre of the Galaxy, $U_G(\mathbf{X}_B)$ is the gravitational potential of the Galaxy (excluding the Solar system's potential) evaluated in the Solar system's barycentre. The term with $D(T)$ remains practically constant over a long interval of time. For the Earth the term with R_B^k results in an annual sinusoidal curve with an amplitude of 0.22 s. Currently, galactic time T is not actually used. But for some problems as, for instance, pulsar timing, this time scale may be preferred over the BRS time.

8.6 Relationships between the time scales obtained here are universal and do not depend on a specific gravitational theory (Will, 1981) nor on the choice of coordinate conditions. This is explained by our use of the Newtonian approximation of GRT, which is valid for all kinds of gravitational theories. But the contributions to relations between TB, TT and τ resulting from different gravitational theories or different coordinate conditions are inevitable in relativistic corrections of higher order. Therefore, in constructing time scales it is necessary to indicate the type of coordinate conditions and the form of metric tensor. For example, in the post-Newtonian approximation of GRT the transformation law between BRS and GRS coordinate times has the form (Kopejkin, 1988; 1989a)

$$u = t - c^{-2}(A(t) + v_E^k R_E^k) + c^{-4}(B(t) + O(R_E)) + O(c^{-5}), \tag{8.7}$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = -\tfrac{1}{8}v_E^4 - \tfrac{3}{2}v_E^2 \bar{U}(\mathbf{x}_E) + 4v_E^k \bar{U}^k(\mathbf{x}_E) + \tfrac{1}{2}\bar{U}^2(\mathbf{x}_E) - \bar{W}(\mathbf{x}_E) \tag{8.8}$$

with

$$\bar{U}^i(\mathbf{x}) = \sum_{A \neq E} \frac{GM_A}{R_A} v_A^i + O(R_A^{-2}), \tag{8.9}$$

$$\bar{W}(\mathbf{x}) = \tfrac{3}{2} \sum_{A \neq E} \frac{GM_A}{R_A} v_A^2 - \sum_{A \neq E} \sum_{C \neq A} \frac{G^2 M_A M_C}{R_A R_{AC}} + $$

$$+ \tfrac{1}{2} \frac{\partial^2}{\partial t^2} \sum_{A \neq E} GM_A R_A + O(R_A^{-2}), \tag{8.10}$$

$$R_{AC} = (R_{AC}^k R_{AC}^k)^{1/2}, \qquad R_{AC}^k = x_A^k - x_C^k.$$

The term $B(t)$ yields corrections comparable in order of magnitude with the magnitude of the tidal terms in the transformation law between τ and TT.

8.7 Equations of motion of celestial bodies are always described in some specific reference system. The coordinate time employed in these equations as an independent argument is also called the dynamical time. Thus, from the theoretical point of view, the notions of coordinate and dynamical times are equivalent. But the practical realization of these time scales may be different. At present, BRS and GRS coordinate times are derived from the atomic time scale TAI with the aid of functional relations (6.7), (6.27) and (6.28). Therefore, TB and TT are actually reduced to atomic time. Its scale, given by the physical processes in the atoms of chemical elements, is proportional to the atomic constants (Planck constant, electron charge). Dynamical time may be obtained directly from observations of the Solar system bodies (ephemeris time) without having recourse to atomic clocks. In this case the dynamical time determined by the gravitation laws is proportional to the universal gravitational constant. Such dynamical time might be called gravitational time. So far we have tacitly assumed that atomic and gravitational times are equivalent. But this assumption is valid only if the fundamental constants of our world do not change over time. In a world with changing physical constants the equivalence of two time scales may be disturbed. In principle, this may be tested by permanently supporting and comparing both scales (Canuto and Goldman, 1982; Canuto *et al.*, 1984). For these reasons we have dropped the adjective 'dynamical' in the definitions of TB and TT as suggested by Guinot and Seidelmann (1988).

It is of interest that the pulsar time scale produced by the angular rotational velocity of pulsars is determined neither by the gravitation laws nor by the atomic processes. This scale depends essentially on the equation of state of matter in neutron stars. Hence, in a world with changing constants the pulsar time will not be equivalent to gravitational or atomic time scales. This may also be tested by astronomical observations of pulsars referred, for instance, to the atomic time scale (Counselman and Shapiro, 1968; Mansfield, 1976).

9. Recommendations

Based on the results and considerations presented above we should like to formulate our recommendations as follows:

(1) the relativistic time scales are referred to the specific four-dimensional reference systems: BRS, GRS, TRS and SRS;

(2) the reference systems indicated above are constructed by solving the Einstein field equations of GRT using harmonic coordinate conditions;

(3) the metric tensors of the appropriate reference systems present the solutions of the Einstein equations with physically adequate boundary conditions;

(4) all reference systems indicated above are dynamically non-rotating;

(5) the transformation to the rotating reference system is performed by introducing kinematically rigid rotation of the spatial axes of the corresponding dynamically non-rotating reference system without changing the time scale;

(6) the BRS origin is the Solar system's barycentre; the GRS origin is the geocentre; the TRS origin is a terrestrial observatory; the SRS origin is the centre of mass of an Earth satellite;

(7) the time scales are designated by TB, TT, TAI and τ; TB is the BRS coordinate time (up to constant factor k_B); TT is the GRS coordinate time (up to constant factor k_E); τ is the TRS (SRS) coordinate time coinciding with the proper time τ_0 of an observer at the TRS (SRS) origin; TAI is the idealization of the atomic time and presents the physical realization of the coordinate time TT; the functional relations between the time scales are given by Equations (6.7), (6.27) and (6.28); constant factors k_B and k_E should be adopted by the IAU;

(8) the unit of measurement of TB, TT, TAI and τ is the SI second on the surface of the geoid in rotation;

(9) the velocity of light and the gravitational constant are fundamental quantities which do not depend on the reference system employed;

(10) the unit of measurement of length in BRS and GRS is the SI metre on the surface of the geoid in rotation, determined as the product of the velocity of light and the SI second.

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