Gravitational field of one uniformly moving extended body and N arbitrarily moving pointlike bodies in post-Minkowskian approximation

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High precision astrometry, space missions and certain tests of General Relativity, require the knowledge of the metric tensor of the solar system, or more generally, of a gravitational system of N extended bodies. Presently, the metric of arbitrarily shaped, rotating, oscillating and arbitrarily moving N bodies of finite extension is only known for the case of slowly moving bodies in the post-Newtonian approximation, while the post-Minkowskian metric for arbitrarily moving celestial objects is known only for pointlike bodies with mass-monopoles and spin-dipoles.

As one more step towards the aim of a global metric for a system of N arbitrarily shaped and arbitrarily moving massive bodies in post-Minkowskian approximation, two central issues are on the scope of our investigation:

(i) We first consider one extended body with full multipole structure in uniform motion in some suitably chosen global reference system. For this problem a co-moving inertial system of coordinates can be introduced where the metric, outside the body, admits an expansion in terms of Damour-Iyer moments. A Poincaré transformation then yields the corresponding metric tensor in the global system in post-Minkowskian approximation.

(ii) It will be argued why the global metric, exact to post-Minkowskian order, can be obtained by means of an instantaneous Poincaré transformation for the case of pointlike mass-monopoles and spin-dipoles in arbitrary motion.

I. INTRODUCTION

Since exact solutions of Einstein's field equations are available only for highly idealized systems usually one is forced to resort to approximation schemes. One of the most powerful and most important approximation schemes is linearized gravity, where the field equations in harmonic coordinates are simplified to an inhomogeneous wave equation [1, 2]. As it has been shown in [3–5] in the post-Newtonian approximation (weak-field slow-motion approximation) the metric outside the matter distribution can be expanded in terms of two families of multipole moments: mass multipole moments M_L and spin multipole moments S_L. Later, in post-Minkowskian approximation (weak-field approximation) such a set of multipole moments has been introduced by *Damour* & *Iyer* [6].

For many purposes, for instance for high precision astrometry or fundamental tests of relativity, the knowledge of the global metric of an N-body system in post-Minkowskian approximation is of fundamental importance. Presently the post-Minkowskian metric for arbitrarily moving celestial objects is known only for pointlike bodies with mass-monopoles and spin-dipoles. The metric of arbitrarily shaped, rotating, oscillating and moving bodies is a highly sophisticated and complex problem and is only known for the case of slowly moving bodies in the post-Newtonian approximation [7]. One reason for this complexity is, that one might want to define the multipole moments of a single body in its own rest-frame, with origin close to the body's center of mass; however, if the acceleration of such a 'local' co-moving system is taken into account corresponding multipole moments have been defined only to post-Newtonian order [7, 8].

Thus, in order to study the global metric field in terms of locally defined multipoles of a realistic N-body system such as the solar system, one has to apply further approximations. Accordingly, this will be the strategy of this paper: we will first consider an arbitrarily shaped, rotating and oscillating body first in uniform motion, and then we treat the problem of N arbitrarily moving pointlike bodies with mass-monopoles and spin-dipoles.

The article is organized as follows: the metric for an extended body with arbitrary Damour-Iyer moments, defined in a co-moving system, in uniform motion is derived in section II in post-Minkowskian approximation. In section III we consider the post-Minkowskian metric for N arbitrarily moving pointlike bodies (mass-monopoles and spin-dipoles) and show that our results agree with corresponding results from the literature. Throughout the article we use fairly standard notation:

- G is the gravitational constant and c is the speed of light.
- Lower case Latin indices $a, b, ...$ take values $1, 2, 3$.
- Lower case Greek indices $\alpha, \beta, ...$ take values $0, 1, 2, 3$.
- repeated Greek indices mean Einstein summation from 0 to 3.
- $\delta_{ab} = \delta^{ab} = \delta_a^b = \text{diag}\,(+1, +1, +1)$ is the three-dimensional Kronecker delta.
- $\delta_{\alpha\beta} = \delta^{\alpha\beta} = \delta_\alpha^\beta = \text{diag}\,(+1, +1, +1, +1)$ is the four-dimensional Kronecker delta.
- ϵ_{abc} is the Levi-Civita symbol.
- L is a Cartesian multi-index, that means $L = a_1 \ldots a_l$.

- $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ is the Lorentz factor.
- parentheses surrounding a group of Roman indices mean symmetrization with respect to these indices: $A^{(ab)} = \frac{1}{2}\left(A^{ab} + A^{ba}\right)$.
- $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1)$ is the metric tensor of Minkowski space.

II. A UNIFORMLY MOVING BODY: MULTIPOLE EXPANSION TO POST-MINKOWSKIAN ORDER

A. Multipole expansion for a body at rest

Consider a single massive body in some inertial system of harmonic coordinates $X^\mu = (cT\,,\,\boldsymbol{X})$. For weak gravitational fields the metric differs only slightly from flat space metric, that means $G_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$, where $|H_{\mu\nu}| \ll 1$; the metric signature is $(-,+,+,+)$. Weak gravitational fields are governed by the equations of linearized gravity, in harmonic gauge given by [1] (from now on all relations will be valid to first order in G, even if this is not indicated explicitly):

$$\Box_X \overline{H}^{\mu\nu}\,(T,\boldsymbol{X}) = -\frac{16\,\pi\,G}{c^4}\,T^{\mu\nu}\,(T,\boldsymbol{X})\,, \tag{1}$$

where $\Box_X = \eta^{\mu\nu}\,\dfrac{\partial^2}{\partial X^\mu\,\partial X^\nu}$ is the d'Alembert operator, the stress-energy tensor of matter is $T^{\mu\nu}$, and $\overline{H}_{\mu\nu}$ is the trace-reversed metric perturbation; definitions and relations are given in Appendix A.

Damour & *Iyer* [6] succeeded to show that outside the body the metric in (skeletonized) harmonic gauge admits an expansion in terms of two families of multipole moments: mass-moments M_L and spin-moments S_L. Their canonical form of the metric perturbation in the exterior region of the matter field can be written as follows:

$$\begin{aligned}
H^{\mu\nu}_{\rm can}\,(T,\boldsymbol{X}) = &+\frac{2\,G}{c^2}\,\delta_{\mu\nu}\sum_{l\ge0}\frac{(-1)^l}{l!}\mathcal{D}_L\left[\frac{M_L\,(T_{\rm ret})}{R}\right]\\
&-\frac{8\,G}{c^3}\,\delta_{0\,(\mu}\,\delta_{\nu)\,i}\sum_{l\ge1}\frac{(-1)^l}{l!}\,\mathcal{D}_{L-1}\left[\frac{\dot{M}_{i\,L-1}\,(T_{\rm ret})}{R}\right]\\
&-\frac{8\,G}{c^3}\,\delta_{0\,(\mu}\,\delta_{\nu)\,k}\sum_{l\ge1}\frac{(-1)^l\;l}{(l+1)!}\,\epsilon_{i\,j\,k}\,\mathcal{D}_{i\,L-1}\left[\frac{S_{j\,L-1}\,(T_{\rm ret})}{R}\right]\\
&+\frac{4\,G}{c^4}\,\delta_{\mu i}\,\delta_{\nu j}\sum_{l\ge2}\frac{(-1)^l}{l!}\mathcal{D}_{L-2}\left[\frac{\ddot{M}_{ij\,L-2}\,(T_{\rm ret})}{R}\right]\\
&+\frac{8\,G}{c^4}\,\delta_{\mu i}\,\delta_{\nu j}\sum_{l\ge2}\frac{(-1)^l\;l}{(l+1)!}\,\mathcal{D}_{k\,L-2}\left[\frac{\epsilon_{km\,(\,i}\,\dot{S}_{j\,)\,m\,L-2}\,(T_{\rm ret})}{R}\right].
\end{aligned} \tag{2}$$

In Eq. (2), an overdot denotes the derivative with respect to $T_{\rm ret}$; e.g. $\dot{F}\left(T_{\rm ret}\right) = \dfrac{dF\left(T_{\rm ret}\right)}{dT_{\rm ret}}$ for any function F, and all multipole moments are taken at the retarded instance of time,

$$T_{\rm ret}\left(T, \boldsymbol{X}\right) = T - \frac{R}{c}\,, \tag{3}$$

with $R = |\boldsymbol{X}|$. The multipole moments, M_L and S_L, are Cartesian symmetric and trace-free (STF) tensors; $\mathcal{D}_L = \partial^l/(\partial X^{a_1}\,\partial X^{a_2}\,...\,\partial X^{a_l})$. Explicit expressions for the multipole moments, M_L and S_L, in post-Minkowskian approximation are given by Eqs. (5.33) - (5.35) in [6].

B. Multipole expansion for a uniformly moving body

Considering a single body in uniform motion we will now attach our inertial coordinates $X^{\mu} = (cT\,,\,\boldsymbol{X})$ to the body, by choosing its origin near the body's center of mass. The spatial coordinate $\boldsymbol{X}^{\rm CoM} = 0$ of center of mass can be defined by the vanishing of the corresponding Damour-Iyer mass-dipole moment $\boldsymbol{M}_a = 0$, but we consider the more general case with $\boldsymbol{M}_a \neq 0$ and $\boldsymbol{X}^{\rm CoM} \neq 0$. This coordinate system will be called co-moving in the following (or 'local' in case that the body's velocity is time dependent).

We now consider another inertial (global) system of coordinates $x^{\mu} = (ct\,,\,\boldsymbol{x})$ in which our body moves with constant velocity $\boldsymbol{v}$. The transformation from local coordinates $X^{\mu} = (cT, \boldsymbol{X})$ to global coordinates $x^{\mu} = (ct, \boldsymbol{x})$ for a massive body in uniform motion is given by a Poincaré transformation,

$$x^{\mu}\left(X^{\alpha}\right) = b^{\mu} + \Lambda^{\mu}_{\alpha}\,X^{\alpha}\,, \tag{4}$$

with $\Lambda^0_0 = \gamma$, $\Lambda^i_0 = \Lambda^0_i = \gamma\,\frac{v_i}{c}$, $\Lambda^j_i = \delta_{ij} + (\gamma - 1)\,\frac{v_i\,v_j}{v^2}$, and $b^{\mu} = (b^0, \boldsymbol{b})$ is a constant four-vector, where $\boldsymbol{b}$ points from the origin $\mathcal{O}$ of global frame to the origin of the co-moving frame at time $T = 0$. Transforming the events $(T, \boldsymbol{0})$ into the global reference system $(t, \boldsymbol{x})$ yields

$$\boldsymbol{x}_A\left(t\right) = \boldsymbol{x}_A\left(t_0\right) + \boldsymbol{v}\left(t - t_0\right), \quad x^i_A\left(t_0\right) = b^i\,, \quad c\,t_0 = b^0\,, \tag{5}$$

where $\boldsymbol{x}_A\left(t\right)$ points from the origin of the global system to the origin at the co-moving frame at any time t, and the initial is t_0. The distance R which appears in the co-moving metric (2), can be written in Lorentz invariant form ρ as (cf. Eq. (4.42) in [7], Eq. (10) in [9], Eq. (B.4) in [10])

$$\rho = \frac{\left|\eta_{\mu\nu}\,u^{\mu}\,\left(x^{\nu} - x^{\nu}_A\left(t_{\rm ret}\right)\right)\right|}{c}\,, \tag{6}$$

where $u^{\mu} = \gamma\left(c, \boldsymbol{v}\right)$ are the contravariant components of four-velocity of $\mathcal{O}$, and the retarded time is defined by Eq. (8) below. The Lorentz invariant distance (6) can also be written as:

$$\rho = \gamma\,\left(r\left(t_{\rm ret}\right) - \frac{\boldsymbol{v}\cdot\boldsymbol{r}\left(t_{\rm ret}\right)}{c}\right) = \sqrt{r^2\left(t\right) + \gamma^2\,\frac{\left(\boldsymbol{v}\cdot\boldsymbol{r}\left(t\right)\right)^2}{c^2}}\,, \tag{7}$$

where $\boldsymbol{r}\left(t_{\rm ret}\right) = \boldsymbol{x} - \boldsymbol{x}_A\left(t_{\rm ret}\right)$, $\boldsymbol{r}\left(t\right) = \boldsymbol{x} - \boldsymbol{x}_A\left(t\right)$ and $\boldsymbol{x}_A\left(t\right)$ is given by Eq. (5); the absolute values are $r\left(t_{\rm ret}\right) = \left|\boldsymbol{r}\left(t_{\rm ret}\right)\right|$ and $r\left(t\right) = \left|\boldsymbol{r}\left(t\right)\right|$.

The latter form in (7) is sometimes preferable and can be obtained by means of the relation $X^i = \Lambda^i_a \, r^a(t)$; for a very similar consideration see [10]. The retarded time in global coordinates reads for arbitrary wordlines

$$t_{\rm ret}\,(t, \boldsymbol{x}) = t - \frac{|\boldsymbol{x} - \boldsymbol{x}_A\,(t_{\rm ret})|}{c}\,. \tag{8}$$

Let us consider a series expansion of (8), which yields: $t_{\rm ret} = t - \dfrac{r\,(t)}{c} - \dfrac{\boldsymbol{v}\,(t)\cdot\boldsymbol{r}\,(t)}{c^2} + \mathcal{O}\left(c^{-3}\right)$, where $\boldsymbol{r}\,(t) = \boldsymbol{x} - \boldsymbol{x}_A\,(t)$, and $\boldsymbol{x}_A\,(t)$ is arbitrarily, hence $\boldsymbol{v}\,(t) = \dot{\boldsymbol{x}}_A\,(t)$ is time-dependent. In general, Eq. (8) is an implicit relation which cannot be resolved analytically for arbitrary worldlines $\boldsymbol{x}_A\,(t)$ of a massive body. However, for the case of a body in uniform motion one can obtain an exact analytical solution:

$$t_{\rm ret}\,(t, \boldsymbol{x}) = t - \gamma^2\,\frac{\boldsymbol{r}\,(t)\cdot\boldsymbol{v} + \sqrt{c^2\,r^2\,(t) - \left(\boldsymbol{r}\,(t)\times\boldsymbol{v}\right)^2}}{c^2}\,. \tag{9}$$

Here, $\boldsymbol{r}\,(t) = \boldsymbol{x} - \boldsymbol{x}_A\,(t)$, and $\boldsymbol{x}_A\,(t)$ is given by Eq. (5). Let us compare (9) with the post-Newtonian approximation. A series expansion of (9) yields the following expression for the retarded time: $t_{\rm ret} = t - \dfrac{r\,(t)}{c} - \dfrac{\boldsymbol{v}\cdot\boldsymbol{r}\,(t)}{c^2} + \mathcal{O}\left(c^{-3}\right)$. This expression agrees with the series expansion given above (for $\boldsymbol{v} = {\rm const}$) which has been obtained directly from the definition (8).

Now we consider a relation among the retarded time $T_{\rm ret}$ in the co-moving system of the body and the retarded time $t_{\rm ret}$ in the global system. The retarded time in the co-moving and global system are defined by Eqs. (3) and (8), respectively. In order to find a relation between $T_{\rm ret}$ and $t_{\rm ret}$, we note that the global coordinates of event $(t_{\rm ret}, \boldsymbol{x}_A\,(t_{\rm ret}))$ correspond to the coordinates $(T_{\rm ret}, \boldsymbol{0})$ of the same event in the co-moving frame. The Poincaré transformation of the coordinates of this event yields

$$T_{\rm ret} = \gamma^{-1}\,(t_{\rm ret} - t_0)\ . \tag{10}$$

Relation (10) can also be obtained directly from the definitions of $T_{\rm ret}$ and $t_{\rm ret}$.

To get the metric in the global system we will transform the spatial derivatives with respect to the co-moving coordinates to derivatives with respect to global coordinates. One obtains

$$\mathcal{D}_L\left[\frac{F\,(T_{\rm ret})}{R}\right] = \Lambda^{\mu_1}_{a_1}\ldots\Lambda^{\mu_l}_{a_l}\,\partial_{\mu_1\ldots\mu_l}\left[\frac{F\left(\gamma^{-1}\,(t_{\rm ret} - t_0)\right)}{\rho\,(t_{\rm ret})}\right]\,, \tag{11}$$

where F stands for any of the mass or spin multipoles in co-moving coordinates, and $\partial_\mu = \partial/\partial x^\mu$. By means of the invariant form of the distance (6) and with the aid of the derivative operation (11), we are in the position to obtain the global metric in terms of local multipoles for a massive body in uniform motion. Using $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}\,(G^2)$, $G^{\alpha\beta} = \eta^{\alpha\beta} - H^{\alpha\beta} + \mathcal{O}\,(G^2)$ and relation $\Lambda^\mu_\alpha\,\Lambda^\nu_\beta\,\eta^{\alpha\beta} = \eta_{\mu\nu}$, we obtain from (4) the transformation law of metric perturbation:

$$h^{\mu\nu}_{\rm can}\,(t, \boldsymbol{x}) = \Lambda^\mu_\alpha\,\Lambda^\nu_\beta\,H^{\alpha\beta}_{\rm can}\,(T, \boldsymbol{X})\,. \tag{12}$$

Applying the general transformation law (12) to the local metric (2), using the invariant form of the distance (6), the derivative operation (11), we obtain for the metric in global coordinates $(t, \boldsymbol{x})$ the following expression:

$$
\begin{aligned}
h^{\mu\nu}_{\rm can}\,(t,\boldsymbol{x}) = & \frac{2\,G}{c^2}\,\Lambda^{\mu}_{\alpha}\,\Lambda^{\nu}_{\beta}\,\delta_{\alpha\beta}\sum_{l\geq 0}\frac{(-1)^l}{l!}\,\Lambda^{\mu_1}_{a_1}\dots\Lambda^{\mu_l}_{a_l}\,\partial_{\mu_1\dots\mu_l}\left[\frac{M_{a_1\dots a_l}\,(T_{\rm ret})}{\rho}\right]\\
& - \frac{8\,G}{c^3}\,\Lambda^{(\mu}_{0}\,\Lambda^{\nu)}_{i}\sum_{l\geq 1}\frac{(-1)^l}{l!}\,\Lambda^{\mu_1}_{a_1}\dots\Lambda^{\mu_{l-1}}_{a_{l-1}}\,\partial_{\mu_1\dots\mu_{l-1}}\left[\frac{\dot{M}_{ia_1\dots a_{l-1}}\,(T_{\rm ret})}{\rho}\right]\\
& - \frac{8\,G}{c^3}\,\Lambda^{(\mu}_{0}\,\Lambda^{\nu)}_{k}\sum_{l\geq 1}\frac{(-1)^l\,l}{(l+1)!}\,\epsilon_{i\,j\,k}\,\Lambda^{\lambda}_{i}\,\Lambda^{\mu_1}_{a_1}\dots\Lambda^{\mu_{l-1}}_{a_{l-1}}\,\partial_{\lambda\mu_1\dots\mu_{l-1}}\left[\frac{S_{ja_1\dots a_{l-1}}\,(T_{\rm ret})}{\rho}\right]\\
& + \frac{4\,G}{c^4}\,\Lambda^{\mu}_{i}\,\Lambda^{\nu}_{j}\sum_{l\geq 2}\frac{(-1)^l}{l!}\,\Lambda^{\mu_1}_{a_1}\dots\Lambda^{\mu_{l-2}}_{a_{l-2}}\,\partial_{\mu_1\dots\mu_{l-2}}\left[\frac{\ddot{M}_{ija_1\dots a_{l-2}}\,(T_{\rm ret})}{\rho}\right]\\
& + \frac{8\,G}{c^4}\,\Lambda^{\mu}_{i}\,\Lambda^{\nu}_{j}\sum_{l\geq 2}\frac{(-1)^l\,l}{(l+1)!}\,\Lambda^{\lambda}_{k}\,\Lambda^{\mu_1}_{a_1}\dots\Lambda^{\mu_{l-2}}_{a_{l-2}}\,\partial_{\lambda\mu_1\dots\mu_{l-2}}\left[\frac{\epsilon_{km\,(\,i}\,\dot{S}_{j\,)\,m\,a_1\dots a_{l-2}}\,(T_{\rm ret})}{\rho}\right],
\end{aligned}
\tag{13}
$$

where $T_{\rm ret}$ can be expressed in terms of global coordinates by means of (10), and an overdot denotes the derivative with respect to $T_{\rm ret}$. The multipoles in (13) are the local multipoles defined in the co-moving frame of the body under consideration, and they are functions of the retarded time $T_{\rm ret}$. Expression (13) describes the metric of an arbitrarily shaped and arbitrarily oscillating and rotating single massive body in uniform motion.

C. Monopole in uniform motion

Let us consider the simplest case of an extended body with monopole structure. According to Eq. (13), the metric perturbation of a uniformely moving mass-monopole in global coordinates $x^{\mu} = (ct, \boldsymbol{x})$ is given by ($l = 0$ in Eq. (13)):

$$
h^{\mu\nu}_{(M)}\,(t,\boldsymbol{x}) = \frac{2\,G\,M}{c^2}\,\frac{\Lambda^{\mu}_{\alpha}\,\Lambda^{\nu}_{\beta}\,\delta_{\alpha\beta}}{\rho}\,,
\tag{14}
$$

where M is the invariant rest mass of the body. For the invariant distance ρ we insert expression (7), then we use the relation $c^2\,\Lambda^{\mu}_{\alpha}\,\Lambda^{\nu}_{\beta}\,\delta_{\alpha\beta} = 2\,u^{\mu}\,u^{\nu} + c^2\,\eta^{\mu\nu}$, and obtain

$$
h^{\mu\nu}_{(M)}\,(t,\boldsymbol{x}) = \frac{4\,G\,M}{c^2}\,\frac{1}{\gamma\left(r\,(t_{\rm ret}) - \dfrac{\boldsymbol{v}\cdot\boldsymbol{r}\,(t_{\rm ret})}{c}\right)}\left(\frac{u^{\mu}}{c}\frac{u^{\nu}}{c} + \frac{1}{2}\,\eta^{\mu\nu}\right).
\tag{15}
$$

This expression coincides with the metric of a pointlike body of mass M, cf. Eq. (15) with Eq. (11) in [11] for the case of uniform motion, i.e. $\boldsymbol{v} = {\rm const}$.

D. Spin-dipole in uniform motion

Next we consider an extended massive body with mass monopole M and spin dipole S_i. According to Eq. (13), the metric perturbation of a uniformely moving mass-monopole in global coordinates $x^\mu = (ct, \boldsymbol{x})$ consists of two contributions ($l = 1$ in Eq. (13)):

$$h^{\mu\nu}(t, \boldsymbol{x}) = h^{\mu\nu}_{(M)}(t, \boldsymbol{x}) + h^{\mu\nu}_{(S)}(t, \boldsymbol{x}), \tag{16}$$

where for simplicity we assume in this section that the co-moving system is located at the center of mass of this body, so that $M_a = 0$. The monopole part $h^{\mu\nu}_{(M)}$ is given by Eq. (15), and the spin part $h^{\mu\nu}_{(S)}$, according to (13), is given by

$$h^{\mu\nu}_{(S)}(t, \boldsymbol{x}) = \frac{4\,G}{c^3}\,\Lambda^{(\mu}_0\,\Lambda^{\nu)}_k\,\epsilon_{i\,j\,k}\,\Lambda^\lambda_i\,\partial_\lambda\,\frac{S_j}{\rho}\,, \tag{17}$$

where S_j is the spin in the local frame $(cT, \boldsymbol{X})$ of the body.

The massive bodies of an N-body system exert a torque on each other leading to a time dependent spin of a body A in the local A-system. Here, we follow the arguments of [12, 13] and will assume that such a local time-dependence is only caused by gravitational interactions and, therefore, are proportional to $\mathcal{O}(G)$. Accordingly, the spin of each individual body in its own co-moving system is here assumed to be time independent. The metric of an arbitrarily moving pointlike body with monopole structure and a time-independent spin has been given by Eq. (16) in [11]. Here, we will compare our result with the results in [11] in case of a body in uniform motion.

Because the spin is time-independent in the local frame, the derivative ∂_λ in (17) does not act on the spin vector, and we obtain

$$\Lambda^\alpha_i\,\partial_\alpha\,\frac{S_j}{\rho} = -S_j\,\frac{r^i(t) + (\gamma - 1)\,\frac{v^i}{v^2}\,(\boldsymbol{v}\cdot\boldsymbol{r}(t))}{\rho^3}\,. \tag{18}$$

In order to obtain (18), we have used the second expression in (7), the explicit form for the Lorentz matrices, and $\frac{\partial}{\partial t}\boldsymbol{r}(t) = -\boldsymbol{v}$. By inserting (18) into (17) we obtain

$$h^{\mu\nu}_{(S)}(t, \boldsymbol{x}) = -\frac{4\,G}{c^3}\,\Lambda^{(\mu}_0\,\Lambda^{\nu)}_k\,\epsilon_{i\,j\,k}\;S_j\,\frac{r^i(t) + (\gamma - 1)\,\frac{v^i}{v^2}\,(\boldsymbol{v}\cdot\boldsymbol{r}(t))}{\rho^3}\,. \tag{19}$$

Furthermore, we note the relation

$$r^i(t) = r^i(t_{\rm ret}) - r(t_{\rm ret})\,\frac{v^i}{c}\,, \tag{20}$$

which follows from $r^i(t) = x^i - x^i_A(t)$, $r^i(t_{\rm ret}) = x^i - x^i_A(t_{\rm ret})$, that means $r^i(t) = r^i(t_{\rm ret}) + v^i(t_{\rm ret} - t)$, and then using relation (8). Thus, by means of (20), we can rewrite (19) as follows:

$$h^{\mu\nu}_{(S)}(t, \boldsymbol{x}) = -\frac{4\,G}{c^3}\,\Lambda^{(\mu}_0\,\Lambda^{\nu)}_k\,\epsilon_{i\,j\,k}\;S_j\,\frac{r^i(t_{\rm ret}) + (\gamma - 1)\,\frac{v^i}{v^2}\,(\boldsymbol{v}\cdot\boldsymbol{r}(t_{\rm ret})) - \gamma\,r(t_{\rm ret})\,\frac{v^i}{c}}{\rho^3}\,. \tag{21}$$

Let us briefly note, that from (21) one easily finds that $h^{00}_{(S)} = \mathcal{O}\left(c^{-4}\right)$, $h^{ij}_{(S)} = \mathcal{O}\left(c^{-4}\right)$, while

$$h^{0i}_{(S)}\left(t, \boldsymbol{x}\right) = \frac{2\,G}{c^3}\,\epsilon_{ijk}\,S_j\,\frac{r_k\left(t\right)}{r^3\left(t\right)} + \mathcal{O}\left(c^{-5}\right) \tag{22}$$

gives rise to the famous Lense-Thirring effect. Now we will show an agreement of (21) with Eq. (16) in [11], where some anti-symmetric spin tensor in global coordinates $S^{\alpha\beta}_{\rm global}$ has been employed. To this end we consider each component of the metric tensor (21) separately.

Accordingly, the strategy for the comparison is, first to perform a Lorentz transformation of the spin-part of the metric from co-moving to global frame and second to rewrite the results in terms of the global spin tensor $S^{\mu\nu}_{\rm global}$, Eqs. (B4) and (B5).

1. *Calculation of* $h^{00}_{(S)}$

For the components $\mu = \nu = 0$ we obtain from (21) the following expression:

$$h^{00}_{(S)}\left(t, \boldsymbol{x}\right) = \frac{4\,G}{c^3}\,\gamma^2\,\left(\frac{\boldsymbol{v}}{c} \times \boldsymbol{S}\right)^i\,\frac{r_i\left(t_{\rm ret}\right)}{\rho^3}\,. \tag{23}$$

Now we use the following relation between the spin vector $\boldsymbol{S}$ in the co-moving system and the anti-symmetric spin tensor $S^{\alpha\beta}_{\rm global}$ in the global system, which is shown in Appendix B:

$$\gamma\left(\frac{\boldsymbol{v}}{c} \times \boldsymbol{S}\right)^i = S^{i0}_{\rm global}\,. \tag{24}$$

Inserting (24) into (23) yields

$$h^{00}_{(S)}\left(t, \boldsymbol{x}\right) = \frac{4\,G}{c^3}\,\frac{r_\alpha\left(t_{\rm ret}\right)\;S^{\alpha 0}_{\rm global}\,u^0}{\rho^3}\,, \tag{25}$$

where the four-vector $r^\alpha = (r, \boldsymbol{r})$ has been introduced. In (25) we have formally extended the summation $r_i\,S^{i0}_{\rm global} = r_\alpha\,S^{\alpha 0}_{\rm global}$, because $S^{00}_{\rm global} = 0$ due to the anti-symmetry of the spin tensor; note also $\gamma = u^0/c$ and $S^{\alpha 0}_{\rm global}\,u^0 = S^{\alpha (0}_{\rm global}\,u^{0)}$.

2. *Calculation of* $h^{0\,a}_{(S)}$

Now let us consider the component $\mu = a$ and $\nu = 0$ in (21), which we separate into two terms as follows,

$$h^{a0}_{(S)}\left(t, \boldsymbol{x}\right) = h^{a0}_1\left(t, \boldsymbol{x}\right) + h^{a0}_2\left(t, \boldsymbol{x}\right), \tag{26}$$

$$h^{a0}_1\left(t, \boldsymbol{x}\right) = -\frac{2\,G}{c^3}\,\Lambda^a_0\,\Lambda^0_k\,\epsilon_{i\,j\,k}\;S_j\;\frac{r^i\left(t_{\rm ret}\right) + \left(\gamma - 1\right)\,\dfrac{v^i}{v^2}\,\left(\boldsymbol{v}\cdot\boldsymbol{r}\left(t_{\rm ret}\right)\right) - \gamma\,r\left(t_{\rm ret}\right)\dfrac{v^i}{c}}{\rho^3}\,, \tag{27}$$

$$h^{a0}_2\left(t, \boldsymbol{x}\right) = -\frac{2\,G}{c^3}\,\Lambda^0_0\,\Lambda^a_k\,\epsilon_{i\,j\,k}\;S_j\;\frac{r^i\left(t_{\rm ret}\right) + \left(\gamma - 1\right)\,\dfrac{v^i}{v^2}\,\left(\boldsymbol{v}\cdot\boldsymbol{r}\left(t_{\rm ret}\right)\right) - \gamma\,r\left(t_{\rm ret}\right)\dfrac{v^i}{c}}{\rho^3}\,. \tag{28}$$

For the expression (27) we obtain

$$h_1^{a0}(t,\boldsymbol{x}) = \frac{2\,G}{c^4}\,\frac{r_\alpha\left(t_{\rm ret}\right)\,S^{\alpha 0}_{\rm global}\,u^a}{\rho^3}\,, \tag{29}$$

where we used (24) and again extended the summation $r_i\,S^{i0}_{\rm global} = r_\alpha\,S^{\alpha 0}_{\rm global}$; note also $u^a = \gamma\,v^a$. For the term (28) we obtain

$$h_2^{a0}(t,\boldsymbol{x}) = \frac{2\,G}{c^4}\,\frac{r_\alpha\left(t_{\rm ret}\right)\,S^{\alpha a}_{\rm global}\,u^0}{\rho^3}\,. \tag{30}$$

The proof of relation (30) is a bit involved; it can be found in Appendix C. According to (26) we add both terms (29) and (30) together, and obtain by means of symmetrization notation:

$$h_{(S)}^{a0}(t,\boldsymbol{x}) = \frac{4\,G}{c^4}\,\frac{r_\alpha\left(t_{\rm ret}\right)\,S^{\alpha (a}_{\rm global}\,u^{0)}}{\rho^3}\,. \tag{31}$$

We remark that $h_{(S)}^{a0} = h_{(S)}^{0a}$ as it follows from (21).

3. Calculation of $h_{(S)}^{ab}$

According to (21), we obtain the following components for the spin part of metric tensor,

$$h_{(S)}^{ab}(t,\boldsymbol{x}) = -\frac{4\,G}{c^3}\,\Lambda_0^{(a}\,\Lambda_k^{b)}\,\epsilon_{i\,j\,k}\;S_j\;\frac{r^i\left(t_{\rm ret}\right) + \left(\gamma-1\right)\,\dfrac{v^i}{v^2}\,\left(\boldsymbol{v}\cdot\boldsymbol{r}\left(t_{\rm ret}\right)\right) - \gamma\,r\left(t_{\rm ret}\right)\,\dfrac{v^i}{c}}{\rho^3}\,. \tag{32}$$

If we compare expression (32) with expression (28), we recognize that:

$$h_{(S)}^{ab}(t,\boldsymbol{x}) = h_2^{a0}(t,\boldsymbol{x})\,\frac{v^b}{c} + h_2^{b0}(t,\boldsymbol{x})\,\frac{v^a}{c}\,. \tag{33}$$

In view of relation (33) and by means of (30), we immediately conclude

$$h_{(S)}^{ab}(t,\boldsymbol{x}) = \frac{4\,G}{c^4}\,\frac{r_\alpha\left(t_{\rm ret}\right)\,S^{\alpha (a}_{\rm global}\,u^{b)}}{\rho^3}\,. \tag{34}$$

4. Collection of terms

Now we collect the results (25), (31) and (34) together and obtain finally

$$h_{(S)}^{\mu\nu}(t,\boldsymbol{x}) = \frac{4\,G}{c^4}\,\frac{r_\alpha\left(t_{\rm ret}\right)\,S^{\alpha (\mu}_{\rm global}\,u^{\nu)}}{\gamma^3\left(r\left(t_{\rm ret}\right) - \dfrac{\boldsymbol{v}\cdot\boldsymbol{r}\left(t_{\rm ret}\right)}{c}\right)^3}\,, \tag{35}$$

where we have used for the distance ρ the form given by relation (7). The metric (35) for the spin part coincides with the metric given by Eq. (16) in [11] for the case of uniform motion, besides an additional factor γ^{-1} which is missing in Eq. (16) of [11], as it has been noted already in [14]. We note, that the use of a *spin tensor* or *spin vector* is more or less a matter of taste and allows for a more compact notation, but from the physical point of view it is not important at all. However, it is important that the metric (35) is given in terms of *global* spin parameters, while our metric (17) for the spin is given in terms of *local* spin parameters. Here, we have shown that both expressions are equivalent.

III. ARBITRARILY MOVING POINTLIKE BODIES TO POST-MINKOWSKIAN ORDER

A. Instantaneous Poincaré transformation and classical electrodynamics

Let us consider the equations of classical electrodynamics in the Lorentz-gauge [19],

$$\Box_x A^{\mu}(t,\boldsymbol{x}) = -\mu_0\, j^{\mu}(t,\boldsymbol{x})\,, \tag{36}$$

where $\Box_x = \eta^{\mu\nu}\dfrac{\partial^2}{\partial x^{\mu}\,\partial x^{\nu}}$ is the d'Alembert operator, $A^{\mu} = (\varphi/c, \boldsymbol{A})$ is the four-potential with scalar-potential φ and vector-potential $\boldsymbol{A}$, and $j^{\mu} = (c\,\rho\,, \boldsymbol{j})$ is the four-current with electric charge density ρ and electric current density $\boldsymbol{j}$; the vacuum permeability μ_0 and vacuum permittivity ϵ_0 are related via $c^{-2} = \epsilon_0\,\mu_0$.

The equations of linearized gravity (1) and the equations of classical electrodynamics (36) have the same mathematical structure. Thus we can use some arguments of classical electrodynamics for our purposes. Especially, we will show that the problem of an arbitrarily moving pointlike body in linearized gravity is similar to the problem of an arbitrarily moving pointlike charge Q in electromagnetism.

Let us consider a pointlike charge Q which in the global inertial system $x^{\mu} = (ct, \boldsymbol{x})$ is moving along an arbitrary timelike worldline parametrized by $x_Q^{\mu}(T)$. At each instant of time we introduce an inertial system $X^{\mu} = (cT, \boldsymbol{X})$ along the worldline $x_Q^{\mu}(T)$ which is comoving with the pointlike charge with the instantaneous velocity of the charge. The transformation from the global inertial coordinate system $x^{\mu} = (ct, \boldsymbol{x})$ to the inertial system $X^{\mu} = (cT, \boldsymbol{X})$ which is comoving with the charge is then given by an instantaneous Poincaré transformation, e.g. [18]:

$$x^{\mu}(X^{\alpha}) = b^{\mu} + \Lambda^{\mu}_{\alpha}(t)\; X^{\alpha}\,, \tag{37}$$

with $\Lambda^0_0(t) = \gamma(t)$, $\Lambda^i_0(t) = \Lambda^0_i(t) = \gamma(t)\,\dfrac{v_i(t)}{c}$, $\Lambda^j_i(t) = \delta_{ij} + (\gamma(t) - 1)\,\dfrac{v_i(t)\; v_j(t)}{v(t)^2}$. Like in (4) we take $b^{\mu} = (b^0, \boldsymbol{b})$, and $\boldsymbol{b}$ points from the origin of global frame to the origin of the inertial frame at time $T = 0$.

We assume the point-charge Q to be located at the origin of the comoving inertial system and then the four-potential in this coordinate system is given by

$$A^{\mu}(T,\boldsymbol{X}) = \left(\frac{1}{4\,\pi\,\epsilon_0}\,\frac{Q}{R}\,,\,\boldsymbol{0}\right), \tag{38}$$

where $R = |\boldsymbol{X}|$, and the four-velocity of the charge in the local system is $u^{\mu}_Q = (c, \boldsymbol{0})$.

Now we want to determine the four-potential in the global coordinate system. As is well-known the Liénard-Wiechert potentials for a moving point-charge expressed in terms of retarded time are independent of acceleration. Accordingly, it has been argued in [2, 20] that one might introduce an instantaneous local rest-system as described above and with the point-charge at its origin at retarded time $t_{\rm ret} = t - \left|\boldsymbol{x} - \boldsymbol{x}_Q\left(t_{\rm ret}\right)\right|/c$, and where the four-potential is given by (38). Then, an instananeous Poincaré transformation (37) at $t = t_{\rm ret}$ yields

$$A^{\mu}\left(t, \boldsymbol{x}\right) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{Q\,u^{\mu}_Q\left(t\right)}{\left|u_{\mu}^Q\left(t\right)\,\left(x^{\mu} - x^{\mu}_Q\left(t\right)\right)\right|}\Bigg|_{t=t_{\rm ret}}\,, \tag{39}$$

where $u^{\mu}_Q\left(t\right) = \gamma\left(t\right)\left(c, \boldsymbol{v}_Q\left(t\right)\right)$ is the four-velocity of Q in the global system and all time-dependent quantities on the right-hand side have to be taken at retarded time $t_{\rm ret}$. Furthermore, in (39) the local coordinate distance R has been replaced by the Lorentz-invariant distance, cf. Eq. (6):

$$\rho = \frac{\left|u_{\mu}^Q\left(t\right)\,\left(x^{\mu} - x^{\mu}_Q\left(t\right)\right)\right|}{c}\Bigg|_{t=t_{\rm ret}} = \gamma\left(t\right)\,\left(r_Q\left(t\right) - \frac{\boldsymbol{v}_Q\left(t\right)\cdot\boldsymbol{r}_Q\left(t\right)}{c}\right)\Bigg|_{t=t_{\rm ret}}\,, \tag{40}$$

where $\boldsymbol{r}_Q\left(t\right) = \boldsymbol{x} - \boldsymbol{x}_Q\left(t\right)$ and $r_Q\left(t\right) = \left|\boldsymbol{r}_Q\left(t\right)\right|$. The solution (39) which has been obtained by an instananeous Poincaré transformation is nothing else than the well-known Liénard-Wiechert potentials in classical electrodynamics.

B. Arbitrarily moving mass-monopoles

Now we are going to determine the metric of a pointlike body A moving arbitrarily along a time-like trajectory $x^{\mu}_A\left(T\right)$ in the global system with the aid of the same approach as described in the previous section. According to (2), the metric of a pointlike body without spin and in its local rest frame $X^{\mu} = \left(cT, \boldsymbol{X}\right)$ is given by

$$H^{\alpha\beta}_{(M)}\left(T, \boldsymbol{X}\right) = \frac{2\,G\,M}{c^2\,R}\,\delta_{\alpha\beta}\,, \tag{41}$$

where M is the mass monopole M_L defined by Eq. (5.33) in [6] for the special case $l = 0$. For the case of an arbitrarily moving pointlike charge we perform an instantaneous Poincaré transformation (37) of the metric field (41) at the retarded instant of time defined by Eq. (8), and obtain the global metric

$$h^{\mu\nu}_{(M)}\left(t, \boldsymbol{x}\right) = \frac{2\,G\,M}{c^2}\;\frac{\Lambda^{\mu}_{\alpha}\left(t\right)\,\Lambda^{\nu}_{\beta}\left(t\right)\,\delta_{\alpha\beta}}{\gamma\left(t\right)\,\left(r\left(t\right) - \dfrac{\boldsymbol{v}\left(t\right)\cdot\boldsymbol{r}\left(t\right)}{c}\right)}\Bigg|_{t=t_{\rm ret}}\,, \tag{42}$$

where $\boldsymbol{r}\left(t\right) = \boldsymbol{x} - \boldsymbol{x}_A\left(t\right)$ and $r\left(t\right) = \left|\boldsymbol{r}\left(t\right)\right|$ and for the distance R we have used the invariant expression (40). Now we use the relation $c^2\,\Lambda^{\mu}_{\alpha}\,\Lambda^{\nu}_{\beta}\,\delta_{\alpha\beta} = 2\,u^{\mu}\,u^{\nu} + c^2\,\eta^{\mu\nu}$, and obtain

$$h^{\mu\nu}_{(M)}\left(t, \boldsymbol{x}\right) = \frac{4\,G\,M}{c^2}\;\frac{1}{\gamma\left(t\right)\,\left(r\left(t\right) - \dfrac{\boldsymbol{v}\left(t\right)\cdot\boldsymbol{r}\left(t\right)}{c}\right)}\;\left(\frac{u^{\mu}\left(t\right)}{c}\,\frac{u^{\nu}\left(t\right)}{c} + \frac{\eta^{\mu\nu}}{2}\right)\Bigg|_{t=t_{\rm ret}}\,, \tag{43}$$

where $u^\mu(t) = \gamma(t)(c, \boldsymbol{v}(t))$ is the four-velocity of the body and $\boldsymbol{v}(t)$ being the three-velocity of the body in the global system. The expression (43) is the contribution to the metric of one arbitrarily moving pointlike body in post-Minkowskian approximation. The metric for the case of N pointlike bodies is simply obtained by a summation over N individual contributions (43), in agreement with Eq. (10) in [9] or Eq. (11) in [11].

For many situations, the slow-motion approximation ($v \ll c$) is of sufficient accuracy, e.g. [15, 16]. Therefore, we will compare the metric (43) with previous results in the literature in the slow-motion approximation. A corresponding series expansion of (43) yields

$$h^{00}_{(M)}(t,\boldsymbol{x}) = \frac{2\,G\,M}{c^2}\,\frac{1}{r(t)}\left(1 + \frac{\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)}{c\,r(t)} + \frac{\left(\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)\right)^2}{c^2\,r^2(t)} + \frac{3}{2}\,\frac{v^2(t)}{c^2}\right)\Bigg|_{t=t_{\rm ret}} + \mathcal{O}\left(c^{-5}\right), \tag{44}$$

$$h^{0i}_{(M)}(t,\boldsymbol{x}) = \frac{4\,G\,M}{c^2}\,\frac{1}{r(t)}\,\frac{v_i(t)}{c}\left(1 + \frac{\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)}{c\,r(t)}\right)\Bigg|_{t=t_{\rm ret}} + \mathcal{O}\left(c^{-5}\right), \tag{45}$$

$$h^{ij}_{(M)}(t,\boldsymbol{x}) = \frac{2\,G\,M}{c^2}\,\frac{1}{r(t)}\,\delta_{ij}\left(1 + \frac{\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)}{c\,r(t)} + \frac{\left(\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)\right)^2}{c^2\,r^2(t)} - \frac{1}{2}\,\frac{v^2(t)}{c^2}\right)\Bigg|_{t=t_{\rm ret}}$$
$$+ \frac{4\,G\,M}{c^2}\,\frac{1}{r(t)}\,\frac{v_i(t)\,v_j(t)}{c^2}\Bigg|_{t=t_{\rm ret}} + \mathcal{O}\left(c^{-5}\right). \tag{46}$$

The retarded time-argument in (44) - (46) has to be replaced by the global coordinate time using the following relations:

$$\boldsymbol{r}(t_{\rm ret}) = \boldsymbol{r}(t) + \frac{\boldsymbol{v}(t)}{c}\,r(t) + \frac{\boldsymbol{v}(t)}{c}\,\frac{\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)}{c} + \mathcal{O}\left(c^{-3}\right) + \mathcal{O}(G), \tag{47}$$

$$r(t_{\rm ret}) = r(t)\left(1 + \frac{\boldsymbol{r}(t)\cdot\boldsymbol{v}(t)}{c\,r(t)} + \frac{1}{2}\,\frac{v^2(t)}{c^2} + \frac{1}{2}\,\frac{\left(\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)\right)^2}{c^2\,r^2(t)}\right) + \mathcal{O}\left(c^{-3}\right) + \mathcal{O}(G), \tag{48}$$

$$\frac{v_i(t_{\rm ret})}{c} = \frac{v_i(t)}{c} + \mathcal{O}\left(c^{-3}\right) + \mathcal{O}(G), \tag{49}$$

where we have taken into account that for a system of N pointlike masses the acceleration is proportional to gravitational constant due to the equations of motion; see also text below Eq. (23) in [15]. Then, to order G we obtain:

$$h^{00}_{(M)}(t,\boldsymbol{x}) = \frac{2\,G\,M}{c^2}\,\frac{1}{r(t)}\left(1 - \frac{1}{2}\,\frac{\left(\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)\right)^2}{c^2\,r^2(t)} + 2\,\frac{v^2(t)}{c^2}\right) + \mathcal{O}\left(c^{-5}\right), \tag{50}$$

$$h^{0i}_{(M)}(t,\boldsymbol{x}) = \frac{4\,G\,M}{c^2}\,\frac{1}{r(t)}\,\frac{v_i(t)}{c} + \mathcal{O}\left(c^{-5}\right), \tag{51}$$

$$h^{ij}_{(M)}(t,\boldsymbol{x}) = \frac{2\,G\,M}{c^2}\,\frac{1}{r(t)}\left(\delta_{ij} - \frac{1}{2}\,\frac{\left(\boldsymbol{v}(t)\cdot\boldsymbol{r}(t)\right)^2}{c^2\,r^2(t)}\,\delta_{ij} + 2\,\frac{v_i(t)\,v_j(t)}{c^2}\right) + \mathcal{O}\left(c^{-5}\right), \tag{52}$$

which agrees with Eqs. (21) - (23) in [15] or with Eqs. (47) - (49) in [16] (for $\beta = \gamma = \epsilon = 1$ in [16]); recall $h^{0i} = -h_{0i}$, while $h^{00} = h_{00}$ and $h^{ij} = h_{ij}$, and all relations are valid only to first order in G.

C. Arbitrarily moving Spin-dipoles

Now we proceed with the consideration of the metric of a pointlike body with spin. According to (2), the metric of a massive body with monopole and spin is, in its local rest frame $X^\mu = (cT, \boldsymbol{X})$, given by

$$H^{\alpha\beta}\left(T, \boldsymbol{X}\right) = H^{\alpha\beta}_{(M)}\left(T, \boldsymbol{X}\right) + H^{\alpha\beta}_{(S)}\left(T, \boldsymbol{X}\right), \tag{53}$$

where the monopole part has been given by Eq. (41) and the spin part is given by

$$H^{0a}_{(S)}\left(T, \boldsymbol{X}\right) = -\frac{4\,G}{c^3}\,\epsilon_{a\,b\,c}\,\frac{\partial}{\partial X^b}\,\frac{S_c}{R}\,, \tag{54}$$

while all other components of the spin part vanish: $H^{00}_{(S)} = 0$ and $H^{ij}_{(S)} = 0$. Again, for simplicity we assume here that the co-moving system is located at the center of mass of this body, and we neglect the time-dependence of the spin vector in the local system.

Now we perform an instantaneous Poincaré transformation of the local metric (54), and obtain the spin part in global coordinates for an arbitrarily moving pointlike body with spin:

$$h^{\alpha\beta}_{(S)}\left(t, \boldsymbol{x}\right) = \frac{4\,G}{c^3}\,\Lambda^{(\mu}_{0}\left(t_{\rm ret}\right)\,\Lambda^{\nu)}_{a}\left(t_{\rm ret}\right)\,\epsilon_{a\,b\,c}\,\Lambda^{\lambda}_{b}\left(t_{\rm ret}\right)\,S_c\,\frac{\partial}{\partial x^\lambda}\,\frac{1}{\rho}\,, \tag{55}$$

where for R we have used the invariant expression ρ given by (6) for the distance. By performing the very same steps as described in some detail in section II D, we obtain for (55) the following expression:

$$h^{\mu\nu}_{(S)}\left(t, \boldsymbol{x}\right) = \frac{4\,G}{c^4}\left.\frac{r_\alpha\left(t\right)\,S^{\alpha(\mu}_{\rm global}\,u^{\nu)}\left(t\right)}{\gamma^3\left(t\right)\left(r\left(t\right) - \dfrac{\boldsymbol{v}\left(t\right)\cdot\boldsymbol{r}\left(t\right)}{c}\right)^3}\right|_{t=t_{\rm ret}}. \tag{56}$$

Eq. (56) is the result for the spin part of the metric of one arbitrarily moving pointlike massive body with spin, cf. Eq. (16) in [11]. Recall, that (35) was valid for the case of an extended body but in uniform motion. Like in the previous section, the metric of a system of N arbitrarily moving pointlike spin-dipoles is simply obtained by a summation over the contributions (56) of N individual pointlike spin-dipoles. In many situations, the metric for a spinning body in slow-motion ($v \ll c$) is sufficient, e.g. [17]. Hence, like for the case of pointlike monopoles, we will compare (56) with results previously obtained in the literature in the slow-motion approximation. By inserting (47) - (49) into (56) we obtain

$$h^{00}_{(S)}\left(t, \boldsymbol{x}\right) = -\frac{4\,G}{c^4}\,\frac{1}{r^3\left(t\right)}\,r_a\left(t\right)\,S_b\,\epsilon_{abc}\,v_c\left(t\right) + \mathcal{O}\left(c^{-5}\right), \tag{57}$$

$$h^{0i}_{(S)}\left(t, \boldsymbol{x}\right) = -\frac{2\,G}{c^3}\,\frac{1}{r^3\left(t\right)}\,r_a\left(t\right)\,\epsilon_{iab}\,S_b + \mathcal{O}\left(c^{-5}\right), \tag{58}$$

$$h^{ij}_{(S)}\left(t, \boldsymbol{x}\right) = -\frac{4\,G}{c^4}\,\frac{1}{r^3\left(t\right)}\,r_a\left(t\right)\,S_b\,\epsilon_{ab\,(\,i}\,v_{j\,)}\left(t\right) + \mathcal{O}\left(c^{-5}\right), \tag{59}$$

in agreement with Eqs. (C.17) - (C.19) in [17]. Recall, that (58) generates the Lense-Thirring effect, the spin in the local frame is time-independent, and all relations are valid to first order in G.

IV. CONCLUSIONS

Extremely high precision astrometry, high precision space missions and certain tests of General Relativity, require the knowledge of the metric tensor of the solar system, or more generally, of a gravitational N-body system in post-Minkowskian approximation. So far, the metric outside of massive and moving bodies in only known in post-Newtonian approximation. In our study, we have considered the metric of massive bodies in motion in post-Minkowskian approximation, that is valid to any order in velocity v/c. Two different scenarios were on the scope of our investigation: (i) the case of one body with full mass and spin multipole structure in uniform motion ($\boldsymbol{v}$ = const) in post-Minkowskian approximation, and (ii) the case of N arbitrarily moving pointlike bodies with time-dependent speed $\boldsymbol{v}\,(t)$ in post-Minkowskian approximation.

For the first problem, a co-moving inertial system of coordinates has been introduced and the starting point is the local metric given in terms of Damour-Iyer moments. A Poincaré transformation then yields the metric tensor in the global system (13) in post-Minkowskian approximation. We have demonstrated that our results are in agreement with known results for pointlike masses having monopole and spin structure and moving uniformly.

Then we have derived the global metric for pointlike massive bodies in arbitrary motion having monopole structure (43) and spin structure (56). We have shown that our results are exact to post-Minkowskian order for the problem of pointlike mass-monopoles and spin-dipoles in arbitrary motion.

The problem to find a global metric for a system of N arbitrarily moving and arbitrarily shaped bodies in post-Minkowskian approximation is highly complex and one encounters many subtle difficulties. Especially (in contrast to the case of pointlike bodies), such a metric cannot be obtained by a simple instantaneous Poincaré transformation of the metric (2) for extended bodies. Moreover, it is obvious that for this problem a corresponding accelerated local reference system has to be constructed. It is clear that such a local system can be defined in many different ways (e.g., Fermi normal coordinates or special harmonic ones). As is well known, however, that even in the case of vanishing gravitational fields, i.e., in Minkowski space, such a construction is highly problematic; the reader is referred to [21–28]. At the moment being, we consider our study as one more step towards the aim of a global metric for a system of N arbitrarily shaped and arbitrarily moving massive bodies in post-Minkowskian approximation.

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Appendix A: Notation for the metric tensors

All relations given here will be valid to first order in G, without explicit indication. For weak gravitational fields the metric differs only slightly from flat space metric, that means

$$G_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}\;, \quad G^{\mu\nu} = \eta^{\mu\nu} - H^{\mu\nu}\;, \tag{A1}$$

where $\eta_{\mu\nu} = \eta^{\mu\nu}$ is the metric of Minkowski space, and $|H_{\mu\nu}| \ll 1$ and $|H^{\mu\nu}| \ll 1$.

The equations of linearized gravity take a simple form in the gothic metric [2, 18, 29], defined by

$$\frac{G_{\mu\nu}}{\sqrt{-G}} = \eta_{\mu\nu} + \overline{H}_{\mu\nu}\;, \quad \sqrt{-G}\,G^{\mu\nu} = \eta^{\mu\nu} - \overline{H}^{\mu\nu}\;, \tag{A2}$$

where $G = \det\left(G_{\mu\nu}\right)$ is the determinant of metric tensor. The factor $\sqrt{-G}$ implies that the gothic metric is not a tensor but a tensor density. Let us further note the following relations for the trace-reversed metric perturbation:

$$\overline{H}_{\mu\nu} = H_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,H\;, \quad \overline{H}^{\mu\nu} = H^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}\,H\;, \tag{A3}$$

where $H = \eta^{\alpha\beta}\,H_{\alpha\beta}$. The inverse relation reads

$$H_{\mu\nu} = \overline{H}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\overline{H}\;, \quad H^{\mu\nu} = \overline{H}^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}\,\overline{H}\;, \tag{A4}$$

where $\overline{H} = \eta^{\alpha\beta}\,\overline{H}_{\alpha\beta}$. Finally we note $H = -\overline{H}$, and we find

$$\sqrt{-G} = 1 - \frac{1}{2}\,\overline{H}\;, \quad \sqrt{-G} = 1 + \frac{1}{2}\,H\;. \tag{A5}$$

Appendix B: Some relations for the Spin

1. Lorentz transformation of Spin

In the local frame the spin four-vector is denoted by $S^{\mu} = (0, \boldsymbol{S})$, while in the global system the spin four-vector is denoted by $S^{\mu}_{\rm global} = \left(S^0_{\rm global}, \boldsymbol{S}_{\rm global}\right)$. The Lorentz transformation for the spin between the co-moving frame co-moving with the massive body and the global frame reads

$$S^i_{\rm global} = \Lambda^i_{\mu}\,S^{\mu} = S^i + (\gamma - 1)\;\frac{\boldsymbol{v}\cdot\boldsymbol{S}}{v^2}\,v_i\;, \tag{B1}$$

$$S^0_{\rm global} = \Lambda^0_{\mu}\,S^{\mu} = \gamma\left(\frac{\boldsymbol{v}\cdot\boldsymbol{S}}{c}\right)\;. \tag{B2}$$

Note, that the spin four-vector in any Lorentz frame has three independent components only. The transformation (B1) and (B2) agree with Eq. (8) in [11]. The inverse transformation can easily be deduced from Eqs. (B1) and (B2) and is given by

$$S^i = S^i_{\rm global} + \frac{1-\gamma}{v^2}\,\frac{c}{\gamma}\,S^0_{\rm global}\,v_i\;. \tag{B3}$$

Of course, relation (B3) can also be obtained from the inverse Lorentz transformation.

2. Proof of relation (24)

In [11], some anti-symmetric spin tensor in global coordinates $S^{\alpha\beta}_{\rm global}$ has been employed. Due to the anti-symmetry of this tensor and because of the orthogonality relation $S^{\alpha\beta}_{\rm global}\,u_\beta = 0$, this spin tensor has three independent degrees of freedom like the spin four-vector $S^\mu_{\rm global}$, thus both mathematical expressions are on an equal footing. Therefore, the anti-symmetric spin tensor $S^{\alpha\,\beta}_{\rm global}$ and the spin four-vector $S^{\rm global}_\mu$ in global coordinates are related to each other by the following relation, cf. Eq. (5) in [11] and cf. Eq. (3.9) in [14]:

$$S^{\alpha\,\beta}_{\rm global} = \eta^{\alpha\,\beta\,\gamma\,\delta}\,S^{\rm global}_\delta\,\frac{u_\gamma}{c}\,, \tag{B4}$$

$$S^{\rm global}_\alpha = \frac{1}{2}\,\eta_{\alpha\,\beta\,\gamma\,\delta}\,\frac{u^\beta}{c}\,S^{\gamma\,\delta}_{\rm global}\,, \tag{B5}$$

where (B5) is the inverse of (B4). Here, $\eta^{\alpha\beta\gamma\delta} = -\dfrac{1}{\sqrt{-g}}\,\epsilon_{\alpha\beta\gamma\delta}$ and $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}$ are the contravariant and covariant components of the Levi-Civita tensor, respectively, and $\epsilon_{\alpha\beta\gamma\delta}$ is the Minkowskian Levi-Civita tensor with $\epsilon_{0123} = 1$. Let us note the following relations of this tensor:

$$\epsilon^{i\,j\,0\,k} = \epsilon^{0\,i\,j\,k} = -\epsilon_{0\,i\,j\,k} = -\epsilon_{i\,j\,k}\,. \tag{B6}$$

In harmonic coordinates $g = -1 + \mathcal{O}\,(G)$, we obtain from Eq. (B4), up to order $\mathcal{O}\,(G)$,

$$S^{a\,0}_{\rm global} = \epsilon^{a\,0\,\gamma\,\delta}\,\frac{u_\gamma}{c}\,S^{\rm global}_\delta = \epsilon^{a\,0\,k\,l}\,\frac{u_k}{c}\,S^{\rm global}_l\,. \tag{B7}$$

And by means of (B6) we finally arrive at

$$S^{a\,0}_{\rm global} = \epsilon_{a\,k\,l}\,\gamma\,\frac{v_k}{c}\,S^{\rm global}_l = \gamma\,\left(\frac{\boldsymbol{v}\times\boldsymbol{S}}{c}\right)^a\,, \tag{B8}$$

where in the last term we have used (B1), i.e. $\boldsymbol{v}\times\boldsymbol{S}_{\rm global} = \boldsymbol{v}\times\boldsymbol{S}$. Eq. (B8) is nothing but relation (24); cf. Eq. (D1) in [11].

Appendix C: Prof of relation (30)

In order to show (30), we insert into Eq. (28) the explicit form for the Lorentz matrix and obtain

$$h^{a0}_B\,(t,\boldsymbol{x}) = -\frac{2\,G}{c^3}\,\gamma\,\left(\epsilon_{ija}\,S_j + (\gamma-1)\,\frac{v_a\,v_k}{v^2}\,\epsilon_{ijk}\,S_j\right)\,\frac{X_i}{\rho^3}\,, \tag{C1}$$

where we have used the abbreviation

$$X_i = r_i\,(t_{\rm ret}) + (\gamma-1)\,\frac{v_i}{v^2}\,(\boldsymbol{v}\cdot\boldsymbol{r}\,(t_{\rm ret})) - \gamma\,r\,(t_{\rm ret})\,\frac{v_i}{c}\,. \tag{C2}$$

The metric (C1) is still given in terms of the local spin $S^\mu = (0,\boldsymbol{S})$ comoving with the massive body, and we have to transform it into the spin tensor in global coordinates. For the first term in the parentheses of Eq. (C1) we will use the following relation (a proof is given below):

$$\gamma\, \epsilon_{ija}\, S_j = S^{ai}_{\rm global} + \frac{\gamma - 1}{v^2}\, (\boldsymbol{v} \cdot \boldsymbol{S})\, \epsilon_{aij}\, v_j\,, \tag{C3}$$

while for the second term in the parentheses of Eq. (C1) we will use relation (24), and then we obtain

$$h^{a0}_B\,(t, \boldsymbol{x}) = -\frac{2\,G}{c^3} \left(S^{a\,i}_{\rm global} + \frac{\gamma - 1}{v^2}\, \frac{c}{\gamma}\, S^0_{\rm global}\, \epsilon_{a\,i\,j}\, v_j + \frac{1 - \gamma}{v^2}\, v_a\, c\, S^{i\,0}_{\rm global}\right) \frac{X_i}{\rho^3}\,. \tag{C4}$$

For the last term in (C3) we have also used relation (B2). The metric (C4) is now given in terms of global spin variables. But we still have to express the second term in (C4) by the global spin tensor. Therefore, we use the following relation, cf. Eq. (B5),

$$S^0_{\rm global} = \frac{1}{2}\, \epsilon_{k\,l\,m}\, \frac{u_k}{c}\, S^{l\,m}_{\rm global}\,. \tag{C5}$$

Inserting (C5) into (C4) yields (recall the anti-symmetry of spin-tensor):

$$h^{a0}_B\,(t, \boldsymbol{x}) = -\frac{2\,G}{c^3}\, \frac{X_i}{\rho^3}$$

$$\times \left(S^{a\,i}_{\rm global} + \frac{\gamma - 1}{v^2}\, v_a\, v_b\, S^{i\,b}_{\rm global} + (\gamma - 1)\, S^{a\,i}_{\rm global} + \frac{\gamma - 1}{v^2}\, v_i\, v_b\, S^{b\,a}_{\rm global} + \frac{1 - \gamma}{v^2}\, v_a\, c\, S^{i\,0}_{\rm global}\right), \tag{C6}$$

where for the second term in the parentheses of Eq. (C4) after inserting (C5) we have used

$$\epsilon_{aij}\, \epsilon_{klm} = \begin{vmatrix} \delta_{ak} & \delta_{al} & \delta_{am} \\ \delta_{ik} & \delta_{il} & \delta_{im} \\ \delta_{jk} & \delta_{jl} & \delta_{jm} \end{vmatrix}\,.$$

We recognize that the second and last term in the parentheses of Eq. (C6) cancel each other, as one can see by using the relation $v_b\, S^{a\,b}_{\rm global} = c\, S^{a\,0}_{\rm global}$ due to $S^{\alpha\,\beta}_{\rm global}\, u_\beta = 0$. For the fourth term in the parentheses of Eq. (C6) we use $v_b\, S^{b\,a}_{\rm global} = -c\, S^{a\,0}_{\rm global}$ and obtain

$$h^{a0}_B\,(t, \boldsymbol{x}) = -\frac{2\,G}{c^3}\, \frac{X_i}{\rho^3} \left(\gamma\, S^{a\,i}_{\rm global} + \frac{1 - \gamma}{v^2}\, v^i\, c\, S^{a\,0}_{\rm global}\right). \tag{C7}$$

Now we reinsert (C2) and obtain, recall $c\,\gamma = u^0$ and $v_i\, S^{a\,i} = c\, S^{a\,0}$,

$$h^{a0}_B\,(t, \boldsymbol{x}) = \frac{2\,G}{c^4}\, \frac{r_\gamma\,(t_{\rm ret})\; S^{\gamma a}_{\rm global}\, u^0}{\rho^3}\,, \tag{C8}$$

where we have used the anti-symmetry of the spin-tensor; note $r_\gamma\,(t_{\rm ret}) = (-r(t_{\rm ret}), \boldsymbol{r}(t_{\rm ret}))$, and $r^\gamma = (r(t_{\rm ret}), \boldsymbol{r}(t_{\rm ret}))$. Eq. (C8) is just relation (30).

Finally let us proof relation (C3). We insert the Lorentz transformations (B1) and (B2) into relation (B4) and obtain up to order G,

$$\begin{aligned} S^{ij}_{\rm global} &= \epsilon^{ij\gamma\delta}\, S^{\rm global}_{\delta}\, u_{\gamma} \\ &= \epsilon^{ijk}\, S_k\, \gamma + \epsilon^{ijk}\, \frac{\gamma-1}{v^2}\, (\boldsymbol{v}\cdot\boldsymbol{S})\; v^k\, \gamma - \epsilon^{ijk} \left(\frac{\boldsymbol{v}\cdot\boldsymbol{S}}{c}\right) \gamma^2\, \frac{v_k}{c}\,, \end{aligned} \tag{C9}$$

where we have also used $u_0 = -c\,\gamma$, $u^0 = c\,\gamma$ and $u_k = u^k = \gamma\, v_k$; note $S_0^{\rm global} = -S^0_{\rm global}$, $\epsilon_{ijk} = \epsilon^{ijk}$, and $\epsilon^{0ijk} = -\epsilon_{0ijk}$. Then, by using the relation $(\gamma-1)\,\gamma - \frac{v^2}{c^2}\,\gamma^2 = 1-\gamma$, we obtain from Eq. (C9),

$$S^{ij}_{\rm global} = \gamma\, \epsilon_{ijk}\, S_k + \frac{1-\gamma}{v^2}\, (\boldsymbol{v}\cdot\boldsymbol{S})\; \epsilon_{ijk}\, v_k\,, \tag{C10}$$

which is just relation (C3); cf. Eq. (D2) in [11].

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[1] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, 1973, W.H. Freeman and Company, San Francisco, USA.

[2] L.D. Landau, F.M. Lifschitz, *The Classical Theory of Fields*, 1975, Fourth Edition, Addison-Wesley, Oxford, UK.

[3] K.S. Thorne, *Multipole expansions of gravitational radiation*, Rev. Mod. Phys. **52** (1980) 299.

[4] L. Blanchet, T. Damour, *Radiative gravitational fields in general relativity: I. General structure of the field outside the source*, Phil. Trans. R. Soc. London A **320** (1986) 379.

[5] L. Blanchet, T. Damour, *Post-Newtonian generation of gravitational waves*, Annales de I'lnstitut Henri Poincaré I Physique Theorique **50** (1989) 377.

[6] T. Damour, B.R. Iyer, *Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors*, Phys. Rev. D **43** (1991) 3259.

[7] T. Damour, M. Soffel, Ch.Xu, *General-relativistic celestial mechanics. I. Method and definition of reference systems*, Phys. Rev. D **43** (1991) 3273.

[8] T. Damour, M. Soffel, Ch.Xu, *General-relativistic celestial mechanics. II. Translational equations of motion*, Phys. Rev. D **45** (1992) 1017.

[9] S.M. Kopeikin, G. Schäfer, *Lorentz covariant theory of light propagation in gravitational fields of arbitrary-moving bodies*, Phys. Rev. D **60** (1999) 124002.

[10] S.A. Klioner, *Light propagation in the gravitational field of moving bodies by means of Lorentz transformation I. Mass monopole moving with a constant velocity*, Astronomy and Astrophysics **404** (2003) 783.

[11] S.M. Kopeikin, B. Mashhoon, *Gravitomagnetic-effects in the propagation of electromagnetic waves in variable gravitational fields of arbitrary-moving and spinning bodies*, Phys. Rev. D **65** (2002) 064025.

[12] J. Ibañez, J. Martin, *Gravitational interaction of two spinning particles in general relativity*, Gen. Rel. Grav. **14** (1982) 439.

[13] J. Ibañez, J. Martin, E. Ruiz, *Gravitational interaction of two spinning particles in general relativity II*, Gen. Rel. Grav. **16** (1984) 225.

[14] M.H. Brügmann, *Light deflection in the post-linear gravitational field of bounded pointlike masses*, PhD Thesis, Friedrich-Schiller Universität Jena (2006).

[15] M.H. Brügmann, *Light deflection in the post-linear gravitational field of bounded pointlike masses*, Phys. Rev. D **72** (2005) 024012.

[16] X.M. Deng, Y. Xie, *Two-post-Newtonian light propagation in the scalar-tensor-theory: An N-point mass case*, Phys. Rev. D **86** (2012) 044007.

[17] Y. Xie, *Astronomical relativistic reference systems with multipolar expansion: the global one*, arXiv: gr-qc/1405.0882v2.

[18] S. Kopeikin, M. Efroimsky, G. Kaplan, *Relativistic Celestial Mechanics of the Solar System*, Wiley-VCH, Signapure (2012).

[19] J.D. Jackson *Classical Electrodynamics*, John Wiley & Sons, Third Edition, New York (1998).

[20] F. Rohrlich, *Classical Charged Particles*, World Scientific, Third Edition (2007).

[21] F.K. Manasse, C.W. Misner, *Fermi normal coordinates and some basic concepts in differential geometry*, J. Math. Phys. **4** (1963) 735.

[22] L.M. Marsh, *Relativistic accelerated systems*, Am. J. Phys. **33** (1965) 934.

[23] E.A. Desloge, R.J. Philpott, *Uniformly accelerated reference frames in special relativity*, Am. J. Phys. **55** (1987) 252.

[24] W.T. Ni, M. Zimmermann, *Inertial and gravitational effects in the proper reference frame of an accelerated, rotating observer*, Phys. Rev. D **17** (1978) 1473.

[25] K.P. Marzlin, *The physical meaning of Fermi coordinates*, Gen. Rel. Grav. **26** (1994) 619.

[26] K.P. Marzlin, *Fermi coordinates for weak gravitational fields*, Phys. Rev. D **50** (1994) 888.

[27] M. Pauri, M. Vallisneri, *Maerzke-Wheeler coordinates for accelerated observers in special relativity*, Found. Phys. Lett **13** (2000) 401.

[28] E. Gourgoulhon, *Special relativity in general frames*, Springer (2013).

[29] E. Poisson, C.M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*, Cambridge University Press (2014).