On Radiation by Moving Charhe

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The problem of electromagnetic field of a charge moving arbitrarily, is studied. The standard representation of the field in the form of retarded potential is analyzed critically. It is shown that, in spite of formally correct construction, the term of this expression adopted as the radiation part of the intensity corresponds in fact to some non-zero charge density. An exact solution of Maxwell equations for the field of a charge describing so-called hyperbolic motion, is presented. It is shown that, due to the exact solution, the field does not contain any radiation part. A new approach to deriving the radiation part of the field is suggested. The result is obtained as explicit form of density of current of displacement in the space-time which can be considered as the source of the radiation part of an arbitrarily motving charge. It is shown that the form of the current density depends on curvature and torsion of the charge world line. PACS 03.50.De

Chapter 1

Introduction

So-called retarded potentials have been proposed as formal solution of equations for electromagnetic field about a century ago as a possible representation of the field of a moving charge [1]. Later on they were adopted as the main representation of arbitrary non-stationary electromagnetic field produced by any source. Due to the conventional views the expressions for the field intensities obtained from the retarded potentials satisfy the Maxwell equations by construction, so it seems to be pointless to check them. It should be noted, however, that another point of view encounters from time to time in literature. In particular, M. Born proposed an alternative to the potentials in 1909 [2] and A. Sommerfeld called them "Lienard-Wiechert approximation" ("Lienard-Wiechertsche Näherung") [3]. He specified the origin of possible discrepancies pointing out that it is power minus two of interval which should be taken as the Green function of a massless scalar field, whereas the retarded Green function used when deriving retarded potentials, is δ-function in form. Thus, due to Sommerfeld results, the retarded potentials underlying the adopted theory of radiation are not exact solutions of the Maxwell equations, whereas composing an exact theory looks an important task of theoretical physics.

Consider a particle carrying a charge e and moving with constant acceleration $\vec{a}$ and let in the frame of reference chosen its velocity is zero at the moment of time $t = 0$. Since within some short period of time the particle shift $\vec{a}t^2/2$ and its velocity $\vec{a}t/c$ are negligeably small compared with distances uip to ct on which the radiation emitted can be considered, and velocity of light respectively, we restrict the consideration with terms of order c^{-2}. Then, electric and magnetic field intensities can be taken in the form [1]

$$\vec{E}(\vec{r}) = \frac{e\vec{n}}{r^2} + \frac{e}{c^2}\frac{\vec{n}\times\vec{n}\times\vec{a}}{r}, \quad \vec{H}(\vec{r}) = \vec{n}\times\vec{E}$$

where $\vec{n} = \vec{r}/r$, and the retardance is omitted due the approximation used. The second term of $\vec{E}$ is adopted as representation of the radiation part of the intensity [1]. Therefore, its divergence is to be zero.

Consider the intensities in a spherical coordinate system $\{r, \theta, \varphi\}$ with r being distance from the charge and θ being the angle between $\vec{a}$ and $\vec{r}$. In this system the vector $\vec{n} \times \vec{a}$ has only one component

$$[\vec{n} \times \vec{a}]_\varphi = r^{-1} |\vec{a}| \sin\theta,$$

note that it is purely toroidal, thus, the "radiation" part of $\vec{E}$ also consists of one component

$$[\vec{E}_{rad}]_\theta = \frac{e|\vec{a}| \sin\theta}{c^2 r^2}$$

which is tangent to spheres $r = const$. Now it is easy to evaluate its divergence:

$$\mathrm{div}\vec{E}_{rad} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\frac{e|\vec{a}| \sin^2\theta}{c^2 r^2} \right) = \frac{2e|\vec{a}| \cos\theta}{c^2 r^2}$$

that is not zero. Consequently, the Lienard-Wiechert potentials do not satisfy the source-free field equations. An obvious non-zero divergence of electric field intensity found from retarded potentials was obtained also in V. M., Dubovik and S. V. Shabanov in their review [6]. In fact, retarded potential generally accepted as the field of a charge describing hyperbolic motion corresponds to the charge density just evaluated. A non-zero current density can also be evaluated similarly from rot$\vec{H}$. The aim of the present work is to develop another approach to radiation by moving charge, based solely on exact solutions of Maxwell equations.

Chapter 2

Field of a uniformly accelerated charge

2.1 Qualitative aspects of the problem

As was shown by M. Born [2], the world line of a uniformly accelerated motion is hyperobla in form, specified in Cartesian coordinates as

$$z^2 - t^2 = a^2, \tag{2.1}$$

where a is reciprocal of the acceleration, and forms an orbit of one-parametric group of Lorentz transformations in the z-direction. Since the corresponding Killing vector is time-like the symmetry of uniformly accelerated motionis a kind of stationarity. This stationarity allowed M. Born to obtain an expression for the field of uniformly accelerated charge which differs from that derived by the method of retarded potentials. The main difference is that, due to the Born's solution the magnetic field vanishes manifestly at $t = 0$, when the charge is instantaneously in rest, whereas the expression obtained from retarded potentials reads that it is equal to $\vec{n} \times \vec{E}$ multiplied by a non-zero scalar factor. There are also some differences in dependence of $\vec{E}$ on coordinates. One of consequences of the Born's expressions is that the Poynting vector is zero and hence, the charge does not radiate. Due to stationarity of the phenomenon mentioned above, the charge does not radiate also at any other moment of time.

Another qualitative apect of the phenomenon in question deals with causality. Consider a pair of oppositely moving charges with world lines given in (2.1), namely, $z = \pm\sqrt{t^2 + a^2}$. Two asymptotic isotrope planes $t = \pm z$ divide the space-time into four domains: 'future' $t > |z|$, positive z ($z > |t|$), negative z ($z < |t|$), and 'past' $t < |z|$. Apparently, the fields produced by both charges are zero in the 'past' domain because of lack of causal connection between their world lines and its points. Similarly, the field produced by the charge with negative z's is zero in the positive z domain where the world line of another charge lies wholly, and vice versa. Thus, if a solution of Maxwell equations found in one of these domains consists of exactly even and exactly odd on z terms and displays thereby presence of the second charge, it, nevertheless, describes the field produced by only one charge in each domain.

As H. Bondi and T. Gold noted in their work [7], because of symmetry of the phenomenon in question it is impossible to assume that the field is non-zero in the 'future' domain and vanish in the 'past' one. Indeed, electric lines of force pass through semiplanes $z = |\pm t|$ similarly, there exist equal possibilities to prolonge them into both 'past' and 'future' domains. This should be done for the 'future' domain because the field there is non-zero, but not for the 'past' one because it is zero there. On the other hand, if the lines of force are cut there arises non-zero divergence of $\vec{E}$ that does not correspond to any charge distribution.

One can think that in such a situation a source-free field sould be added in the domain, which cancels the normal component of $\vec{E}$ on its border such that the non-zero divergence vanishes. However, the result looks to contradict natural physical considerations. Indeed, consider an observer which rests at constant value of z. Before the moment of time $t = -z$ he does not see any field and at this moment an intensity suddenly appears. Thus, the observer detects endpoints of lines of force without any charge presence. The lines are cut and their endpoints move in the free propagation regime. It should be noted that the endpoints occure only if the charge motion is permanenly accelerated and in this case it produces electric field on the $z = -t$ plane only when passing infinite values of $-t$ and z. Therefore the endpoints of the lines of force constitute a mathematical abstraction of the same sort as the phenomenon of permanently accelerated motion.

2.2 The coordinate system

A coordinate system of with one of coordinates being the parameter of Lorentz transformations along one spatial direction (z-axis here), was proposed by W. Rindler [8], and some its generalizations are found in our work [9]). Let $\{t, z, \rho, \varphi\}$ be the standard circular cylinder coordinate system with the metric

$$ds^2 = dt^2 - (dz^2 + d\rho^2 + \rho^2 d\varphi^2).$$

The substitutions

$$\xi = \pm arctanh\frac{t}{z}, \quad \zeta = \sqrt{s^2 - t^2} \tag{2.2}$$

$$t = \pm\zeta \sinh\xi, \quad z = \zeta\cosh\xi$$

specify the coordinate system in question. The metric in these coordinates has the following form:

$$ds^2 = \zeta^2 d\xi^2 - (d\zeta^2 + d\rho^2 + \rho^2 d\varphi^2) \tag{2.3}$$

The surfaces $\xi = const$ are three-dimensional Euclidean spaces equipped with circular cylinder coordinate systems $\{\zeta, \rho, \varphi\}$ related to inertial frames moving with velocities equal to $\tanh\xi$ along the ζ-axis, and the displacements $\xi \to \xi + \alpha$ are nothing but Lorentz transformations mentioned above. Every point with fixed values of ζ, ρ and φ coordinates moves with acceleration equal to ζ^{-1} whereas the value of $\zeta\xi$ equals to its proper time. Evidently, the coordinates ζ and ξ (2.2) are usual polar coordinates in Minkowskian 2-plane with t and z as Cartesian ones. Then, an element of the dimensionless parameter of Lorentz transformations is

$$d\xi = \frac{zdt - tdz}{z^2 - t^2}. \tag{2.4}$$

Now, introducing the well-known bispherical coordinate system $\{u, v, \varphi\}$ [10] in the $\xi = 0$ hyperplanes:

$$e^{-2u} = \frac{(\zeta - a)^2 + \rho^2}{(\zeta + a)^2 + \rho^2}; \quad \tan v = -\frac{2a\rho}{\zeta^2 - a^2 + \rho^2}; \tag{2.5}$$

$$\zeta = \frac{a \sinh u}{\cosh u + \cos v}; \quad \rho = \frac{a \sin v}{\cosh u + \cos v}$$

changes the system into the accelerated bispherical coordinate system $\{\xi, u, v, \varphi\}$. On the semiplanes $\xi = const$, $\varphi = const$, the lines $u = const$ form a family of circles whose exrteme cases are points $\zeta = \pm a$, $\rho = 0$ for $u = \pm\infty$ and the straightline $\zeta = 0$ for $u = 0$, whereas the lines $v = const$ are circular arcs that link the foci $u = \pm\infty$. These arcs are orthogonal to all the cirles $u = const$ [10]. The space-time surfaces $u = const$ represent world tubes of spheres moving with constant acceleration.

The coordinate system covers two causally independent spacetime domains with $|z| \geq |t|$. As was noted above, if the charge moves in one of them the field is zero in the another one, thus, it will not be considered. Inserting the well-known bispherical metric [10] into the expression (2.3) and composing the pseudo-Euclidean generalization of the coordinate system for the domain $|z| < |t|$, one obtains the metric of the system in question:

$$ds^2 = a^2 \frac{\sinh^2 u d\xi^2 - du^2 - dv^2 - \sin^2 v d\varphi^2}{(\cosh u + \cos v)^2}, \quad |z| > |t|$$

$$ds^2 = a^2 \frac{du^2 - dv^2 - \sin^2 u d\xi^2 - \sin^2 v d\varphi^2}{(\cos u + \cos v)^2}, \quad |z| < |t|.$$

The corresponding Levi-Civita symbol components have the following form:

$$\varepsilon_{\xi u v \varphi} = a^4 (\cosh u + \cos v)^{-4} \sinh u \sin v \tag{2.6}$$

$$\varepsilon^{\xi u}{}_{v\varphi} = \varepsilon^{\xi u} v\varphi = a^2 \sinh^{-1} u \sin v$$

$$\varepsilon_{\xi}{}^{u}{}_{v}{}^{\varphi} = \varepsilon_{\xi u}{}^{v\varphi} = a^2 \sinh u \sin^{-1} v$$

In the positive z domain considered below the coordinate u has only non-negative values and its extreme value $u = 0$ corresponds to either infinitely far region or the domain border $\zeta = 0$. Therefore, the boundary condition to be accepted requires the field and its intensities to vanish at $u = 0$.

The function $\tanh u$ expressed in cylindrical coordinates as

$$\tanh u = -\frac{2a\zeta}{\zeta^2 + a^2 + \rho^2}$$

makes it possible to find out radii of spheres $u = const$ in their rest frames:

$$R(u) = \sqrt{(\zeta - a \coth u_0)^2 + \rho^2} = a \sinh^{-1} u \tag{2.7}$$

2.3 The field equation

The coordinate system just introduced admits the complete variables separation in the Maxwell equations, however we will restrict ourselves with considering only axially-symmetric Lorentz-invariant fields, i.e., that independent on ξ and φ. The field equations for source-free electromagnetic field specified as a 1-form α is $d\ {}^*d\alpha = 0$. Then, if a field is tangent to the orbits, i.e., $\alpha = \Phi(u, v)d\xi$ the equation has the following form:

$$0 = d\ {}^*d\alpha =$$
$$= -\frac{\sin v}{\sinh u}\left[\sinh u \left(\frac{\Phi_u}{\sinh u}\right)_u + \frac{1}{\sin v}(\Phi_v \sin v)_v\right] du \wedge dv \wedge d\varphi$$

where we used the Levi-Civita symbol which is $\varepsilon_{\xi uv\varphi} = a^4(\cosh u + \cos v)^{-4} \sinh u \sin v$, where the Hodge conjugation is made with applying the Levi-Civita symbol (2.6) . The general solution of this equation is:

$$\Phi(u, v) = a_0(\cosh u - 1) + \sum_{n=1}^{\infty} a_n \sinh u \frac{d}{du} p_n(\cosh u) P_n(\cos v)$$

where $p_n(x)$ and $P_n(x)$ are Legendre polynomials, a_n are constant coefficients [10]represents the complete multipole expansion. It will be shown below that the field of a charge is specified by the first term in the expansion and a_0 equals to the charge value.

2.4 Gauss-Ostrogradskii theorem and energy of the field

Consider a source-free field of the form $\alpha = q(\cosh u - 1)d\xi$. Then, the integral

$$\oint {}^*d\alpha$$

taken over any sphere $u = const$ is equal to the flux of intensity of electric field through it and, hence, to charge contained in it multiplied by 4π. Since ${}^*d\alpha = a^2 q \sin v dv \wedge d\varphi$ the integration is trivial and the factor q is the charge value. Consequently, a charge q placed in the point $u = \infty$ produces the field

$$\alpha = q(\cosh u - 1)d\xi, \quad \alpha = q(\cos u - 1)d\xi$$

in the domain for $z > |t|$. As follows from the equation (2.4) at the moment $t = 0$ when the charge instataneously rests the magnetic field is zero and that of electric field has only u-component equal to $a^{-1}q(\cosh u + \cos v)$. Its rest energy m evaluated beyond a sphere $u = u_0$ does not depend on the charge acceleration:

$$m = \frac{1}{8\pi}\int \vec{E}^2 d^3x = \frac{q^2}{8\pi a}\int_0^{u_0} du \int_0^{\pi} dv \int_0^{2\pi} d\varphi(\cosh u + \cos v)\sin v =$$

$$\frac{q^2}{2a}\sinh u_0 = (2a^{-1}q^2 \sinh u_0) \equiv \frac{q^2}{2R_0}$$

where due to the equation (2.7) R_0 is radius of the sphere. It is seen explicitly that the field does not contain any radiative part.

To rewrite the field in standard Lorentzian coordinates it suffices to return to the coordinates $\{t, z, \rho, \varphi\}$. It turns out that the vector potential components are

$$A_t = q\frac{z}{z^2 - t^2}\left(\frac{z^2 - t^2 + \rho^2 + a^2}{\sqrt{[(\sqrt{|z^2 - t^2|} - a)^2 + \rho^2][(\sqrt{|z^2 - t^2|} + a)^2 + \rho^2]}} - 1\right)$$

$$A_z = -q\frac{t}{z^2 - t^2}\left(\frac{z^2 - t^2 + \rho^2 + a^2}{\sqrt{[(\sqrt{|z^2 - t^2|} + a)^2 + \rho^2][(\sqrt{|z^2 - t^2|} + a)^2 - \rho^2]}} - 1\right) \tag{2.8}$$

for both the domains. This solution was obtained in our works [11, 12].

Now we evaluate the normal component of electric field on the plane $z = -t$ that moves with velocity of light and cuts the lines of force. For small values of u it is convenient to use the identity

$$\cosh u - 1 = 2\sinh^2 u/2$$

and linearity of the function sinh near the zero value of the argument. The result gives for the normal component the expression

$$a^2/2(a^2 + \rho^2)$$

.

Chapter 3

Infinitesimal radiation processes

3.1 Statement of the problem

So far the theory was adopted which reads that a charge radiates whenever its motion is noninetrial and the radiation part of the field produced by an arbitrarily moving charge can be taken as that of done under uniformly accelerated motion with the same value of instantaneous acceleration. However, the results obtained above show that unioformly accelerated motion is not concerned with any radiation and, hence a charge radiates only when its acceleration changes. Thus, if the charge acceleration is piecewise-constant with single step-like change the radiation is emitted only at the moment of change of the acceleration. The corresponding world line is smooth and consists of two halves of different hyperbolas matched at the moment $t = 0$ in the denotions introduced above. To study the phenomenon it suffices to consider only infinitesimal changes of the acceleration.

Consider a world line composed of two halves of hyperbolas (2.1) with different values of a and lying, in general, in different planes. The fields corresponding to these motions differ in shape and orientation. The Born solution (2.8) describes the field up to the moment of the acceleration change. Applying another solution to the later moments leads to a wrong result describing a non-smooth field which suffers an infinitesimal discontinuity whereas the field really produced is smooth. Evidently, the discontinuity is to be erased by adding a source-free field equal to the difference at the moment of junction and to zero before it. To see that such a source-free field exists and is unique, consider the discontinuity in a frame corresponding to zero particle velocity at the moment of junction. Both the fields have only electric intensities whose divergences are equal to the same charge density. Consequently, their difference constituting the discontinuity in question, is an electric intensity with zero divergence and, hence, it can be considered as a initial value for some electromagnetic wave. Now, inserting the electric intensity into Maxwell equations leads to a wave equation for the field to be added. The discontinuity cancels by the only solution. Apparently, the solution describes some purely radiative field which thus can be considered as radiation emitted by the charge. In the present section we evaluate the field differences for an arbitrary infinitesimally small change of the charge acceleration.

3.2 Matching the coordinate systems

As the field of a charge moving with constant acceleration is found in a special coordinate system whose shape depends on the acceleration vector, we should introduce two such systems for $t > 0$ and $t < 0$ domains. Their foci coincide at $t = 0$ and belong to a smooth world line of the charge while boundaries of their domains at $t = 0$ are different. Indeed, the perpendicular dropped from the focus to the bounding plane is collinear with the vector of acceleration and its length is reciprocal to the value of the acceleration. Therefore, there exist two main cases to be considered. In the first case the acceleration changes only in value and in the second case it does only in direction. Therefore we label the corresponding increments of coordinates and the field components with subscripts $\|$ and $\perp$. For infinitesimal changes of the acceleration all the rest cases are reducible to the two main ones. As the fields at $t > 0$ and $t < 0$ have the same form in both the coordinate systems it is natural first to match the coordinate systems.

In the first case there are two spatial coordinate systems in the 3-plane $t = 0$ specified by the expressions (2.5) with distinct values of the parameter a. Let δa be the small increment of the parameter caused by small change of acceleration. This change causes a small deformation of the coordinate system. Small deformation of the coordinate system results a small change of any object considered in the system. As well-known, the change is equal to its Lie differential with respect to the deformation [13, 14]. The deformation is expressed in changes of values of coordinates u and v for any point of the space. Therefore, we start with evaluating changes of the coordinates and then find out the Lie derivatives of the field. Since Lie derivatives of the field are different in the first and the second cases we denote them as $L_\|$ and $L_\perp$ respectively.

Small differences of the values of u and v coordinates at any point may be obtained from the fact that increment of the s coordinate is zero:

$$0 = \delta s = -\frac{2\delta a}{1 + e^w} + \frac{2ae^w \delta w}{(1 + e^w)^2}.$$

Thus, increment of the w coordinate is equal to

$$\delta w = \frac{1 + e^{-w}}{a} \delta a = -\frac{\partial w}{\partial a}.$$

Apparently, the differences of u and v coordinates constitute the real and imaginary parts of δw respectively. Finally, one can rewrite the results in the following form:

$$L_{\|} u = a^{-1}(1 + e^{-u}\cos v); \quad L_{\|} v = a^{-1}e^{-u}\sin v. \tag{3.1}$$

To match the coordinate systems in the second case we introduce Cartesian coordinates $\{x, y, \eta\}$

$$x = \frac{a \sin v \cos\varphi}{\cosh u + \cos v}; \quad y = \frac{a \sin v \sin\varphi}{\cosh u + \cos v}; \tag{3.2}$$

$$\eta = a\left(\frac{\sinh u}{\cosh u + \cos v} - 1\right)$$

and the vector $\vec{a}$ with components $(0, 0, a)$ in these coordinates. The charge world line intersects the space in the point $\vec{r} = 0$ where $\vec{r}$ has components (x, y, η) and it is easy to check out that

$$e^{2u} = \frac{|\vec{r} + 2\vec{a}|^2}{|\vec{r}|^2}; \quad \cos v = -\frac{(\vec{r} + 2\vec{a}) \cdot \vec{r}}{|\vec{r} + 2\vec{a}||\vec{r}|}. \tag{3.3}$$

The vector $\vec{a}$ is collinear to the charge acceleration and has the length reciprocal to the acceleration. As was pointed out above, the vector $\vec{a}$ represents the perpendicular dropped from the focus to the bounding plane and its increment is orthogonal to it. Let the small change of the vector $\vec{a}$ be denoted as $\delta\vec{a}$. For any given vector $\delta\vec{a}$ orthogonal to $\vec{a}$ the φ coordinate may be established such that the $\delta\vec{a}$ vector has only one Cartesian component:

$$\delta\vec{a} = (\delta_{\perp} a, 0, 0)$$

As follows from the formulas (3.2), the expression of $\vec{r}^{\ 2}$ in terms of bi-spherical coordinates has the form

$$\vec{r}^{\ 2} = \frac{2a^2 e^{-u}}{\cosh u + \cos v}.$$

Now, evaluating the increments of the expressions (3.3) with $\delta\vec{r} = 0$ leads to the following results:

$$-2e^{-2u} L_{\perp} u = \frac{2x}{\vec{r}^{\ 2}} = 2a^{-1}e^{u}\sin v\cos\varphi;$$

$$-\sin v L_{\perp} v = \sin v \cdot a{-}1(1 + e^{-u}\cos v)\cos\varphi.$$

Extracting the Lie derivatives from the last two expressions one finds that they and the formulas (3.1) form Cauchy-Riemann–like equations for Lie derivatives of the coordinates:

$$L_{\|} u = a^{-1}(1 + e^{-u}\cos v); \quad L_{\|} v = -a^{-1}e^{-u}\sin v \tag{3.4}$$

$$L_{\perp}u = -a^{-1}e^{-u}\sin v\cos\varphi; \quad L_{\perp}v = -a^{-1}(1+e^{-u}\cos v)\cos\varphi.$$

These relations make it possible to find out the Lie derivatives of 1-forms like du. Indeed, since the operation of Lie derivation commutes with that of exterior derivation [14], the explicit form of Ldu is

$$L_{\|}du = a^{-1}d(1+e^{-u}\cos v); \quad L_{\perp}du = -a^{-1}d(e^{-u}\sin v\cos\varphi). \tag{3.5}$$

3.3 Matching the fields

As was shown above, the electric intensity produced by a charge moving with constant acceleration can be expressed in its rest frame as the following 1-form:

$$E = E_i dx^i = \frac{q}{a}(\cosh u + \cos v)du \tag{3.6}$$

where q denotes the charge value. Increments of the intensities are Lie derivatives $L_{\|}E$ and $L_{\perp}E$ of the 1-form E on the deformation of the coordinate system. As was pointed out above, both the expressions to be constructed represent sourceless electromagnetic fields. Consequently, they are to satisfy the equation

$$d*(LE) = 0 \tag{3.7}$$

i.e., $div\ \vec{E} = 0$ in the standard denotions. Below this equation will be used when checking the results out.

When evaluating the expression $L_{\|}E$ one must take into account the explicit dependence of the intensity on the parameter a (3.6), i.e. to add the usual partial derivative $\partial E/\partial a$ to the expression. Derivating the 1-form E (3.6) and substituting the Lie derivatives (3.4)(8) and (3.5) (9) one obtains

$$L_{\|}E = qa^{-2}\{[-(\cosh u + \cos v) + (1+e^{-u}\cos v)\sinh u + e^{-u}\sin^2 v -$$

$$-e^{-u}\cos v(\cosh u + \cos v)]du - e^{-u}\sin v(\cosh u + \cos v)dv].$$

This result can be simplified by the following identity:

$$(1+e^{-u}\cos v)\sinh u + e^{-u}\sin^2 v = (1-e^{-u}\cos v)(\cosh u + \cos v).$$

After that the result takes the following form:

$$L_{\|}E = -qa^{-2}e^{-u}(\cosh u + \cos v)(2\cos v du + \sin v dv). \tag{3.8}$$

The analogous evaluation in the second case yields

$$L_{\perp}E = \tag{3.9}$$

$$= qa^{-2}e^{-u}(\cosh u + \cos v)(2\sin v\cos\varphi du + \cos v\cos\varphi dv + \sin v\sin\varphi d\varphi).$$

3.4 Checking out and the final form of the results

As was pointed out above, the results may be checked out by inserting them into the equation (3.7). To do this, we start with constructing the conjugated 1-forms $*(L_\parallel E)$ and $*(L_\perp E)$. The Levi-Civita symbol components corresponding to the three-dimensional bi-spherical metric [10]

$$ds^2 = \frac{du^2 + dv^2 + \sin^2 v d\varphi^2}{(\cosh u + \cos v)^2}$$

have the following form:

$$\varepsilon^u{}_{v\varphi} = \varepsilon_u{}^v{}_\varphi = \frac{a \sin v}{\cosh u + \cos v}; \;\; \varepsilon_{uv}{}^\varphi = \frac{a}{\sin v(\cosh u + \cos v)}$$

Now, transforming the components of the 1-forms (3.8) and (3.9) one obtains the following 2-forms:

$$*(L_\parallel E) = qa^{-2}e^{-u}(-\sin^2 v du \wedge d\varphi + 2 \sin v \cos v dv \wedge d\varphi)$$
$$*(L_\perp E) = qa^{-2}e^{-u}(\sin\varphi du \wedge dv + \sin v \cos v \cos\varphi du \wedge d\varphi$$
$$+2 \sin^2 v \cos\varphi dv \wedge d\varphi).$$

It is easy to check out that both of them satisfy the equation (3.7), for example:

$$d * (L_\perp E) =$$
$$= qa^{-2}e^{-u}[-2\sin^2 v - \frac{d}{dv}(\sin v \cos v) + 1] \cos\varphi du \wedge dv \wedge d\varphi$$

By definition, the vector $\vec{a}$ is related to the acceleration vector $\vec{\kappa}$ as follows:

$$\vec{\kappa} = a^{-2}\vec{a}.$$

Denoting increments of the charge acceleration components as $\delta\vec{w}_\parallel$ and $\delta\vec{w}_\perp$ one can derive the following relations for their increments:

$$\delta\vec{w}_\parallel = -a^{-2}\delta a_\parallel; \quad \delta\vec{w}_\perp = -a^{-2}\delta a_\perp.$$

Now it is possible to find out the strength increment related to that of acceleration. Denoting them as $\varepsilon_\parallel$ and $\varepsilon_\perp$ respectively one obtains the final result in the form

$$\varepsilon_\parallel = q\delta w_\parallel e^{-u}(\cosh u + \cos v)(2 \cos v du + \sin v dv)$$
$$\varepsilon_\perp = q\delta w_\perp e^{-u}(\cosh u + \cos v)(2 \sin v \cos\varphi du - \cos v \cos\varphi dv + \sin v \sin\varphi d\varphi).$$

Apparently, for an arbitrary change of the charge acceleratrion the intensity increment is sum of ε's. It is seen that the radiation field grows as r^2 near the charge. This means that in a small enough neighbourhood of the charge the radiation field is negligeably small and cannot yield large corrections for the charge self-energy. The results of this section have been obtained in our work [15].

Chapter 4

Differential geometry of an arbitrary world line

4.1 The co-moving tetrahedron

In this section we consider Minkowskian space-time as a vector space and use the standard vector denotions for its points. Let $\vec{x}(s)$ be a time-like curve pasrametrized with its length. We introduce the moving tetrahedron $\{\vec{e}_a\},\ a = 0, ..., 3$, defined similarly to the moving trihedron for Euclidean 3-space. Thus, the unit vector $\vec{e}_0$ is tangent to the curve:

$$\vec{e}_0 = \frac{d}{ds}\vec{x}(s), \tag{4.1}$$

and the rest ones are space-like: the unit vector $\vec{e}_1$ represents the principal normal to te curve

$$\frac{d^2}{ds^2}\vec{x}(s) = -\kappa\vec{e}_1, \tag{4.2}$$

where κ is curvature of the world line; the unit vector $\vec{e}_2$ is binormal:

$$\frac{d\vec{e}_1}{ds} = -\kappa\vec{e}_0 + \tau\vec{e}_2, \tag{4.3}$$

and the unit vector $\vec{e}_3$ is orthogonal to the rest ones. The equations (4.1), (4.2)and (4.3) are nothing but the Serret-Frenet formulas for Minkowskian space-time. Unlike the standard Serret-Frenet formulas the curvature term in the equation (4.2) has opposite sign due to the Minkowskian metric. The point is that a time-like curve always is convex towards its curvature center. As for the torsion, its action is the same as in the classical case, i.e., the vector $\vec{e}_1$ rotates in the 2-plane established on the vectors $\vec{e}_1$ and $\vec{e}_2$ with angular velocity τ referred to the proper time s.

A 3-plane normal to the curve in a point $\vec{x}(s)$ is specified by the equation

$$< \vec{y} - \vec{x}(s), \vec{e}_0 >= 0$$

where $< \vec{a}, \vec{b} >$ denotes scalar product of vectors $\vec{a}$ and $\vec{b}$, and $\vec{y}$ means an arbitrary point of the plane. The center of curvature for a point $\vec{x}(s)$ lies at the distance κ^{-1} from this point on the principal normal dropped from it.Thus, if $\vec{y}(s)$ is the center of curvature for $\vec{x}(s)$ then

$$\vec{y}(s) = \vec{x}(s) + \kappa^{-1}\vec{e}_1(s) \tag{4.4}$$

By definition, the centers of curvature form a curve $\vec{y}(s)$ called the evolute of $\vec{x}(s)$. The 3-planes normal to a world line are tangential to its evolute.The intersection of such a 3-plane with an infinitesimally close one forms a 2-plane normal to the vectors $\vec{e}_0$ and $\vec{e}_1$ which is the two-dimensional characteristic of the curve in the space-time. These 2-planes constitute a three-dimensional ruled surface which forms the envelope of normal planes. The evolute lies in this surface and, likely, forms its cuspidal edge. One may check it easily that the evolute is space-like. Indeed, derivating the equation (4.4) with help of equations (4.1) and (4.3) one obtains that

$$\frac{d}{ds}\vec{y}(s) = \kappa^{-1}\tau\vec{e}_2 - \frac{d\kappa}{ds}\kappa^{-2}\vec{e}_1(s) \tag{4.5}$$

When considering an inertial moving object one introduces its world line as a time axis and family of 3-planes normal to the world line as “spaces” concerned with its frame of reference. It is possible because the 3-planes are parallel and form a chronologically ordered set. A family of 3-planes normal to an arbitrary world line do not form such a set and, hence, one cannot consider them as “spaces” concerned with non-inertial frame of reference. Chronological ordering may be completed only for halves of the 3-planes bounded by the envelope. Therefore, the notion of “space” cannot be introduced for a non-inertial motion. One can introduce only the notion of “semispace” as the half of normal 3-plane lying on one side from the envelope. Apparently, the “space” as it defined for inertial motion represents a particular case of the semispace just defined in which the the envelope is infinitely far. As was shown in Section 2, the initial conditions for the radiation field are defined only in the semispace under consideration.

The semispaces concerned with a world line is being specified by a system consisting of one equation and one inequality

$$< \vec{y} - \vec{x}(s), \vec{e}_0 >= 0 \tag{4.6}$$

$$\kappa < \vec{y} - \vec{x}(s), \vec{e}_1 >\geq -1$$

It is seen that for any given smooth enough time-like curve $\vec{x}(s)$ the semispace is defined exhaustively by the equations (4.1), (4.2) and (4.3) and the inequality (4.6). Now we shall consider a couple of simplest examples in which the evolutes and concerned semispaces will be constructed explicitly.

4.2 Examples

Hereafter t, x, y, z denotes a standard Cartesian coordinate system for Minkowskian space-time.

A world line of hyperbolic motion may be specified as follows: $t = a\sinh s/a, \quad z = a\cosh s/a$. Its evolute consists of one point lying in the origin of coordinates. The semispaces defined in the Section 2 form a family of three dimensional semiplanes $t - z\tanh s/a = 0, z = 0$. In this case the world line curvature is constant and equal to a^{-1} and the torsion is zero. As was shown in the Section 2, a charge moving so does not radiate.

Consider a world line of a point whose acceleration suffers a step-like change at $t = 0$:

$$t = \begin{cases} a\sinh s/a, & s < 0 \\ b\sinh s/b, & s > 0 \end{cases} \tag{4.7}$$

$$z = \begin{cases} a\cosh s/a, & s < 0 \\ a + b(\cosh s/b - 1)\cos\psi, & s > 0 \end{cases}$$

$$x = \begin{cases} 0, & s < 0 \\ b(\cosh s/b - 1)\sin\psi, & s > 0 \end{cases}$$

Its evolute forms a straight line segment with endpoints in the origin of cordinates and in the point $(0, b\cos\psi - a, 0, b\sin\psi)$. The semispaces draw two families of semiplanes, one specified as $t - z\tanh s/a, z > 0$ for $s < 0$ and one did as $t - (z\cos\psi + x\sin\psi + b)\tanh s/b, z > 0$ for $s > 0$. In this case the world line curvature is a^{-1} under $s < 0$ and b^{-1} under $s > 0$ and torsion behaves as $\delta(t)$. As was shown in the Section 3, in this case the radiation is being emitted only at $s = 0$.

In the case of zero torsion the world line and its evolute lies wholly in a plane, say, $x = y = 0$, as for example, the curve just considered with $\psi = 0$. Therefore, generalizing the classical theory of evolutes and evolvents to the case of pseudo-Euclidean plane gives a ready to use model of the "semispace" which is nothing but a tangent ray dropped from a point of evolute. Now x and y coordinates forms supply the plane naturally up to the Minkowskian space-time. It must be pointed out that in the case of zero torsion the semispaces concerned with the world line appear as halves of rectifying 3-planes for the evolute. In the next example it will be shown that they may also be that of osculating planes for the evolute.

Consider a world line of uniform circular motion:

$$t = s\sqrt{1 + R^2\omega^2}; \;\; x = R\cos\omega s; \;\; y = R\sin\omega s; \;\; z = 0. \tag{4.8}$$

It has constant curvature and torsion:

$$\kappa = R\omega^2; \quad \tau = \omega\sqrt{1 + R^2\omega^2}$$

and as they are constant, derivating of the equation (4.5) gives:

$$\frac{d^2}{ds^2}\vec{y} - \kappa^{-1}\tau\frac{d}{ds}\vec{e}_2.$$

Then, the scalar product

$$< \frac{d^2}{ds^2}\vec{y}, \vec{e}_0 >= -\kappa^{-1}\tau < \frac{d}{ds}\vec{e}_2, \vec{e}_0 >= \kappa^{-1}\tau < \vec{e}_2, \frac{d}{ds}\vec{e}_0 >$$

is zero due to the equation (4.2). Thus, both $d\vec{y}/ds$ and $d^2\vec{y}/ds^2$ are orthogonal to $\vec{e}_0$ and, hence, lie in the normal plane. Consequently, the 3-plane normal to the world line is osculating plane for its evolute.

In this case both the world line and its evolute are helicoids having the same helicity and with the same angular velocity referred to the coordinate time t. It is to be pointed out that the radius of the evolute helicoid is greater than that of the world line. This property of the evolute, though somewhat unexpected, reflects actually the properties of pseudo-Euclidean orthogonality. Indeed, they are these properties which require the evolute to turn the same way as the world line helicoid does, and, as it is space-like, it may have the same period only if it has greater radius. Inserting the expressions (4.8) into the equations (4.1), (4.2), (4.3)and (4.4) one founds the following equations of the evolute:

$$t = s\sqrt{1 + R^2\omega^2}; \;\; x = (R+R^{-1}\omega^{-2})\cos\omega s; \;\; y = (R+R^{-1}\omega^{-2})\sin\omega s; \;\; z = 0.$$

Due to the equation (4.3) the normal 3-planes are being specified as follows:

$$t + \frac{xR\omega}{\sqrt{1 + R^2\omega^2}}\sin\omega s - \frac{yR\omega}{\sqrt{1 + R^2\omega^2}}\cos\omega s - s\sqrt{1 + R^2\omega^2} = 0.$$

The semispace boundary, as spcified by the inequality (4.6):

$$x\cos\omega s + y\sin\omega s \leq R + R^{-1}\omega^{-2} = 0,$$

is tangent to the evolute. It is seen that the semispace concerned to the world line is stretched from this 2-plane towards the common axis of both helicoids.

4.3 Coordinate systems for the concerned semispace

By construction, the vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ constitute an orthonormal vector basis for the semispace and their directions are the same as the principal ones of the world line. Thereby, it is natural to equip the semispace with Cartesian coordinates $\{\zeta, \eta, \xi\}$ whose coordinate lines are parallel to the vectors. It is convenient to establish their origin in the intersection point of the semispace and the world line. Then the first coordinate ζ runs the ray $[-\kappa^{-1}, \infty)$ and others do the complete numerical line. Here the 2-plane $\zeta = -\kappa^{-1}$ represents the boundary of the semispace.

Another natural coordinate system for the semispace is a generalization of the spherical one. To define it let us introduce radius vectors in the space: $\vec{r} = (\zeta, \eta, \xi)$ and the vector $\vec{a}$ defined as the nearest point of the boundary to the origin of coordinates: $\vec{a} = (-\kappa^{-1}, 0, 0)$. Then the system in question denoted as $\{u, v, \varphi\}$ may be defined as follows:

$$e^{2u} = \frac{|\vec{r} + 2\vec{a}|^2}{|\vec{r}|^2}; \quad \cos v = -\frac{(\vec{r} + 2\vec{a}) \cdot \vec{r}}{|\vec{r} + 2\vec{a}||\vec{r}|} \quad \tan \varphi = \xi/\eta$$

Here the coordinate u runs the semiline $[0, \infty)$ with the value $u = 0$ corresponding to the boundary and $u = \infty$ doing to the point $\vec{r} = 0$. The rest coordinate surfaces $u = const$ are spheres such that te system constitutes the semispace restriction of the well-known bispherical coordinate system [10].

Chapter 5

The source of radiaton emitted by an arbitrarily moving charge

In the Section 3 the source of radiation was expressed through infinitesimal changes of the vector $\vec{a}$ caused by small changes of the charge acceleration. In the case of hyperbolic motion this vector is constant, otherwise it suffers longitudinal and transversal changes which yield different initial conditions for the radiation field. Due to the Serret-Frenet formula (4.3) the changes have the following form:

$$\delta_{\|}\vec{a} = -\kappa^{-2}\frac{d\kappa}{ds}\vec{e}_1 ds, \quad \delta_{\perp}\vec{a} = -\kappa^{-1}\tau\vec{e}_2 ds$$

where the term $\kappa\vec{e}_0$ does not contribute in the charge rest frame.

Consider, oncemore the simplest case of radiation emitting under a motion with world line of the form (4.8). As was mentioned above, the radiation part appears only at the moment $t = 0$ and, hence, its time derivative behaves as δ-function of time. Then, due to the Maxwell equations the radiation part may be considered as the field of a source specified as a space-like current of displacement coinciding with the initial field of electric strength on the 3-plane $t = 0$ and running in this 3-plane. Thus, although the whole of electromagnetic field of a moving charge has not other sources than the time-like current defined on its world line only and tangent to it its radiation part considered separately has its own source in the form of space-like current of displacement running in 3-planes normal to the world line. In the simplest case the current of displacement runs only the plane $t = 0$ and depends on time as the δ-function.

As was mentioned above, in the case of arbitrary motion the process of radiation emitting becomes continuous. Then, the current of displacement occures on each normal 3-plane and forms a continuous in time space-like current density. By construction, the current density consists of two parts one formed by longitudional and one did by transversal increments of the vector $\vec{a}$, i.e., $\vec{e}_1$ and $\vec{e}_2$ respectively, each of which is factorized as a (co-)vector field specified in a normal plane multiplied by functions of s, i.e.,

$\kappa^{-2}\dot{\kappa}$ and $\kappa^{-1}\tau$ respectively. The explicit form of the current density in the coordinates $\{u, v, \varphi\}$ introduced in the normal 3-planes may be found from the expressions obtained in the Section 3 (3.8) and (3.9):

$$\varepsilon_{\parallel} = q\kappa^{-2}\dot{\kappa}ds \wedge e^{-u}(\cosh u + \cos v)(2\cos v du + \sin v dv)$$

$$\varepsilon_{\perp} = q\kappa^{-1}\tau ds \wedge e^{-u}(\cosh u + \cos v)(2\sin v \cos\varphi du - \cos v \cos\varphi dv + \sin v \sin\varphi d\varphi)$$

where the 1-forms denoted as ε are non-relativistic expressions of electric field in the charge frame of reference defined as $\varepsilon = E_i ds \wedge dx^i$. As was mentioned above these initial values of the radiation part specify the form of space-like current as the source of radiation, so that the current 1-form is

$$^*I_{\parallel} = q\kappa^{-2}\dot{\kappa}e^{-u}(\cosh u + \cos v)(2\cos v du + \sin v dv)$$

$$^*I_{\perp} = q\kappa^{-1}\tau e^{-u}(\cosh u + \cos v)(2\sin v \cos\varphi du - \cos v \cos\varphi dv + \sin v \sin\varphi d\varphi)$$

where the current 1-form $^*I$ is defined as

$$^*I = J_i dx^i.$$

This expression is obtained in our work [16].

Chapter 6

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