The field of a uniformly accelerated charge, with special reference to the problem of gravitational acceleration

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Physically significant limiting processes are employed to derive the correct field of a uniformly accelerated charge. This solution is used to discuss a difficulty associated with the field of a charge statically supported against gravity.

1. The field of a uniformly accelerated charge was first discussed by Born (1909) and since then by several other authors, notably by Schott (1912) and Milner (1921). The field given by Schott differs greatly from that given by Born, which, as Milner showed, is in fact a particular field associated with a pair of oppositely charged and oppositely moving uniformly accelerated charges.

In spite of this work, most discussions (Pauli 1920; Drukey 1949) have been based on Born's solution. In the present paper the problem is considered again, and it is shown that two physically significant limiting processes having the mathematical abstraction of the (permanently) uniformly accelerated charge as limit both lead to the same solution, which differs from those previously given. As there is no reason to believe that there are any fundamentally different processes with this motion as limit, this result is taken to establish that this solution is the *only* correct description of the field of a uniformly accelerated charge.

If the velocity of light is put equal to unity and the motion of the charge is supposed to take place along the x-axis (i.e. velocity and acceleration are parallel), then the position of the charge can be represented by

$$x = \left[\frac{1}{f^2}+t^2\right]^{\frac{1}{2}}, \tag{1}$$

where f is the acceleration. The motion may be represented diagrammatically by one branch ABC of a hyperbola in the (x, t) plane (figure 1). Since the charge is always on one side of the light ray AOA', it follows that the field due to the charge must be similarly confined, i.e. that there can be no field in the space $x+t<0$.

In our notation the Born field is given by

$$\left.\begin{aligned} E_x &= -[1+f^2(t^2-x^2+y^2+z^2)]\,U, & H_x &= 0,\\ E_y &= 2f^2xyU, & H_y &= -2f^2tzU,\\ E_z &= 2f^2xzU, & H_z &= 2f^2tyU,\\ \text{where}\quad U &= 4ef^2\{[f^2(t^2-x^2-y^2-z^2)+1]^2+4f^2(y^2+z^2)\}^{-\frac{3}{2}}. \end{aligned}\right\} \tag{2}$$

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The electric lines of force at any instant are the family of circles passing through the instantaneous position of the charge and through its mirror image in the plane $x = 0$.

Although this is the field derived by the method of retarded potentials, it clearly does not satisfy the condition that the field must vanish for $x+t<0$. Nor is it possible to assume that the field is given by (2) for $x+t \geqslant 0$ and vanishes for $x+t<0$, since E_x as given by (2) does not vanish on $x+t = 0$. As E_y and E_z are finite there, the condition $\operatorname{div} \mathbf{E} = 0$ would not be satisfied. This cut-off Born field, violating Maxwell's equations, has, however, been given by Schott.

It may be worth pointing out that the field given by (2) contains a charge e moving on ABC and the image charge $-e$ on $C'B'A'$, but is *not* the field due to these two charges only, as such a field would vanish below AOC' (i.e. $t< -|x|$). This field, considered as a field associated with two charges, was discussed by Milner (1921).

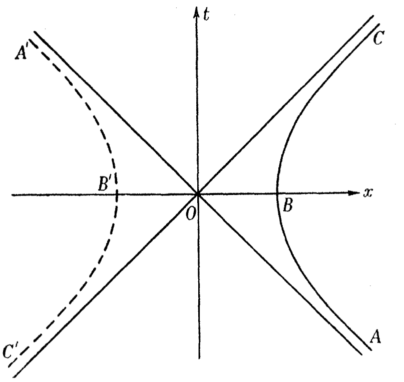

FIGURE 1

2. The failure of the method of retarded potentials to give the correct field is hardly surprising. The solution of the wave equation by retarded potentials is valid only if the contributions due to distant regions fall off sufficiently rapidly with distance. But in our case, as we consider a succession of points with t diminishing towards $-x$, while the relevant section of the trajectory of the particle recedes to infinity (at A), the velocity of approach becomes closer and closer to the velocity of light. It is easily seen that the contribution in fact diverges. This does not of course affect the validity of the Born field in the space $t+x \geqslant \epsilon > 0$. In this region the Born field is the field of a charge of maximum velocity v ($v<c$). For such cases the use of the method of retarded potentials has been fully vindicated by experiment, and this method leads uniquely to the Born field.

In order to derive the correct field of a uniformly accelerated charge, it is therefore necessary to employ some limiting process avoiding the part of the trajectory stretching to infinity at A.

Two such processes would appear to be particularly suitable:

(i) Before $t = -\tau$ the charge was moving with constant velocity, after $t = -\tau$ with constant acceleration.

(ii) At $t = -\tau$ a charge e and a charge $-e$ are simultaneously created. Charge e moves with constant acceleration, charge $-e$ in some trajectory that always remains a distance of order τ from O.

Although both methods tend to the same limit as τ tends to infinity, method (ii) seems to be more instructive. The chief reason for this is that in method (i), throughout the whole limiting process, the field extends through all space-time. The limitation of the field to one side of a wave front is such a characteristic property of the field of a uniformly accelerated charge that it would seem to be more appropriate to employ a limiting process such as (ii) in which the property applies at all stages.

3. Although, no doubt, the characteristics of classical fields due to pair creation are well understood, the absence of any explicit reference in the literature makes a brief discussion desirable.

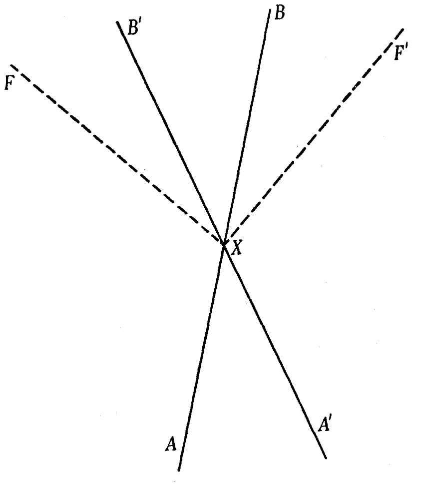

Figure 2

It is clear from the law of conservation of charge that the two members of the pair must be created at the same point at the same time. The following argument shows that the two charges must also be created with the same velocity; or at any rate that the creation with different velocity requires an additional radiation field of the type that would be associated with an infinite acceleration just after the event, in order for the resultant field to be admissible.

To show this, consider the field of a pair created with different velocities. Let X in a space-time diagram (figure 2) be the event of pair creation and let XB, XB' be the world lines describing the motions of the two charges. Since later accelerations do not concern the present argument, assume that the charges move with constant (but different) velocities, and therefore that XB, XB' are sections of straight lines.

Consider now the light cone corresponding to an event at X. In a space-time diagram this will be a cone of 45° semivertical angle (XFF' in figure 2). The field due to a pair created at X must vanish *outside* the light cone XFF'. At any point *inside* this light cone the field, as derived by the method of retarded potentials, depends only on sections of the world-lines after creation and hence later than X. The field is therefore identical with the field due to permanent charges e, $-e$, moving like the created pair after X and in any manner before X, say each with the same constant velocity as after X. The world-lines of these charges are AXB, $A'XB'$. This field is easily calculated, but cannot be combined (without recourse to infinities) with the previous requirement of zero field outside the light cone. The normal electric intensity of the 'inside' field does not vanish just inside the wave front XFF', while it would have to vanish just outside. The condition that the divergence of the electric vector should vanish can therefore not be satisfied, and the combination of fields is therefore inadmissible. (This conclusion is changed only by the presence at the wave front of a δ-function field such as would be due to an infinite acceleration having been suffered by one of the charges immediately after the event at X.)

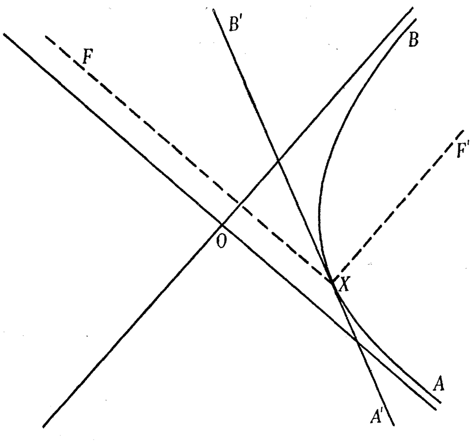

FIGURE 3

Next, consider pair creation with the same velocity, but different accelerations. The simplest case is given by a uniformly accelerated particle $+e\,(XB)$ and a particle $-e$ moving with constant velocity XB' (figure 3). In order to examine the effects of pair creation, consider zero field outside the light cone XFF' and the sum of the field of $A'XB'$ and the Born field of AXB inside the light cone. The use of the Born field is permissible since only regions with $t+x \geqslant \epsilon > 0$ are considered within the light cone. It can readily be shown that the normal component of the electric field vanishes on the light cone, and indeed that all the necessary conditions are satisfied on the wave front. This follows directly from the fact that all the components of the 4-potential vanish linearly on the wave front. The field is therefore the correct one.

It is easily seen by superposition that the field of any pair created with the same velocity will be given in the same way even if both are accelerated.

4. A model for a suitable limiting process may now be set up (figure 4). AXB is a world line of constant acceleration f and $A'XB'$ is a world line of constant acceleration g. The two world lines have a common tangent at X, and g is greater than f. Now let a pair of charges e, $-e$ be created at X, e moving along XB and $-e$ along XB'.

Let $-\tau$ be the time co-ordinate of X. The field of the two charges is then given by the superposition of their Born fields inside the light cone XFF', and it vanishes outside. If τ tends to infinity, with g fixed or increasing, the field approximates to that of the single uniformly accelerated charge e.*

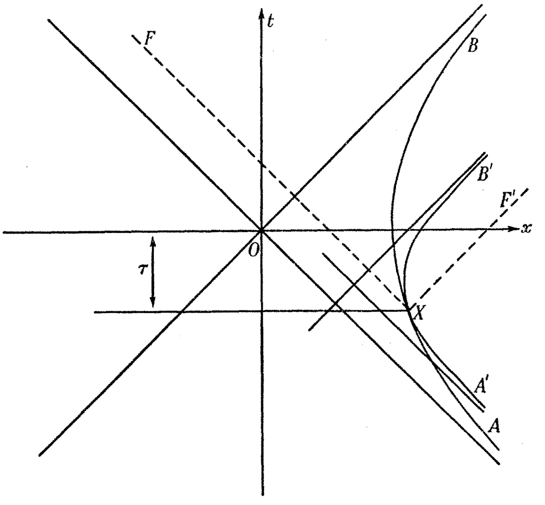

FIGURE 4

The calculations are readily made and are given in the appendix. If τ is large, the field of the second charge is negligible except near the wave front XF. Near the wave front the effect of the field of the second charge is to reduce the normal electric field of $+e$ to nil and to produce strong transverse electric fields and corresponding magnetic fields. In the limit the transverse fields become Dirac δ-functions. The complete field of a uniformly accelerated charge then becomes, for $t+x \geqslant 0$:

$$\left.\begin{aligned} E_x &= -[1+f^2(t^2-x^2+y^2+z^2)]\,U, \\ E_y &= y[2f^2xU+W], \\ E_z &= z[2f^2xU+W], \\ H_x &= 0, \\ H_y &= -z[2f^2tU+W], \\ H_z &= y[2f^2tU+W], \\ U &= 4ef^2\{[f^2(t^2-x^2-y^2-z^2)+1]^2+4f^2(y^2+z^2)\}^{-\frac{3}{2}}, \\ W &= -2ef^2[1+f^2(y^2+z^2)]^{-1}\,\delta(x+t), \end{aligned}\right\} \quad (3)$$

while for $t+x<0$ the field vanishes.

* In the present paper the forces responsible for the motions of the charges are not investigated. This is appropriate in a study of Maxwellian electrodynamics, which admits of other forces. If this were not so, and if an inconsistency could be introduced in that way, it would imply that Maxwellian electrodynamics itself was capable of making statements about the construction of matter. If a more physical picture is desired, a system of static charges may be found producing the desired motions, or alternatively these may be thought of as being due to the interplay of nuclear forces.

5. It has been argued on the basis of the Born field that a uniformly accelerated charge does not radiate (Pauli 1920) and the opposite point of view has also been held (Drukey 1949).

The two arguments in favour of the non-radiation hypothesis were:

(i) That the Born field is symmetric between past and future.

(ii) That at $t = 0$, i.e. when the charge is instantaneously at rest, the magnetic field vanishes everywhere and hence the Poynting vector vanishes. Since a Lorentz transformation can reduce any point of the world line to rest, the field simply moves with the particle.

It will be seen that with our field (3) neither of these properties applies. From the present work it appears therefore that a radiation field is necessarily associated with the accelerating charge. It is not generally permissible, of course, to consider a dissected path and apportion the radiative contribution of each part, but it must be pointed out that in our field (3) all the travelling energy at $t = 0$ is in the δ-function wave at a distance of f^{-1} from the charge. It may be worth mentioning here that the following is a known property of the Lorentz transformation: if a uniformly accelerated point is chasing a point travelling at the speed of light, and if there is a distance f^{-1} between them, as reckoned in one Lorentz frame which is instantaneously coincident in velocity with the accelerated point, then the same distance will be measured in all subsequent such Lorentz frames. (The rate at which the point moving at the speed of light gains on the accelerated point is just balanced by the increasing contraction of the successive Lorentz frames.) The system of charge and radiation field is therefore similar in successive Lorentz frames; but it could not fairly be said that the radiation field moves with the charge.

6. The discussion given in the present paper has an interesting application to a problem in the theory of gravitation. The principle of equivalence states that it is impossible to distinguish between the action on a particle of matter of a constant acceleration or of static support in a gravitational field (Einstein 1923). This might be thought to raise a paradox when a charged particle, statically supported in a gravitational field, is considered, for it might be thought that a radiation field is required to assure that no distinction can be made between the cases of gravitation and acceleration. But if the charge had an associated radiation field, this would reach distant regions unaffected by the gravitating body, where Maxwell's equations apply without any modification due to gravitational effects. According to these equations there can be no static radiation field; and as the whole system is static the electromagnetic field cannot depend upon time.

There is, however, no paradox where the principle of equivalence is considered with its limitations. The surveying of a sufficiently large region can always reveal the presence of a gravitational field through the inhomogeneity that is associated with it, which is absent for an acceleration field. The presence or absence of radiation from the charge could only be established by surveying the space out to a distance f^{-1} from the charge; but at that distance the presence of a gravitational field can in general be inferred from its inhomogeneity, and there is hence no requirement for the electromagnetic effects to conceal the distinction between the two cases. In the special case where a gravitational field is homogeneous over the entire distance f^{-1}, the potential difference is c^2 over that interval. That case is known to result in several anomalies (cf. Eddington 1930; Curtis 1950), and it appears from other considerations that it must be excluded from any physical discussion. For this reason we do not intend to deal with this case here, although we appreciate that it is sometimes instructive to enquire into the way in which paradoxes arise when conditions are assumed that have no counterpart in the physical world.

We are grateful to the referee for drawing our attention to the relation of the formulae for the field derived here to those obtained earlier by Schott.

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Appendix

The calculation of the field of the charge $-e$ may be made by using the Born field (2). The world line of $-e$ is

$$x = x_0 + \left[\frac{1}{g^2} + (t+t_0)^2\right]^{\frac{1}{2}}, \tag{A 1}$$

where $(-t_0, x_0)$ are the co-ordinates of the centre of the hyperbola that represents the world line of the charge, and g is its acceleration. The condition that the curves (1) and (A 1) touch at $t = -\tau$ determines t_0 and x_0 to be

$$x_0 = \left(1 - \frac{f}{g}\right)\left[\frac{1}{f^2} + \tau^2\right]^{\frac{1}{2}}, \qquad t_0 = \left(1 - \frac{f}{g}\right)\tau \qquad (g > f). \tag{A 2}$$

Since the co-ordinates of X are $\left[-\tau, \left(\frac{1}{f^2} + \tau^2\right)^{\frac{1}{2}}, 0, 0\right]$, the equation of the light cone FXF' (figure 4) is

$$(t+\tau)^2 = \left[x - \left(\frac{1}{f^2} + \tau^2\right)^{\frac{1}{2}}\right]^2 + y^2 + z^2. \tag{A 3}$$

For given t, y, z, let the smaller value of x determined by (A 3) be called x_w. The point (t, x_w, y, z) is on the part of the light cone referred to as XF in figure 4. Since τ will be made to tend to infinity, while t, x, y, z are finite, $XF'$is of no interest. Also we may change co-ordinates to (t, ξ, y, z), where

$$\xi = \xi(t, x, y, z) = x - x_w(t, y, z). \tag{A 4}$$

We shall only be concerned with $\xi \geqslant 0$, since the field vanishes outside FXF'.

If the expression U of equation (2) corresponding to $-e$ is called $\tilde{U}$ (and similarly we shall have $\tilde{E}$, etc.), then

$$\tilde{U} = -4eg^2\{\{g^2[(t+t_0)^2-(x-x_0)^2-y^2-z^2]+1\}^2+4g^2(y^2+z^2)\}^{-\frac{3}{2}}. \tag{A 5}$$

Now

$$\begin{aligned}(t+t_0)^2-(x_0-x)^2 &= (t+t_0)^2-(x_0-x_w-\xi)^2\\ &= \left[\left(1-\frac{f}{g}\right)\tau+t\right]^2-\left\{\left(1-\frac{f}{g}\right)\left(\frac{1}{f^2}+\tau^2\right)^{\frac{1}{2}}-\left(\frac{1}{f^2}+\tau^2\right)^{\frac{1}{2}}+[(\tau+t)^2-y^2-z^2]^{\frac{1}{2}}-\xi\right\}^2\\ &= \left[\left(1-\frac{f}{g}\right)\tau+t\right]^2-\left\{\left(1-\frac{f}{g}\right)\left(\tau+\frac{1}{2f^2\tau}\right)-\left(\tau+\frac{1}{2f^2\tau}\right)+\tau+t-\frac{y^2+z^2}{2\tau}-\xi+O\left(\frac{1}{\tau^2}\right)\right\}^2\\ &= 2\xi\tau\left(1-\frac{f}{g}\right)+2\xi t-\xi^2+\left(1-\frac{f}{g}\right)\left(y^2+z^2+\frac{1}{fg}\right)+O\left(\frac{1}{\tau}\right)\end{aligned}$$

for large τ.

Accordingly

$$\tilde{U} = -\frac{4e}{g}\left\{\left[2\xi\tau(g-f)+g\xi(2t-\xi)-f(y^2+z^2)+\frac{1}{f}\right]^2+4(y^2+z^2)\right\}^{-\frac{3}{2}}, \tag{A 6}$$

neglecting terms that tend to zero as τ tends to infinity. It will be observed that the expression in curly brackets does not vanish in the region in question, since ξ is non-negative and since we are not interested in values of ξ of order τ, as these correspond to regions close to XF' in figure 4. It is clear then that, for large τ, $\tilde{U}$ is small unless ξ is small. Hence the term $g\xi(2t-\xi)$ in (A 6) may be neglected. Accordingly

$$\left.\begin{aligned}\tilde{E}_x &= 4e\left[2\xi\tau(g-f)+(2g-f)(y^2+z^2)+\frac{1}{f}\right]\left\{\left[2\xi\tau(g-f)-f(y^2+z^2)+\frac{1}{f}\right]^2+4(y^2+z^2)\right\}^{-\frac{3}{2}},\\ \tilde{H}_x &= 0,\\ \frac{1}{y}\tilde{E}_y = -\frac{1}{y}\tilde{H}_z = \frac{1}{z}\tilde{E}_z = \frac{1}{z}\tilde{H}_y &= 8eg\left[\tau\left(1-\frac{f}{g}\right)+t\right]\left\{\left[2\xi\tau(g-f)-f(y^2+z^2)+\frac{1}{f}\right]^2+4(y^2+z^2)\right\}^{-\frac{3}{2}},\end{aligned}\right\} \tag{A 7}$$

to the same order of approximation.

It will be seen that the variables ξ and τ enter E_x only in the combination $\xi\tau$, whereas the leading term of the transverse field variables is the product of τ and a function of $\xi\tau$ (and of y and z).

For fixed τ, all the functions become small if $\xi\tau$ is large. For large τ, the fields are therefore negligible outside a narrow zone just behind the wave front, but while the maximum value of $\tilde{E}_x$ is bounded, the maxima of the other field variables are proportional to τ. Therefore the only effect on the total E_x field will be to produce the step function that is already known to occur at the wave front. The effect of $-e$ on the transverse field will, however, be large though localized. Now

$$\int_0^\infty \tilde{E}_y \,\mathrm{d}\xi = 4ey \int_0^\infty \frac{2(g-f)\,\mathrm{d}(\xi\tau)}{\left\{\left[2(g-f)\,\xi\tau - f(y^2+z^2) + \dfrac{1}{f}\right]^2 + 4(y^2+z^2)\right\}^{\frac{3}{2}}}$$

$$= \frac{ey}{y^2+z^2}\left[1 + \frac{f(y^2+z^2) - \dfrac{1}{f}}{f(y^2+z^2) + \dfrac{1}{f}}\right]_0^\infty = \frac{2eyf^2}{1+f^2(y^2+z^2)}.$$

This expression is independent of τ. Since $\tilde{E}_y$ is non-negligible only in a narrow zone, $\tilde{E}_y$ is a multiple of a δ-function. Formulae (3) follow.

It should also be pointed out that in checking the vanishing of the normal electric field on the wave front, the deviation of the normal from the x-direction must be allowed for, since $\tilde{E}_y$ is of higher order than $\tilde{E}_x$.