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INTRODUCTION OF A RIEMANNIAN GEOMETRY ON A TRIANGULABLE 4-MANIFOLD[1]

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In this note, the writer completes the establishment of the following result.

Theorem. *If M is a triangulable m-manifold, $m \leqq 4$, then there exists an analytic Riemannian manifold homeomorphic to M.*

Since every analytic, indeed every differentiable, manifold is known to be triangulable,[2] this theorem permits us to say that the classes of triangulable m-manifolds and of analytic Riemannian manifolds are topologically equivalent ($m \leqq 4$) in the sense that an arbitrary member of either class can be mapped topologically on some member of the other class.

A topological equivalence theorem for two classes of manifolds, or other objects, one of which has a more restricted definition than the other, is of obvious potential usefulness, for it may carry over to the less restricted class certain results previously proved for the more restricted. For an example, note the application below of a topological equivalence theorem due to Whitehead.

In an earlier paper,[3] the writer obtained sufficient conditions for the possibility of introducing an analytic Riemannian geometry on a topological manifold of any dimensionality [H, Theorems I and V]. For the case of a triangulable m-manifold, $m \leqq 3$, the fulfillment of these conditions is there verified [H, Theorem III and §11]. For the case of a 4-dimensional Brouwer manifold, the problem of verifying these conditions is reduced to a certain deformation problem [H, p. 808] pertaining to geodesic triangulations of a 2-sphere. The writer has since generalized and solved this problem.[4] Hence the present theorem is justified with the restriction, in the case $m = 4$, that M have a Brouwer triangulation. This restriction, however, can be removed, as a consequence of a recent theorem due to Whitehead.[5] Although non-Brouwer triangulated manifolds had been shown to exist,[6] Whitehead proved that every triangulated manifold can be

[1] Presented in part to the American Mathematical Society, October 25, 1941, under the title *Topological mapping of a Brouwer 4-manifold on an analytic Riemannian 4-manifold* (Bulletin of the Society, vol. 47, 1941, p. 711).

[2] S. S. Cairns, *Triangulation of the manifold of class one*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 549–552.

[3] S. S. Cairns, *Homeomorphisms between topological manifolds and analytic Riemannian manifolds*, these Annals, vol. 41 (1940), pp. 796–808. This paper is hereafter referred to as H.

[4] *Isotopic deformations of geodesic complexes on the 2-sphere and on the plane*, these Annals, immediately preceding the present article.

[5] J. H. C. Whitehead, *Note on manifolds*, Quarterly Journal of Mathematics, Oxford series, 12 (1941), pp. 26–29.

[6] S. S. Cairns, *Triangulated manifolds which are not Brouwer manifolds*, these Annals, vol. 41 (1940), pp. 792–795. This reference also contains definitions of all the types of manifolds involved in the present discussion. In the terminology there used, we note that

subdivided into a Brouwer manifold. Hence statement (A) at the end of reference H holds for all triangulable manifolds, and the argument is complete, in view of the solution of the deformation problem.[4]

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Whitehead's theorem establishes the topological equivalence of triangulable, star, and Brouwer manifolds, reducing the writer's hierarchy of five classes of manifolds to topological, triangulable ($\approx$ star $\approx$ Brouwer) and differentiable ($\approx$ analytic $\approx$ analytic Riemannian), where $\approx$ indicates topological equivalence. All the classes are known to be equivalent for dimensions less than three, and the triangulable are now known to be equivalent to the differentiable for dimensions less than five.