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CR REGULAR EMBEDDINGS AND IMMERSIONS OF 6-MANIFOLDS INTO COMPLEX 4-SPACE

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ABSTRACT. We provide necessary and sufficient conditions in terms of characteristic classes for a closed smooth orientable 6-manifold to admit a CR regular immersion/embedding into $\mathbb{C}^4$.

1. Introduction and main results

Consider a closed smooth real $2n$-manifold M of real dimension $\dim_{\mathbb{R}}(M) = 2n$, a complex manifold (X, J) of complex dimension $\dim_{\mathbb{C}}(X) = n+1$, an immersion

$$f : M \to X, \tag{1}$$

and the bundle

$$f_*T_pM \cap Jf_*T_pM \subset T_pM. \tag{2}$$

A point $p \in M$ is said to be CR regular provided that

$$\dim_{\mathbb{C}}(f_*T_pM \cap Jf_*T_pM) = n - 1. \tag{3}$$

The points of M whose complex tangent space has complex dimension equal to n are called complex or CR singular (see Section 2.2, [S13, S15] and the references there for further details).

Definition 1. An immersion (embedding) $f : M \to X$ for which every point $p \in M$ is CR regular is said to be a CR regular immersion (embedding). An embedding is denoted by $f : M \hookrightarrow X$.

Slapar [S15, Theorem 1.1] has shown that a closed real orientable 4-manifold admits a CR regular immersion into $\mathbb{C}^3$ if and only if its first Pontrjagin class p_1 and its Euler characteristic χ are zero. In particular, such a 4-manifold admits an almost-complex structure and its Euler class equals its second Chern class c_2. Slapar also showed that the existence of a CR regular embedding requires the vanishing of the second Stiefel-Whitney class ω_2. The purpose of this note is to determine necessary and sufficient conditions for the existence of such an immersion/embedding of smooth closed 6-manifolds into $\mathbb{C}^4$. The precise statement of our main result is as follows.

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Theorem 1. *Let M be a closed smooth real orientable 6-manifold with torsion-free homology and second Stiefel-Whitney class equal to zero.*

(A) There is a CR regular immersion

$$M \to \mathbb{C}^4 \tag{4}$$

if and only if $\chi(M) = 0 = p_1(M) + X^2$ for some $X \in 2H^2(M;\mathbb{Z})$.

(B) There is a CR regular embedding

$$M \hookrightarrow \mathbb{C}^4 \tag{5}$$

if and only if $\chi(M) = 0 = p_1(M)$.

A manifold as in the hypothesis of Theorem 1 admits an almost-complex structure and the identities $\chi(M) = \langle c_3(M), [M]\rangle$ and $2c_2(M) = p_1(M) + X^2$ hold [W66, Theorem 9] (cf. Section 3.1). Applications of our theorem are given in the next result. The connected sum of two manifolds X and Y is denoted by $X\#Y$ and the connected sum of k copies of X by $k(X)$. The symbols S^m and Σ_g stand for the m-dimensional sphere and a closed smooth orientable real 2-manifold of genus g, respectively.

Corollary 1. *(A) The real 6-manifolds*

$$(k-1)(S^4 \times S^2)\#k(S^3 \times S^3) \tag{6}$$

admit a CR regular embedding into $\mathbb{C}^4$ for every $k \in \mathbb{N}$.

(B) Let N be a closed smooth orientable real 5-manifold with torsion-free homology. The product manifold

$$N \times S^1 \tag{7}$$

admits a CR regular embedding into $\mathbb{C}^4$ if and only if $\omega_2(N) = 0$.

(C) Let G be a finitely presented torsion-free group. There exists a closed smooth orientable real 6-manifold

$$M(G \times \mathbb{Z}) \tag{8}$$

whose fundamental group is $G \times \mathbb{Z}$ and which admits a CR regular embedding into $\mathbb{C}^4$.

(D) Let N be a closed smooth real 4-manifold with torsion-free homology and zero second Stiefel-Whitney class, and suppose $g \neq 1$. The product manifold

$$N \times \Sigma_g \tag{9}$$

admits a CR regular embedding into $\mathbb{C}^4$ if and only if the tangent bundle TN is trivial.

(E) There exists a finite set $\{M_i : i \in \{1, \ldots, k\}\}$ of cardinality $k > 1$ that consists of closed smooth orientable real 6-manifolds that are homotopy equivalent and admit CR regular embeddings

$$f_i : M_i \hookrightarrow \mathbb{C}^4 \tag{10}$$

such that f_i is not isotopic to f_j for $i \neq j$ for $j \in \{1, \ldots, k\}$.

Corollary 1 exhibits severe differences between the six dimensional scenario and the four dimensional one that had been previously studied in [S15], and we mention some of them now. While no closed real simply connected 4-manifold admits a CR regular immersion into $\mathbb{C}^3$ [S15, Example 1.1], the real 6-manifolds with trivial fundamental group in (6) can be CR regularly embedded into complex 4-space. A necessary and sufficient condition for a closed real orientable 4-manifold to admit such an embedding is for its tangent bundle to be trivial [S15, Corollary 1.1], which in turn restrains the possible choices of fundamental groups. Items (B) and (C) of Corollary 1 display a myriad of choices of fundamental groups of 6-manifolds that can be CR regularly embedded into $\mathbb{C}^4$. Moreover, Item (E) is a consequence of inequivalent smooth structures up to diffeomorphism within a homeomorphism class [W99].

The organization of the paper is as follows. Our proof of Theorem 1 consists of two steps. The first step is establishing necessary and sufficient topological conditions for a 6-manifold to be immersed or embedded into $\mathbb{R}^8$. This step relies on results of C. T. C. Wall, which are stated and discussed in Section 2.1. The second step consists of showing that, under our hypothesis, such an immersion/embedding can be perturbed into a CR regular one. A sufficient and necessary condition for the second step to be implemented is given by a result of M. Slapar, which we present in Section 2.2 along with a discussion on its background. Corollary 1 is proven in Section 3.2.

2. Background results

2.1. Immersing and embedding 6-manifolds into $\mathbb{R}^8$.

Results of C. T. C. Wall regarding immersions and embeddings of real 6-manifolds into Euclidean 8-space are a fundamental ingredient in the proof of Theorem 1. We summarize them in the following statement.

Theorem 2 ([W66, Theorems 10, 12 and 13]). *Suppose M is a closed smooth real oriented 6-manifold with torsion-free homology and $\omega_2(M) = 0$.*

(i) There is an immersion

$$M \to \mathbb{R}^8 \tag{11}$$

if and only if $p_1(M) + X^2 = 0$ for some $X \in 2H^2(M;\mathbb{Z})$.

(ii) There is an embedding

$$M \hookrightarrow \mathbb{R}^8 \tag{12}$$

if and only if $p_1(M) = 0$.

We conclude the section by shedding some light on Theorem 2 given its role in the proof of our main result. We follow the discussion in [W66]. The existence of a codimension two embedding of a manifold into an Euclidean space implies the vanishing of its second Stiefel-Whitney class [MS74, Corollary 11.4]. Hirsch-Smale theory states in full generality that the existence of an immersion $f : M \to \mathbb{R}^{m+k}$ (where $\dim_{\mathbb{R}}(M) = m$) is equivalent to the existence of a k-bundle V over M such that $TM \oplus V$ is trivial. In particular, an immersion as in the hypothesis of Theorem 1 requires the existence of an SO_2-bundle. The topological obstructions for such a V to exist lie within the cohomology groups

$$H^2(M;\pi_1(SO, SO_2)) \quad \text{and} \quad H^4(M;\pi_3(SO, SO_2)).$$

The second cohomology group provides the obstruction $W_3(M) = 0$. Fix a choice of immersion and of a normal 2-plane γ over the 3-skeleton with Euler class X. In the case of an embedding, γ is the normal bundle of M with structure group SO_2, and it has the structure of a vector bundle. The square X^2 is the Pontrjagin class of the extension of γ over the 4-skeleton. For this extension to be inverse to TM, it is required $X^2 + p_1(TM) = 0$. Given that $\pi_3(SO, SO_2) \cong \pi_3(SO) \cong \mathbb{Z}$, this condition is sufficient for the obstruction to vanish. An element $X \in 2H^2(M;\mathbb{Z})$ defines a normal 2-plane bundle γ that is inverse to TM on the 3-skeleton if and only if $\omega_2(M) + \omega_2(\gamma) = 0$. The element X reduces modulo 2 to $\omega_2(\gamma) = 0 = \omega_2(M)$.

2.2. **Complex points.** Slapar [S13] has studied topological obstructions to deform a generic immersion/embedding that may have complex or CR singular points into a CR regular immersion/embedding. Before quoting Slapar's result, we describe the context of the invariants following the exposition in [S13, Introduction]. We first discern among complex points as follows. Let f, M, and X be as in the introduction, and let $p \in M$ be a complex point. A generic immersion/embedding will have isolated complex points. We first discern among complex points as follows. Once an orientation on M is pinned down, the orientation of T_pM can be compared with the induced orientation of $T_pM \subset TX$ as a complex subspace. If the two orientations agree, we say that the complex point p is positive. Otherwise, we say that p is negative. Next consider coordinates $(z, \omega) \in \mathbb{C}^n \times \mathbb{C}$, and n by n matrices A, B with complex entries and such that $B = B^T$. An appropriate choice of coordinates (z, w) and Taylor expansion of f allows for a local expression of $p \in M$ as

$$w = \overline{z}^T Az + \mathrm{Re}(z^T Bz) + o(|z|^2). \tag{13}$$

Complex points can be then classified in terms of the sign of the determinant of the associated matrix

$$\begin{pmatrix} A & \overline{B} \\ B & \overline{A} \end{pmatrix}.$$

The corresponding complex point is said to be elliptic if the determinant is positive. If the determinant is negative, the complex point is said to be hyperbolic. The reader is directed toward [S13, S15] and the references there for details.

We now move into our discussion of the aforementioned topological obstruction. Denote by $e_\pm(M)$ the number of positive/negative elliptic complex points and by $h_\pm(M)$ the number of positive/negative hyperbolic complex points on M. The Lai indices are defined as

$$I_\pm(M) := e_\pm(M) - h_\pm(M) \tag{14}$$

and they are invariant under regular homotopies of immersions/embeddings. Their vanishing $I_\pm(M) = 0$ is a necessary condition for the existence of a regular homotopy between f and a CR regular immersion/embedding. These indices can be computed using the following topological formula [L72]:

$$2I_\pm(M) = \chi(M) + \Big\langle \sum_{k=0}^{n} (\pm 1)^{k+1} e^k(\nu(M)) \cup c_{n-k}(TX|_M), [M] \Big\rangle, \tag{15}$$

where $\nu(M)$ stands for the normal bundle of $M \to X$, e and c_{n-k} are the Euler and $(n-k)$th Chern classes, respectively.

The result of Slapar that is needed in our proof of Theorem 1 can be stated as follows. It is corollary of the Cancellation theorem (cf. [S13, Corollary 1.2], [S15, Proposition 4]).

Proposition 1 (Slapar [S13, S15])**.** *Let M be a closed smooth real and oriented 6-manifold, and let X be a complex manifold of $\dim_{\mathbb{C}}(X) = 4$ equipped with a Riemannian metric h. Suppose $f : M \to X$ is a smooth generic immersion (embedding), and $\epsilon > 0$. If $I_\pm(M) = 0$, then there is a regular homotopy (isotopy)*

$$f_t : M \times [0,1] \to X \tag{16}$$

that satisfies

(i) $h(f_t(p), f(p)) < \epsilon$ for every $t \in [0,1]$ and every point $p \in M$, and
(ii) $f_1 : M \to X$ is a CR regular immersion (embedding).

Without loss of generality, it is assumed in the statement of Proposition 1 that f is a generic immersion for its complex points (if it has any) to be isolated. The proof of the proposition is immediate from [S13, Theorem (Cancellation theorem)].

3. Proofs

3.1. **Proof of Theorem 1.** A closed smooth real orientable 6-manifold with torsion-free homology M admits an almost-complex structure if and only if the image of ω_2 under the Bockstein homomorphism $\beta : H^2(M;\mathbb{Z}/2) \to H^3(M;\mathbb{Z})$ vanishes, i.e., $\beta\omega_2(M) = 0$. The latter condition amounts for the second Stiefel-Whitney class to be $\omega_2(M) = c_1(L) \bmod 2$ for a complex line bundle L over M. In particular, the manifold M has a $\mathrm{Spin}^{\mathbb{C}}(6)$-structure and homotopy classes of almost-complex structures are in one-to-one correspondence with integral lifts $W_2 \in H^2(M;\mathbb{Z})$ of ω_2 [W66, Theorem 9]. For any 6-manifold M as in the hypothesis of Theorem 1 its Euler characteristic is equal to $\chi(M) = \langle c_3(M), [M]\rangle = \int_M c_3$, and its first and second Chern classes satisfy $c_1(M) = W_2$ and $2c_2(M) = p_1(M) + c_1^2(M)$.

We now show that the conditions are necessary (cf. [S15, Section 1]). Suppose that $f : M \to \mathbb{C}^4$ is a CR regular immersion so that

$$V = f^*(f_*TM \cap Jf_*TM) \to M \tag{17}$$

is a complex bundle of rank two (see Section 3) over the 6-manifold M. Denote by $E \to M$ the orthogonal complement of V in TM so that the tangent bundle is expressed as $E \oplus V$. In particular, E is an $SO(2) \cong U(1)$-bundle. Theorem 2 states that $c_2(TM) = c_2(E \otimes \mathbb{C} \oplus V) = 0$ and since the bundle

$$E \otimes \mathbb{C} \oplus V = f^*T\mathbb{C}^4 \tag{18}$$

is trivial, straightforward computations of Chern classes [MS74, p. 104] show $\chi(M) = c_3(TM) = c_3(E \otimes \mathbb{C} \oplus V) = 0$. Thus, the existence of a CR regular immersion $M \to \mathbb{C}^4$ implies $c_3(M) = 0$ and $p_1(M) + c_1^2(M) = 0$. Regarding the case of a CR regular embedding, we proceed as follows. The normal bundle of a closed oriented real manifold that embeds into an Euclidean space as a codimension two submanifold is trivial and its Euler class is zero [MS74, Corollary 11.4]. For a CR regular embedding $f : M \hookrightarrow \mathbb{C}^4$ we conclude that $c_3(M) = 0 = p_1(M)$ holds.

Regarding sufficiency of the conditions, we argue as follows. Theorem 2 implies that there exists an immersion $f' : M \to \mathbb{C}^4$. Proposition 1 will imply our main result once we show that both Lai indices are zero. From (15) it follows that the Lai indices $I_\pm$ are given by the sum of $c_3(M)$ and

$$\big\langle(\pm)c_3(TX|_M) + e(\nu(M)) \cup c_2(TX|_M) + (\pm 1)^3 e^2(\nu(M)) \cup c_1(TX|_M) + e^3(\nu(M)), [M]\big\rangle. \tag{19}$$

Since $TX|M = TM \oplus \nu(M)$ and $\gamma = \nu(M)$ as mentioned in Section 2.1, in the case of $X = \mathbb{C}^4$, we conclude $I_\pm(M) = 0$ from (19). Proposition 1 implies the existence of a regular homotopy between f' and a CR regular immersion. The claim regarding existence of a CR regular embedding follows from a verbatim argument. This concludes the proof of our main result. □

3.2. **Proof of Corollary 1.** The claims in Item (A) and Item (B) follow from Theorem 1 and straightforward calculations of the required characteristic classes. We now prove the claim in Item (C). A classical result of M. Dehn states the existence of a closed smooth orientable real 5-manifold $N(G)$ whose fundamental group is G. The manifold $N(G)$ is constructed from a connected sum $k(S^4 \times S^1)$ by applying surgery to the loops representing the generators of the free group $\pi_1(k(S^4 \times S^1))$, and in particular $N(G)$ has zero second Stiefel-Whitney class. Define $M(G, \mathbb{Z}) := Y(G) \times S^1$, whose Euler characteristic and first Pontrjagin class are zero. Its fundamental group is $G \times \mathbb{Z}$. The existence of the embedding $f : M(G \times \mathbb{Z}) \hookrightarrow \mathbb{C}^4$ follows from Item (B) of Theorem 1.

The proof of the claim in Item (D) is a direct consequence of the product formulas for characteristic classes [MS74] for $X \times \Sigma_g$ and the fact that the invariants $\{c_2, p_1, \omega_2\}$ completely classify SO_4-bundles over a 4-manifold.

Remark on the hypothesis $g \neq 1$: Let $T^2 := \Sigma_1$ be the real 2 dimensional torus. The Euler characteristic, second Stiefel-Whitney class and first Pontrjagin classes of the 6-manifold $N \times T^2$ are zero for every closed smooth real orientable spin 4-manifold N.

Finally, we present an argument to prove the claim in Item (E). There are real 6-manifolds $\{M_i : i \in \{1, \dots, k\}\}$ with $k > 3$ that are homotopy equivalent to the 6-torus $T^6 = S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1$; although they are homeomorphic to T^6, they are pairwise non-diffeomorphic [W99, Chapter 15A]. The existence of the CR regular embedding f_i follows from Theorem 1. The corresponding normal bundles are different for $i \neq j$ since M_i is not diffeomorphic to M_j, hence f_i is not (smoothly) isotopic to f_j. □

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