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## VOLUME PRESERVING EMBEDDINGS OF OPEN SUBSETS OF $\mathbb{R}^n$ INTO MANIFOLDS

FELIX SCHLENK

(Communicated by Jozef Dodziuk)

ABSTRACT. We consider a connected smooth  $n$ -dimensional manifold  $M$  endowed with a volume form  $\Omega$ , and we show that an open subset  $U$  of  $\mathbb{R}^n$  of Lebesgue measure  $\text{Vol}(U)$  embeds into  $M$  by a smooth volume preserving embedding whenever the volume condition  $\text{Vol}(U) \leq \text{Vol}(M, \Omega)$  is met.

### 1. INTRODUCTION

Consider a connected smooth  $n$ -dimensional manifold  $M$  with or without boundary. A volume form on  $M$  is a smooth nowhere vanishing differential  $n$ -form  $\Omega$ . It follows that  $M$  is orientable. We orient  $M$  such that  $\int_M \Omega$  is positive, and we write  $\text{Vol}(M, \Omega) = \int_M \Omega$ . We endow each open (not necessarily connected) subset  $U$  of  $\mathbb{R}^n$  with the Euclidean volume form

$$\Omega_0 = dx_1 \wedge \cdots \wedge dx_n.$$

A smooth embedding  $\varphi: U \hookrightarrow M$  is called volume preserving if

$$\varphi^* \Omega = \Omega_0.$$

Then  $\text{Vol}(U, \Omega_0) \leq \text{Vol}(M, \Omega)$ . In this note we prove that this obvious condition for the existence of a volume preserving embedding is the only one.

**Theorem 1.** *Consider an open subset  $U$  of  $\mathbb{R}^n$  and a smooth connected  $n$ -dimensional manifold  $M$  endowed with a volume form  $\Omega$ . Then there exists a volume preserving embedding  $\varphi: U \hookrightarrow M$  if and only if  $\text{Vol}(U, \Omega_0) \leq \text{Vol}(M, \Omega)$ .*

If  $U$  is a bounded subset whose boundary has zero measure and if  $\text{Vol}(U, \Omega_0) < \text{Vol}(M, \Omega)$ , Theorem 1 is an easy consequence of Moser's deformation method. Moreover, if  $U$  is a ball and  $M$  is compact, Theorem 1 has been proved in [K]. The main point of this note therefore is to show that Theorem 1 holds true for an arbitrary open subset of  $\mathbb{R}^n$  and an arbitrary connected manifold even in the case that the volumes are equal.

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2. PROOF OF THEOREM 1

Assume first that  $\varphi: U \hookrightarrow M$  is a smooth embedding such that  $\varphi^*\Omega = \Omega_0$ . Then

$$\text{Vol}(U, \Omega_0) = \int_U \Omega_0 = \int_U \varphi^*\Omega = \int_{\varphi(U)} \Omega \leq \int_M \Omega = \text{Vol}(M, \Omega).$$

Assume now that  $\text{Vol}(U, \Omega_0) \leq \text{Vol}(M, \Omega)$ . We are going to construct a smooth embedding  $\varphi: U \hookrightarrow M$  such that  $\varphi^*\Omega = \Omega_0$ .

We orient  $\mathbb{R}^n$  in the natural way. The orientations of  $\mathbb{R}^n$  and  $M$  orient each open subset of  $\mathbb{R}^n$  and  $M$ . We abbreviate the Lebesgue measure  $\text{Vol}(V, \Omega_0)$  of a measurable subset  $V$  of  $\mathbb{R}^n$  by  $|V|$ , and we write  $\overline{V}$  for the closure of  $V$  in  $\mathbb{R}^n$ . Moreover, we denote by  $B_r$  the open ball in  $\mathbb{R}^n$  of radius  $r$  centered at the origin.

**Proposition 2.** *Assume that  $V$  is a non-empty open subset of  $\mathbb{R}^n$ . Then there exists a smooth embedding  $\sigma: V \hookrightarrow \mathbb{R}^n$  such that  $|\mathbb{R}^n \setminus \sigma(V)| = 0$ .*

*Proof.* We choose an increasing sequence

$$V_1 \subset V_2 \subset \dots \subset V_k \subset V_{k+1} \subset \dots$$

of non-empty open subsets of  $V$  such that  $\overline{V_k} \subset V_{k+1}$ ,  $k = 1, 2, \dots$ , and  $\bigcup_{k=1}^\infty V_k = V$ . To fix the ideas, we assume that the sets  $V_k$  have smooth boundaries.

Let  $\sigma_1: V_2 \hookrightarrow \mathbb{R}^n$  be a smooth embedding such that  $\sigma_1(V_1) \subset B_1$  and

$$|B_1 \setminus \sigma_1(V_1)| \leq 2^{-1}.$$

Since  $\overline{V_1} \subset V_2$  and  $\overline{\sigma_1(V_1)} \subset \overline{B_1} \subset B_2$ , we find a smooth embedding  $\sigma_2: V_3 \hookrightarrow \mathbb{R}^n$  such that  $\sigma_2|_{V_1} = \sigma_1|_{V_1}$  and  $\sigma_2(V_2) \subset B_2$  and

$$|B_2 \setminus \sigma_2(V_2)| \leq 2^{-2}.$$

Arguing by induction we find smooth embeddings  $\sigma_k: V_{k+1} \hookrightarrow \mathbb{R}^n$  such that  $\sigma_k|_{V_{k-1}} = \sigma_{k-1}|_{V_{k-1}}$  and  $\sigma_k(V_k) \subset B_k$  and

$$(1) \quad |B_k \setminus \sigma_k(V_k)| \leq 2^{-k},$$

$k = 1, 2, \dots$ . The map  $\sigma: V \rightarrow \mathbb{R}^n$  defined by  $\sigma|_{V_k} = \sigma_k|_{V_k}$  is a well defined smooth embedding of  $V$  into  $\mathbb{R}^n$ . Moreover, the inclusions  $\sigma_k(V_k) \subset \sigma(V)$  and the estimates (1) imply that

$$|B_k \setminus \sigma(V)| \leq |B_k \setminus \sigma_k(V_k)| \leq 2^{-k},$$

and so

$$|\mathbb{R}^n \setminus \sigma(V)| = \lim_{k \rightarrow \infty} |B_k \setminus \sigma(V)| = 0.$$

This completes the proof of Proposition 2. □

Our next goal is to construct a smooth embedding of  $\mathbb{R}^n$  into the connected  $n$ -dimensional manifold  $M$  such that the complement of the image has measure zero. If  $M$  is compact, such an embedding has been obtained by Ozols [O] and Katok [K, Proposition 1.3]. While Ozols combines an engulfing method with tools from Riemannian geometry, Katok successively exhausts a smooth triangulation of  $M$ . Both approaches can be generalized to the case of an arbitrary connected manifold  $M$ , and we shall follow Ozols.

We abbreviate  $\mathbb{R}_{>0} = \{r \in \mathbb{R} \mid r > 0\}$  and  $\overline{\mathbb{R}}_{>0} = \mathbb{R}_{>0} \cup \{\infty\}$ . We endow  $\overline{\mathbb{R}}_{>0}$  with the topology whose base of open sets consists of the intervals  $]a, b[ \subset \mathbb{R}_{>0}$  and the subsets of the form  $]a, \infty[ = ]a, \infty[ \cup \{\infty\}$ . We denote the Euclidean norm on  $\mathbb{R}^n$  by  $\|\cdot\|$  and the unit sphere in  $\mathbb{R}^n$  by  $S_1$ .

**Proposition 3.** *Endow  $\mathbb{R}^n$  with its standard smooth structure, let  $\mu: S_1 \rightarrow \overline{\mathbb{R}}_{>0}$  be a continuous function and let*

$$S = \left\{ x \in \mathbb{R}^n \mid 0 \leq \|x\| < \mu \left( \frac{x}{\|x\|} \right) \right\}$$

*be the starlike domain associated with  $\mu$ . Then  $S$  is diffeomorphic to  $\mathbb{R}^n$ .*

*Remark 4.* The diffeomorphism guaranteed by Proposition 3 may be chosen such that the rays emanating from the origin are preserved.

*Proof of Proposition 3.* If  $\mu(S_1) = \{\infty\}$ , there is nothing to prove. In the case that  $\mu$  is bounded, Proposition 3 has been proved by Ozols [O]. In the case that neither  $\mu(S_1) = \{\infty\}$  nor  $\mu$  is bounded, Ozols’s proof readily extends to this situation. Using his notation, the only modifications needed are: Require in addition that  $r_0 < 1$  and that  $\epsilon_1 < 2$ , and define continuous functions  $\tilde{\mu}_i: S_1 \rightarrow \mathbb{R}_{>0}$  by

$$\tilde{\mu}_i = \min \left\{ i, \mu - \epsilon_i + \frac{\delta_i}{2} \right\}.$$

With these minor adaptations the proof in [O] applies word-by-word. □

In the following we shall use some basic Riemannian geometry. We refer to [KN] for basic notions and results in Riemannian geometry. Consider an  $n$ -dimensional complete Riemannian manifold  $(N, g)$ . We denote the cut locus of a point  $p \in N$  by  $C(p)$ .

**Corollary 5.** *The maximal normal neighbourhood  $N \setminus C(p)$  of any point  $p$  in an  $n$ -dimensional complete Riemannian manifold  $(N, g)$  is diffeomorphic to  $\mathbb{R}^n$  endowed with its standard smooth structure.*

*Proof.* Fix  $p \in N$ . We identify the tangent space  $(T_p N, g(p))$  with Euclidean space  $\mathbb{R}^n$  by a (linear) isometry. Let  $\exp_p: \mathbb{R}^n \rightarrow N$  be the exponential map at  $p$  with respect to  $g$ , and let  $S_1$  be the unit sphere in  $\mathbb{R}^n$ . We define the function  $\mu: S_1 \rightarrow \overline{\mathbb{R}}_{>0}$  by

$$(2) \quad \mu(x) = \inf \{ t > 0 \mid \exp_p(tx) \in C(p) \}.$$

Since the Riemannian metric  $g$  is complete, the function  $\mu$  is continuous [KN, VIII, Theorem 7.3]. Let  $S \subset \mathbb{R}^n$  be the starlike domain associated with  $\mu$ . In view of Proposition 3 the set  $S$  is diffeomorphic to  $\mathbb{R}^n$ , and in view of [KN, VIII, Theorem 7.4 (3)] we have  $\exp_p(S) = N \setminus C(p)$ . Therefore,  $N \setminus C(p)$  is diffeomorphic to  $\mathbb{R}^n$ . □

A main ingredient of our proof of Theorem 1 are the following two special cases of a theorem of Greene and Shiohama [GS].

**Proposition 6.** (i) *Assume that  $\Omega_1$  is a volume form on the connected open subset  $U$  of  $\mathbb{R}^n$  such that  $\text{Vol}(U, \Omega_1) = |U| < \infty$ . Then there exists a diffeomorphism  $\psi$  of  $U$  such that  $\psi^* \Omega_1 = \Omega_0$ .*

(ii) *Assume that  $\Omega_1$  is a volume form on  $\mathbb{R}^n$  such that  $\text{Vol}(\mathbb{R}^n, \Omega_1) = \infty$ . Then there exists a diffeomorphism  $\psi$  of  $\mathbb{R}^n$  such that  $\psi^* \Omega_1 = \Omega_0$ .*

**End of the proof of Theorem 1.** Let  $U \subset \mathbb{R}^n$  and  $(M, \Omega)$  be as in Theorem 1. After enlarging  $U$ , if necessary, we can assume that  $|U| = \text{Vol}(M, \Omega)$ . We set  $N = M \setminus \partial M$ . Then

$$(3) \quad |U| = \text{Vol}(M, \Omega) = \text{Vol}(N, \Omega).$$

Since  $N$  is a connected manifold without boundary, there exists a complete Riemannian metric  $g$  on  $N$ . Indeed, according to a theorem of Whitney [W],  $N$  can be embedded as a closed submanifold in some  $\mathbb{R}^m$ . We can then take the induced Riemannian metric. A direct and elementary proof of the existence of a complete Riemannian metric is given in [NO].

Fix a point  $p \in N$ . As in the proof of Corollary 5 we identify  $(T_p N, g(p))$  with  $\mathbb{R}^n$  and define the function  $\mu: S_1 \rightarrow \overline{\mathbb{R}}_{>0}$  as in (2). Using polar coordinates on  $\mathbb{R}^n$  we see from Fubini's Theorem that the set

$$\tilde{C}(p) = \{\mu(x)x \mid x \in S_1\} \subset \mathbb{R}^n$$

has measure zero, and so  $C(p) = \exp_p(\tilde{C}(p))$  also has measure zero (see [Bo, VI, Corollary 1.14]). It follows that

$$(4) \quad \text{Vol}(N \setminus C(p), \Omega) = \text{Vol}(N, \Omega).$$

According to Corollary 5 there exists a diffeomorphism

$$\delta: \mathbb{R}^n \rightarrow N \setminus C(p).$$

After composing  $\delta$  with a reflection of  $\mathbb{R}^n$ , if necessary, we can assume that  $\delta$  is orientation preserving. In view of (3) and (4) we then have

$$(5) \quad |U| = \text{Vol}(\mathbb{R}^n, \delta^* \Omega).$$

*Case 1.*  $|U| < \infty$ .

Let  $U_i, i = 1, 2, \dots$ , be the countably many components of  $U$ . Then  $0 < |U_i| < \infty$  for each  $i$ . Given numbers  $a$  and  $b$  with  $-\infty \leq a < b \leq \infty$  we abbreviate the "open strip"

$$S_{a,b} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a < x_1 < b\}.$$

In view of the identity (5) we have

$$\sum_{i \geq 1} |U_i| = |U| = \text{Vol}(\mathbb{R}^n, \delta^* \Omega).$$

We can therefore inductively define  $a_0 = -\infty$  and  $a_i \in ]-\infty, \infty]$  by

$$\text{Vol}(S_{a_{i-1}, a_i}, \delta^* \Omega) = |U_i|.$$

Abbreviating  $S_i = S_{a_{i-1}, a_i}$  we then have  $\mathbb{R}^n = \bigcup_{i \geq 1} \overline{S_i}$ .

For each  $i \geq 1$  we choose an orientation preserving diffeomorphism  $\tau_i: \mathbb{R}^n \rightarrow S_i$ . In view of Proposition 2 we find a smooth embedding  $\sigma_i: U_i \hookrightarrow \mathbb{R}^n$  such that  $\mathbb{R}^n \setminus \sigma_i(U_i)$  has measure zero. After composing  $\sigma_i$  with a reflection of  $\mathbb{R}^n$ , if necessary, we can assume that  $\sigma_i$  is orientation preserving. Using the definition of the volume, we can now conclude that

$$\text{Vol}(U_i, \sigma_i^* \tau_i^* \delta^* \Omega) = \text{Vol}(\sigma_i(U_i), \tau_i^* \delta^* \Omega) = \text{Vol}(\mathbb{R}^n, \tau_i^* \delta^* \Omega) = \text{Vol}(S_i, \delta^* \Omega) = |U_i|.$$

In view of Proposition 6 (i) we therefore find a diffeomorphism  $\psi_i$  of  $U_i$  such that

$$(6) \quad \psi_i^*(\sigma_i^* \tau_i^* \delta^* \Omega) = \Omega_0.$$

We define  $\varphi_i: U_i \hookrightarrow M$  to be the composition of diffeomorphisms and smooth embeddings

$$U_i \xrightarrow{\psi_i} U_i \xrightarrow{\sigma_i} \mathbb{R}^n \xrightarrow{\tau_i} S_i \subset \mathbb{R}^n \xrightarrow{\delta} N \setminus C(p) \subset M.$$

The identity (6) implies that  $\varphi_i^* \Omega = \Omega_0$ . The smooth embedding

$$\varphi = \coprod \varphi_i: U = \coprod U_i \hookrightarrow M$$

therefore satisfies  $\varphi^* \Omega = \Omega_0$ .

*Case 2.*  $|U| = \infty$ .

In view of (5) we have  $\text{Vol}(\mathbb{R}^n, \delta^* \Omega) = \infty$ . Proposition 6 (ii) shows that there exists a diffeomorphism  $\psi$  of  $\mathbb{R}^n$  such that

$$(7) \quad \psi^* \delta^* \Omega = \Omega_0.$$

We define  $\varphi: U \hookrightarrow M$  to be the composition of inclusions and diffeomorphisms

$$U \subset \mathbb{R}^n \xrightarrow{\psi} \mathbb{R}^n \xrightarrow{\delta} N \setminus C(p) \subset M.$$

The identity (7) implies that  $\varphi^* \Omega = \Omega_0$ . The proof of Theorem 1 is complete.  $\square$

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#### REFERENCES

- [Bo] W. Boothby. *An introduction to differentiable manifolds and Riemannian geometry*. Second edition. Pure and Applied Mathematics **120**. Academic Press, Orlando 1986. MR **87k**:58001
- [GS] R. Greene and K. Shiohama. Diffeomorphisms and volume preserving embeddings of non-compact manifolds. *Trans. Amer. Math. Soc.* **255** (1979) 403–414. MR **80k**:58031
- [K] A. Katok. Bernoulli diffeomorphisms on surfaces. *Ann. of Math.* **110** (1979) 529–547. MR **81a**:28015
- [KN] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Volume II, Interscience, New York 1969. MR **97c**:53001b
- [M] J. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.* **120** (1965) 286–294. MR **32**:409
- [NO] K. Nomizu and H. Ozeki. The existence of complete Riemannian metrics. *Proc. Amer. Math. Soc.* **12** (1961) 889–891. MR **24**:A3610
- [O] V. Ozols. Largest normal neighborhoods. *Proc. Amer. Math. Soc.* **61** (1976) 99–101. MR **55**:4221
- [W] H. Whitney. Differentiable manifolds. *Ann. of Math.* **37** (1936) 645–680.

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