Geometrical Theorems on Einstein's Cosmological Equations
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Source: *American Journal of Mathematics*, Oct., 1921, Vol. 43, No. 4 (Oct., 1921), pp. 217-221
Published by: The Johns Hopkins University Press
Stable URL: https://www.jstor.org/stable/2370192

GEOMETRICAL THEOREMS ON EINSTEIN'S COSMOLOGICAL EQUATIONS.

By Edward Kasner.

I wish to generalize here some of my results published in the January and April numbers of the American Journal of Mathematics (Vol. 48, 1921, pp. 20, 126), relating to Einstein's original equations of gravitation (in space free from matter),

$$G_{\mu\nu} = 0. \tag{1}$$

Later Einstein introduced a so-called cosmological term involving a constant λ, the equations being then

$$G_{\mu\nu} - \lambda g_{\mu\nu} = 0. \tag{2}$$

More recently* he has employed the form

$$G_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} G = 0, \tag{3}$$

where G is the scalar curvature $g^{\alpha\beta}G_{\alpha\beta}$. I shall refer to (3) simply as the *cosmological equations*. Every solution of the former equations (1) is, of course, a solution of the latter (3), but not vice-versa. The ten equations (3), as Einstein shows, involve one extra dependence as compared with the ten equations (1).

§ 1. Five Dimensions.

I shall first take up the question of dimensionality (that is, class of the quadratic form). In the April paper† it was shown that no solution of (1) can represent a 4-spread imbedded in a 5-flat (except in the trivial case where the 4-spread is euclidean, that is, has zero Riemann curvature). We now inquire what solutions of (3) can be imbedded in a 5-flat, and shall find that there are actually two distinct possibilities.

Using the notation of the April paper, we write our spread in the form

$$w = f(x_1, x_2, x_3, x_4).$$

Referring to the formulas on p. 128, we have, in the standard coördinates there employed, involving the four principal curvatures k_i,

$$G_{12} = 0, \text{ etc.}; \qquad G_{11} = -k_1(k_2 + k_3 + k_4), \text{ etc.}$$

* *Berichte Berlin Akad. d. Wiss.* (1919). The complete equations when matter is present of course involve the energy tensor $T_{\mu\nu}$. See also Kopff, *Grundzüge* (1921), pp. 163–165, where the author refers to the field equations of the first, second, third kinds.

† "The Impossibility of Einstein Fields Immersed in Flat Space of Five Dimensions," Vol. 48, pp. 126–129.

Also by the footnote on the same page, we have

$$G = -2(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4).$$

Substituting in the cosmological equations and simplifying, we have the following set of equations for the determination of the four principal curvatures,

$$(4) \quad k_1(k_2 + k_3 + k_4) = k_2(k_3 + k_4 + k_1) = k_3(k_4 + k_1 + k_2) = k_4(k_1 + k_2 + k_3).$$

Subtracting say the second member from the first, we have

$$(4') \qquad (k_1 - k_2)(k_3 + k_4) = 0,$$

with five similar equations obtained by permuting the subscripts. Hence either

$$(4'') \qquad k_1 = k_2 \text{ or } k_3 = -k_4, \text{ etc.}$$

If the common value of the four expressions in (4) is zero, then we have exactly the system (12) of the April paper, giving merely the trivial case where 3 or 4 of the k's vanish (which means that the manifold is euclidean). Otherwise we find from (4″) two possible types of solution

$$(a) \qquad k_1 = k_2 = k_3 = k_4 \neq 0,$$
$$(b) \qquad k_1 = k_2 = -k_3 = -k_4 \neq 0.$$

In the first case (a) the four principal curvatures are equal at every point, that is, every point is umbilical. It follows then from known theorems that the 4-spread must be a hypersphere. This, of course, checks up since it is known that a 4-dimensional hypersphere is actually imbedded in a 5-flat and is actually a solution of the cosmological equations. (It is sometimes referred to as DeSitter's "Spherical World.")

In the second possibility (b), we have, at every point, the four principal curvatures numerically equal, but two of them are positive and two are negative. We may say then that every point is *semi-umbilical.* It is immediately seen that the Riemann curvature of such a spread is not constant; for we find, in our special coördinates, that the conditions for constant Riemann curvature are

$$(5) \qquad k_1k_2 = k_1k_3 = k_1k_4 = k_2k_3 = k_2k_4 = k_3k_4.$$

For the spherical solutions (a), these products are all equal; but in the new case (b), two of the products are positive and four are negative. We may term a spread of this new type, a *hyperminimal spread,* since we may think of it as a generalization of ordinary minimal surfaces, which have the property that the two principal curvatures are numerically equal, but opposite in sign. The actual existence of such hyperminimal spreads depends on the consistency of a certain set of three partial differential equations, of the second order, which can readily be written down. It may be that no solution exists, or that all the solutions are imaginary, but the possibility still remains open.

THEOREM I. *If a four-spread imbedded in a five-flat is to obey Einstein's cosmological equations (3), then either every point is umbilical (giving a hypersphere), or else every point is semi-umbilical (giving a possible type of hyperminimal spread).*

§ 2. CONFORMAL REPRESENTATION.

I next take up the generalization of a theorem given in the January paper.* It was there shown that the only solutions of Einstein's original equations (1), which have the same light equation as the euclidean or Minkowski spread, are themselves euclidean; that is, if the spread is to be conformally representable on a 4-flat it must have zero Riemann curvature. We shall now prove:

THEOREM II. *The only spreads which obey the cosmological equations (3) and which are conformally representable on a 4-flat are those that have constant Riemann curvature (that is, the spread must be of spherical or pseudospherical character).*

For this purpose we use the notation of the earlier paper, in particular the formulas on pp. 22–23. Using the value of the tensor $G_{\mu\nu}$ there calculated, we find that the scalar curvature is

$$G = \frac{6}{\lambda}\{\Sigma N_{ii} + \Sigma N_i^2\}, \qquad \text{where} \qquad \lambda = e^{2N}.$$

Substituting then in the equations (3) we find the following system of equations for the determination of the unknown function N:

$$N_{12} - N_1 N_2 = 0, \qquad \text{etc.};$$

$$3N_{11} - N_{22} - N_{33} - N_{44} - 3N_1^2 + N_2^2 + N_3^2 + N_4^2 = 0, \qquad \text{etc.}$$

Employing the same transformation $N = -\log M$ as in the earlier paper, we obtain this simple system:†

$$M_{12} = 0, \qquad M_{13} = 0, \qquad \text{etc.};$$
$$M_{11} = M_{22} = M_{33} = M_{44}.$$

The general solution is obviously

$$(6) \quad M = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5.$$

* "Einstein's Theory of Gravitation: Determination of the Field by Light Signals," Vol. 48, pp. 20–28. Apparently without knowledge of this paper, Ogura has recently given this special theorem in *Comptes Rendus*, Nov. 7, 1921.

† In the January paper, by a typographical error, an extra member $\frac{1}{2}M^{-1}\Sigma M_i^2$ was omitted in the second line of the corresponding set (10), p. 23, but the final result there given is correct.

Our differential form is therefore, since $\lambda = M^{-2}$,

$$ds^2 = M^{-2}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2). \tag{7}$$

This is recognized as a manifold of constant Riemann curvature. (The curvature becomes zero only when the relation $a_1^2+a_2^2+a_3^2+a_4^2-4aa_5=0$ holds, verifying the special theorem of the earlier paper.)

Since a hypersphere can be mapped conformally on a 4-flat, it follows directly from Theorem II that the only cosmological solutions which can be represented conformally on a hypersphere are those of constant Riemann curvature.

We may also generalize the discussion of approximately-euclidean manifolds, given on pp. 27, 28 of the January paper, to approximately-spherical manifolds. The final result is

THEOREM III. *If two approximately-spherical spreads, both obeying the cosmological equations (3), admit conformal representation upon each other (thus having the same light equation), then they are necessarily isometric, except for a homothetic transformation.*

§ 3. SOLUTIONS DEPENDING ON ONE VARIABLE.

A simple example of a solution of (1) where the potentials involve only one of the variables is

$$x_4^{-2}dx_4^2 - x_1^4(dx_1^2 + dx_2^2 + dx_3^2). \tag{8}$$

All orthogonal solutions of this type (see *Bull. Amer. Math. Soc.*, vol. 27 (1920), p. 62) are easily shown to be reducible to the form

$$\begin{aligned} ds^2 &= x_1^{2a_1}dx_1^2 + x_1^{2a_2}dx_2^2 + x_1^{2a_3}dx_3^2 + x_1^{2a_4}dx_4^2, \\ a_2 + a_3 + a_4 &= a_1 + 1, \qquad a_2^2 + a_3^2 + a_4^2 = (a_1 + 1)^2. \end{aligned} \tag{9}$$

This can be put in the static form, and is completely determined by its light rays.

Analogous solutions of (3) are stated in an abstract printed in *Science* referred to below. We shall state the general result as

THEOREM IV. *All cosmological solutions for which the four potentials in the orthogonal form are functions of one of the four coördinates can be found explicitly by elementary algebraic and transcendental functions. The corresponding spreads can be imbedded in a 7-flat.*

§ 4. AN ALGEBRAIC SOLUTION.

We also state, omitting the easy proof,

THEOREM V. *If the quaternary form $ds^2 = g_{\mu\nu}dx_\mu dx_\nu$ is to be expressible as the sum of two binary forms, one involving say x_1, x_2, the other involving* *say* x_3, x_4, *and if the cosmological equations* (*3*) *are to be obeyed, then the only solution* (*except for a constant factor*) *is*

$$ds^2 = x_1^{-2}(dx_1^2 + dx_2^2) + x_3^{-2}(dx_3^2 + dx_4^2).$$

This can be imbedded in a 6-flat with cartesian coördinates $(X_1X_2X_3X_4X_5X_6)$, *the finite representation being*

$$X_1^2 + X_2^2 + X_3^2 = 1, \qquad X_4^2 + X_5^2 + X_6^2 = 1.$$

Excluding the obvious flat and spherical solutions, this is apparently the simplest solution of Einstein's equations which has thus far been obtained, and is the first case where the finite solution is an algebraic spread. In the example (8) given in § 3, the potentials $g_{\mu\nu}$ are algebraic, but not necessarily the corresponding finite spread.

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* The theorems of the present paper were first published in *Science*, Vol. 54 (Sept. 30, 1921), pp. 304–305. A typographical error on page 305 should be corrected as in formula (8) above. See also a paper appearing in the *Mathematischen Annalen*, entitled "The Solar Gravitational Field Completely Determined by its Light Rays." An independent proof of Theorem II, together with very elegant proofs of two of my previous results, applying to forms in n variables, is given in the paper by Prof. J. A. Schouten and Dr. D. J. Struick page 213 of this volume of this JOURNAL. The authors were kind enough to send me a copy of their manuscript.