

Études on D–Branes

Clifford V. Johnson

*Department of Physics and Astronomy
177 Chem–Phys Building
Lexington, KY 40506–0055, USA
cvj@pa.uky.edu*

This is a short arrangement of notes on D–branes, offered as an embellishment of five lectures which were presented at the 1998 Trieste Spring School entitled “Non–Perturbative Aspects of String Theory and Supersymmetric Gauge Theory”. There is a good number of collections of pedagogical notes on D–branes in the literature, and since space here is limited, no attempt will be made to cover all of the introductory material again. Instead, the notes cover selected topics and themes in string theory and M–theory, emphasizing how certain technical aspects of D–branes play a role. The subject is developed mainly from the perspective of non–perturbative string theory, touching on aspects of the old matrix model, string duality, the new matrix model, the AdS/CFT correspondence and other gauge theory/geometry correspondences.

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1 Prelude: Remarks on Ten Years of Non–Perturbative Strings

The last ten years have seen remarkable progress in our understanding of the physics of non–perturbative string theory. Some may be surprised that this presentation goes as far back as 1988, but it is not without justification, as will be clear shortly.

The first wave of progress in this area was made in the area of “non–critical” string theory^{1,3}. These are string theories propagating in dimensions outside the critical dimension of 26 (for bosonic strings) or 10 (for superstrings). Such string theories (described in the conformal gauge) are in principle more complicated than “critical” strings, as there is no decoupling of the quantum fluctuations of the two dimensional world–sheet metric (given by the “Liouville field”, φ) from the two dimensional degrees of freedom representing the embedding of the string into spacetime (a two dimensional “matter” theory¹). So in general, the theory is a complicated system involving two dimensional quantum gravity² coupled to a matter sector of central charge c .

Perhaps the main challenge was to find techniques which could facilitate the description of the gravity sector, which required making sense of the path integral sum over random fluctuating surfaces. By the mid–’80’s, the technique of discretizing the world–sheet appeared⁴, with the addition of matter degrees of freedom within that approach following soon after^{5,7}. There was a definite convergence of the results of the discrete approach with those of the continuum approach^{3,8,6}, by about 1988. Certain important toy models were quickly identified: For $c < 1$ the matter theories are the (p, q) conformal “minimal models”, using the standard notation, with $c = 1 - 6(p - q)^2 / pq$. Many results were obtained for the case where the topology of the world sheet was

that of a sphere (string tree level), and perturbative techniques for developing the topological expansion were also available. In 1989 non-perturbative information was obtained⁹. The now traditional explosion of papers followed.

Although these models are drastically simplified string theories, some of the lessons learned from them about non-perturbative string theory are still relevant in today's approaches to string theory, both non-perturbative and perturbative. This has been already pointed out in the current literature to some extent (see later), but one suspects that there are a few more important lessons from those days lying in wait to be appreciated.

The second wave of progress (1994/1995) is what is now called the “Second Superstring Revolution”, in which our understanding of non-perturbative critical superstring theory was greatly improved. That new knowledge is phrased in terms of the dramatic phenomenon known as “strong/weak coupling duality”, and that collection of discoveries deserves to be called a revolution, in contrast (arguably) to the previous non-perturbative breakthrough: In 1988–1990, the severe simplicity of the models made it hard to see just what the general lessons were at the time (although see ref.²³). In this case, however, although probably all agree that we have not fully assimilated the meaning of duality, there are some important conclusions which have changed at least our qualitative expectations of what string theory really is, while at the quantitative level, the relative importance of certain features (like the role of extended objects) has been considerably reevaluated. In addition, there has been a vast number of applications of the duality results to problems not central (as far as we know) to string theory.

Of course, none of these breakthroughs, whether they are called revolutions or not, can happen without the appropriate tools, which will be discussed in turn. In the case of the first wave, the really sharp tools were actually matrix models, while in the second wave, they were D-branes, which will ultimately lead us back to matrix models.

1.1 *The First Wave: 1988/1989 and Beyond*

Matrix models allowed for a complete solution of the problem of coupling two dimensional quantum gravity to various matter systems with central charge $c \leq 1$. This worked very well because the Feynman graphs of certain theories of $N \times N$ matrices M are dual (in a sense to be made clear shortly) to “polygonizations” of two dimensional surfaces. For example, for the $(2, 2m-1)$ minimal models coupled to gravity, the partition function is:

$$Z = \int dM \exp \left(-\frac{N}{\gamma} \text{Tr} V(M) \right) \quad (1)$$

where $V(M) = \sum_{i=2}^k g_i M^i$ is a polynomial in an $N \times N$ Hermitian ^a matrix M of order k (which is at least m), with couplings $\{g_i\}$. At large N , one can organize the graphs, which contain vertices of up to k th order, as an expansion in $1/N$. A graph of V vertices, F faces and E edges comes with a factor $N^{F-E+V} = N^\chi$, where χ is the Euler number of the graph. Replacing each vertex, face and edge of the graph by the face, vertex and edge of dual graph, one arrives at a polygonization of a two dimensional surface of genus h where $\chi = 2 - 2h$, and h is the number of handles. For example, in figure (1), the figure on the left is dual to a tiling of the sphere by six triangles and one square, while the figure on the right is dual to a tiling of the torus with the same shapes.

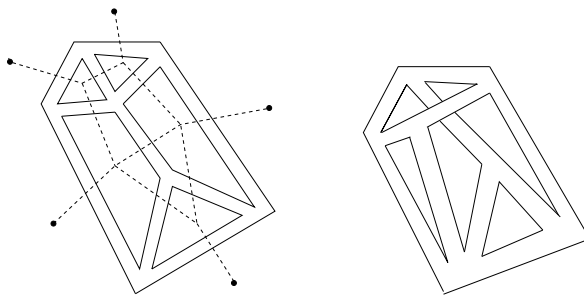


Figure 1: A sphere and a torus, respectively, each tiled by six triangles and a square by the dotted dual diagram. The dots should be identified. The completion of the second diagram is left for the reader's amusement.

In this way, we see that the sum over random graphs represented by the integral (1) is equivalent to a sum over two dimensional discretized random surfaces. Of course, in using this as a model of continuum string world sheets, the physics should not depend upon whether we use triangles ($k=3$), squares ($k=4$), or pentagons ($k=5$), *etc.*, to tile the surface. In the continuum limit (when the size of the polygons shrinks away) we should obtain universal results. For arbitrary couplings $\{g_i\}$ this is not true. However, for the order k potential, there is a family of critical couplings $\{g_1^c, g_2^c, \dots, g_k^c\}$ for which universal physics (earmarked by certain critical exponents³) may be extracted, by tuning them to a place where the free energy of the model (divided by N^2) becomes non-analytic.

^aOne can have other types of matrix ensemble: unitary, general complex, symmetric, *etc.* Complex matrix models can be tuned to give a rich class of models^{25,32} which includes the ones discussed here. Symmetric matrix models²⁸ give unoriented strings, while unitary³¹ ones lead to continuum models which have been shown to be “dual”³⁰ to certain open string models²⁹.

Physically, this means that in the large N limit, not only do the graphs which can be drawn on the sphere dominate because they are an N^2 factor greater than the others, but the *numerical coefficients* of those graphs (which depend on the $\{g_i\}$) are such that the perturbative expansion in the $\{g_i\}$ ceases to converge.

This actually corresponds to tuning the potential and eigenvalue spectrum of the model in the following way^{21,7}: The function describing the density of matrix eigenvalues $u(\lambda)$ (representing the ‘‘Dyson Gas’’ of eigenvalues) is supported on a square root cut of length $2a$ in the complex λ plane and is of the form

$$u(\lambda) = \frac{1}{\pi} P^{(2k-2)}(\lambda) \sqrt{a^2 - \lambda^2} \quad (2)$$

where $P^{(2k-2)}(\lambda)$ is a $(2k-2)$ th order polynomial in λ (derived from $V(\lambda)$), and a is a function of the couplings. (Aficionados of the old, old matrix model days, where the distribution of large nuclei was the context, will recognize the case of Wigner’s semi-circle law for $k=1$. See figure 2(a).)

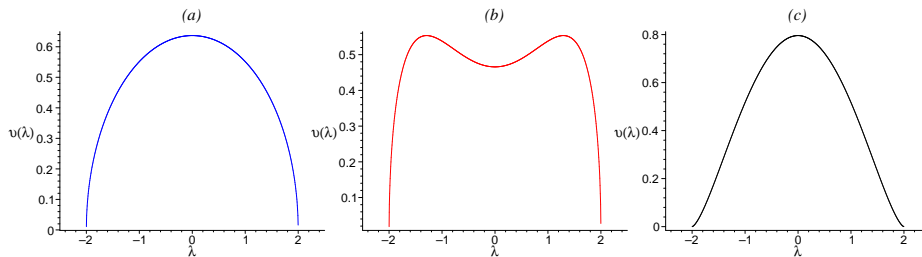


Figure 2: Eigenvalue densities. Case (a) is the Wigner semi-circle behaviour. Case (b) represents an arbitrary point in coupling space, while (c) is the critical $m=2$ point, ultimately representing the case of pure gravity. (See text.)

When one tunes the potential to the critical point there is a family of distinct models which are distinguished by how fast the eigenvalue density function vanishes at the edges. (It turns out that each edge gives a copy of a critical model^{26,27}, so one can focus on one end without loss of generality). The m th critical model is characterized by the vanishing of $u(\lambda)$ with m extra zeros at the endpoint: $u(\lambda) \sim \lambda^{m+1/2}$. Tuning this number of zeros to be at the edge gives the critical couplings $\{g_i^c\}$ of the potential. In this way, one can get a family of critical continuum models in the large N limit, and solve for their behaviour on the sphere. See figure 2(c) for the case $m=2$. Figure 2(b) is a random off-critical density distribution.

It turned out that there is a way of tuning⁹ to critical couplings $\{g_i^c\}$ while

taking the large N limit, and capture the whole genus perturbative expansion in a succinct expression, which even contains non-perturbative information. This careful tuning is called the “double scaling limit”, where all quantities of interest are tuned to their critical values at a rate determined by their length dimension: The amount that they differ from their critical value is proportional to a power of ℓ , the typical side length of a polygon in the tiling. The limit $N \rightarrow \infty$ is accompanied by $\ell \rightarrow 0$ to recover the continuum limit at a critical point.

One must cast the quantities of interest in a form which allows the double scaling limit to be made explicit. An important part of the story was the re-expression of the content of the partition function (1) in terms of its Dyson-Schwinger equations, or equivalently in terms of recursion relations between an infinite family of polynomials \mathcal{P}_n , orthogonal²² with respect to the measure $d\lambda \exp\{-\frac{N}{\gamma}V(\lambda)\}$.

In the double scaling limit, the recursion relations yield a differential equation for (essentially) the partition function. Alternatively, the discrete Dyson-Schwinger equations¹⁸ become an infinite series of constraints (the “Virasoro constraints”) on all of the correlators of the theory, which has equivalent content¹⁹.

For example, in the case $m=2$, the differential equation is the Painlevé II equation:

$$-\frac{\nu^2}{3} \frac{\partial^2 \rho}{\partial \mu^2} + \rho^2 = \mu, \quad (3)$$

where

$$\rho = \nu^2 \frac{\partial^2 F}{\partial \mu^2}. \quad (4)$$

F is the free energy. The correctly scaling part of γ in the limit is μ , the “cosmological constant” of the theory (*i.e.*, it is the variable coupling to the area in the two dimensional gravity theory). ν is a renormalized $1/N$, counting loops in the genus expansion. The string coupling, g_s , is given by the combination $g_s^2 = \nu^2 \mu^{-\frac{5}{2}}$.

Obbligato: The cosmological constant μ has dimensions of (length)⁻² and therefore ν has dimension $-\frac{5}{2}$ in length units. It is the ratio $\nu = \ell^{-\frac{5}{2}}/N$ which is held finite and fixed as $N \rightarrow \infty$ and $\ell \rightarrow 0$. Therefore the dimensionless string coupling g_s is given by $g_s^2 = \nu^2 \mu^{-\frac{5}{2}}$. More generally, for the m th model, ν has dimension $-(2 + \frac{1}{m})$ and the dimensionless string coupling is $g_s^2 = \nu^2 \mu^{-(2 + \frac{1}{m})}$.

The Painlevé equation (3) contains the complete genus expansion of the theory. At tree level we have $\rho=\mu^{\frac{1}{2}}$, while higher orders can be obtained by recursively substituting corrections into the differential equation:

$$\rho = \mu^{\frac{1}{2}} - \frac{1}{24} \frac{\nu^2}{\mu^2} - \frac{49}{1152} \frac{\nu^4}{\mu^{\frac{9}{2}}} + \dots \quad (5)$$

One can integrate twice (and divide by ν^2) to get the free energy:

$$F = g_s^{-2} \frac{4}{15} + \frac{1}{24} \ln \mu - \frac{7}{1440} g_s^2 + \dots \quad (6)$$

representing the sphere, torus and double torus contribution, *etc.*

The differential equation contains more than just the complete string perturbation series, as it has non-perturbative solutions too. Perhaps the most important information that can be deduced from the equation and its series solution is knowledge about the high order behaviour of string perturbation theory, and the nature of non-perturbative effects. It was observed²³ that the high genus behaviour of the h th term in the series expansion is of the form $C^{-2h}(2h)!$. (Here C is not an essential (but computable) constant.) Furthermore an examination of (for example) an attempt to perform a Borel resummation of the series reveals that the resulting failure^b to resum can be expressed in terms of non-perturbative ambiguities of the form $\exp(-C\mu^{\frac{3}{4}}/\nu) = \exp(-C/g_s)$. Tracing this back to the original matrix model definition, we see that this translates²³ into $\exp(-CN)$, as $1/N$ is the string coupling before double scaling. This is to be identified with the exponentiated action for a single matrix eigenvalue, λ , to tunnel from the potential well given by $V(\lambda)$ at the critical value of the couplings which defines the model. In tunneling, it leaves a gas of N eigenvalues and hence has action of order N . That characteristic high genus behaviour, and the associated non-perturbative effects, were correctly predicted²³ to be features of the models which will persist to the case of critical string theories. This was borne out in the physics of D-branes³⁷. Now that D-branes have brought us back to large N models again (where N is the number of D-branes) we should expect to interpret order $\exp(-N)$ effects as the action of a single D-brane (see next sub-section, and also section 4).

^bNote that a failure of Borel re-summability is not on its own a sign of a problem with perturbation theory. Consider for example the case of degenerate double wells in ordinary quantum mechanics. The non-Borel-resummability of the perturbation theory developed there is simply a consequence of its inability to capture the effects of instantons representing tunneling between the degenerate wells. This persists to more complicated examples in quantum field theory¹⁷.

The string equation for the m th critical model is

$$C_m \mathcal{R}_m[\rho] = \mu \quad (7)$$

where the $\mathcal{R}_m[\rho]$ form a family of differential polynomials ^c in ρ and its μ derivatives (a prime denotes $\nu\partial/\partial\mu$):

$$\mathcal{R}_0 = 1, \quad \mathcal{R}_1 = -\rho, \quad \mathcal{R}_2 = 3\rho^2 - \rho'', \quad \mathcal{R}_3 = 10\rho^3 - 10\rho\rho'' - 5(\rho')^2 + \rho'''' , \dots \quad (8)$$

and the C_m are numbers which normalize the coefficients of ρ^m to 1 in the string equations.

One can formally define an interpolating model¹³, flowing from one critical model to the k th one, by adding the operator \mathcal{O}_k to the model with coupling t_k . (In the original matrix model, this is equivalent to adding the k th critical potential and then double scaling.) There is a beautiful underlying organization of this flow in terms of the k th equation of the KdV hierarchy¹⁰ of integrable flow equations ^d:

$$\frac{\partial\rho}{\partial t_k} = \mathcal{R}'_{k+1}[\rho]. \quad (9)$$

Note that from equations (9) and (8) it is evident that $t_0 \equiv -\mu$, and therefore \mathcal{O}_0 is the operator that measures area, known as the “puncture operator” ^e. This identification of \mathcal{O}_0 is correct for the non-trivial unitary member of the series of models, (2,3). For the m th model, the operator \mathcal{O}_{m-2} is the puncture operator.

Therefore, from (4), we see that the string equation is an equation for the two point function $\rho \equiv \langle \mathcal{O}_0 \mathcal{O}_0 \rangle$, while the KdV flow equation (9) is an equation relating the insertion of the operator \mathcal{O}_k , $\langle \mathcal{O}_k \mathcal{O}_0 \mathcal{O}_0 \rangle$, to other insertions, following from the fact that the \mathcal{R}_k obey a recursion relation.

This KdV organization has a number of consequences, and is equivalent to an infinite family of constraints on the correlation functions of the point-like operators \mathcal{O}_k appearing in any of the theories. These constraints form a Virasoro algebra, and may be thought of as an expression of the Dyson-Schwinger equations of the “microscopic loops” (point-like operators) of the theory^{7,16,19}. Unfortunately that is not a story which we can cover here, due to lack of space.

^cThese “Gelfand–Dikki” polynomials¹⁵ arise naturally in expanding the diagonal part of the resolvent of the one dimensional Schrödinger operator $\mathcal{H} \equiv -\nu^2 \partial_\mu^2 + \rho(\mu)$.

^dThis extends to the full family of the (p, q) minimal models in terms of the “generalized” KdV hierarchy of flows. These models may be derived by double scaling two-matrix models^{11,12}

^eThe name arises because one may think of \mathcal{O}_0 as marking a fixed point on the world-sheet. Fluctuations about this point then give a measure of the average world-sheet area.

Incidentally, the even m models with the string equation (7) defined above were shown to be problematic at the non-perturbative level: A unique exact solution to the equation with the sphere-level asymptotic behaviour $\rho \sim \mu^{1/m}$ does not exist for m even²⁴. Furthermore, the flow from the unique, exact solution¹⁴ for an odd m model to an even m model, using the KdV evolution is unstable²⁴. So the even m models (including the unitary model (3, 2), the case of pure gravity) were considered to be sick at the non-perturbative level.

It is worth pointing out that while this is a perfectly acceptable and non-disturbing conclusion (given that these are toy models), there are other string equations closely related to those above (eqn.(7)) which may be derived from slightly different (and no less well motivated) matrix models. That family of models has the same behaviour on the sphere, *and at any order in perturbation theory* and the same underlying KdV flow structure, but have a unique stable solution for all m , and therefore might be considered to be the correct non-perturbative completion of the perturbative series³².

1.2 The Second Wave: 1994/1995 and Beyond

In the case of the second wave of non-perturbative discoveries, the sharp tools are D-branes, which are extended objects with (at least) four extra special properties.

- They have a very good description within weakly coupled type I and type II critical string theory in terms of the inclusion of Dirichlet boundary conditions^{33,34,35,36} into the usual conformal field theory description. Their low energy collective motions have a simple description in terms of gauge field theory^{40,37,81}, a fact which is not just convenient, but very fundamental, as we shall discuss.
- They are extended sources⁴⁵ of a whole family of spacetime fields, the “Ramond–Ramond” (R–R) fields. This was a crucial fact in the string duality⁶³ story, because R–R fields themselves were implicated as important role players as the duality map mixed them with “Neveu–Schwarz–Neveu–Schwarz” (NS–NS) fields. Extended soliton sources of the NS–NS fields were correspondingly mapped⁴⁴ into extended sources¹⁰⁵ of the R–R fields, which were noticed to be much lighter and singular objects in the theory. Those properties were thus shown to be necessary by duality, adding to evidence⁴⁶ that R–R extended objects play a fundamental role in string theory.
- D-branes carry the smallest possible⁴⁵ R–R charges in the theory allowed by Dirac–Nepomechie–Tietelbiom charge quantization⁴², and as

solitons, their collective motions are described at low energy by gauge theory^{40,37,81}. These facts (combined with the BPS property, below) allowed for quantitative checks of duality statements, and led to applications beyond the issue of non-perturbative string theory, for example to the physics of black holes⁵¹ and gauge theory⁵⁰.

- As massive states of the type I and type II strings’ extended space-time supersymmetry algebra in ten dimensions (or fewer by compactification), D-branes are Bogomol’nyi–Prasad–Sommerfeld (BPS) saturated⁴³ states^{45,38}. As the spectrum of such states is the same for all values of the coupling⁶⁵, many statements made about string theory at strong coupling may be tested by following these objects from weak coupling, where we can calculate.

We will uncover some of these properties and connections in later sections, with particular attention to their role in string duality^f. However let us discuss a few applications for now, in order to point out an interesting ten year cycle.

At low energy, branes have a description of their dynamics in terms of an effective field theory living on their world volume. In the case of D p -branes, where p denotes the number of spatial directions into which it extends, the effective field theory is a gauge theory. Indeed, when there are N coincident D p -branes the gauge symmetry is $U(N)$ and^g the field theory is thus a model of $N \times N$ matrix-valued fields: a “matrix model” in a very precise sense, as we shall see.

Since the relevance of D-branes to strong/weak coupling duality was noticed⁴⁵, this convenient low energy description has had many applications, including ones beyond string duality for its own sake (although such a distinction is of course blurring daily).

Two such areas are the subject (at least in part) of the lectures, at this school, of H. Verlinde and M. Douglas, respectively. One is the “matrix theory” (and its relatives, the “matrix string theories”) approach^h to the search for a definition of M-theory, while the other is the AdS/CFT correspondence and

^fFor introductory reviews, and reviews with advanced applications, see for example refs. ^{47,48,49,50,51}.

^gIn order to highlight certain essential parts of the properties of D-branes, the introduction of orientifolds and the objects correspondingly known as “Op-planes” will not be discussed much in this section, or in these notes. They are a necessary component of a complete discussion of the string duality story, and applications to gauge theories with SO or USp gauge groups, orientifold model building, *etc.*, but space here is limited. (A working title for this collection of notes was therefore “Supposed Former Orientifold Junkie”).

^hSee also the lectures of H. Nicolai⁶⁴ on membranes in eleven dimensions, and their relation to M-theory and Matrix theory.

its generalizations. Both were motivated and initially tested using D-branes as the basic tools, but it is largely believed that they are examples of phenomena (examples include “holography”⁵⁵, and a description^{81,57} of spacetime in terms of a non-commutative geometry) which are true beyond the D-brane context.

Matrix theory⁶² might most readily be called the “new matrix model”. It is a supersymmetric quantum mechanics of $N \times N$ matrices, with a very particular form for its potential, dictated by $\mathcal{N}=16$ supersymmetry. In the limit of infinite N , it manages to capture the physics of eleven dimensional supergravity (the low energy limit of “M-theory”) in the infinite momentum frame (often called the “light cone gauge”) and in the “discrete light cone gauge”^{58,59,60} for finite N .

The AdS/CFT correspondence^{52,53,54} is also a sort of new matrix model, but in a more subtle sense. It is founded upon the results of many investigations relating the properties of low energy string theory in curved spacetime backgrounds to gauge theory, tailored to the case of large N . Let us pause here and reflect.

The strong statement of the AdS/CFT correspondence, in the simplest case of 16 supersymmetries, is that string (or M-) theory propagating on a background composed of the product of $p+1$ dimensional anti de-Sitter spacetime (AdS_{p+2}) and a sphere S^{8-p} (or S^{9-p}), is dual to a $p+1$ dimensional conformal field theory, for $p=3$ (the type IIB string), and $p=2, 5$ (M-theory).

Weakening the statement somewhat, the relation speaks to the low energy, and weakly coupled limit of string (or M-) theory, which is simply the classical ten (or eleven) dimensional supergravity limit (type IIB in the ten dimensional case). For $p=3$, the conformal field theory is the $N \rightarrow \infty$ limit of the $SU(N)$ $\mathcal{N}=4$ supersymmetric gauge theory living on the 3+1 dimensional world volume of the D3-brane. The Feynman diagrams of the gauge theory may be organized at large N in terms of a genus expansion²⁰ of discretized string world-sheets, just as we saw before. For $p=2$ and 5, the theories are the exotic conformal field theories with 16 supercharges to be found on the world volumes of the M2- and M5-brane solitons^{97,98} of M-theory, respectively. More details of how to arrive at this correspondence is reviewed and discussed in the lectures of M. Douglas in this school.

Let us focus on the case of $p=3$ for now. The effective string whose world-sheets the gauge theory Feynman diagrams were constructing was the type IIB string compactified on $\text{AdS}_5 \times S^5$. The correspondence works partly because the squared radius of the S^5 and the inverse (spacetime) cosmological constant of the AdS_5 are set by the combination⁵² $R^2 = \alpha' (4\pi g_s N)^{1/2}$, in the limit that $\alpha' \rightarrow 0$.

So in order to make the classical supergravity limit valid, one must keep this radius R well above the Planck length, in order to honestly ignore quantum gravity corrections. So given that we are at weak string coupling, $g_s < 1$, we must take $N \rightarrow \infty$ in order to keep $g_s N \gg 1$. This means that $g_s \sim 1/N$, which should call to mind the beginning of this section. The sphere level of critical string perturbation theory in the limit $\alpha' \rightarrow 0$ is classical supergravity, and it captures the large N physics of the Yang–Mills theory.

The stronger statement is that the $1/N$ and α' corrections to the large N limit are captured by the full type IIB string theory, which of course goes a long way to realizing 't Hooft's expectations set out in 1973 concerning the large N expansion of four dimensional gauge theories²⁰.

It is hopefully clear now that we have indeed come full circle over the past ten years. We have now (1997/1998) a strong statement that the sphere level in critical string perturbation theory is dual to the large N limit of a “matrix model”, which in this case is four dimensional Yang Mills. In 1987/1988, a zero dimensional matrix model was understood to correspond to the sphere level of a non–critical string theory. In both cases, their Feynman graphs are dual to polygonizations of the world sheet of the string theory. Now, the string theory is a string theory propagating on $\text{AdS}_5 \times S^5$ while then it was a string theory propagatingⁱ in $2-6(p-q)^2/pq$ dimensions^j.

1.3 Cadenza: Beyond Large N ?

A next step is of course the understanding of $1/N$ and α' corrections. Unfortunately, we start to run into difficulties. The negative (spacetime) cosmological constant involved in the AdS_5 part of the compactification induces N units of R–R flux supported on the S^5 (carried by the N D3–branes in the dual description). This means that to directly understand the $1/N$ and α' corrections to the description of string theory propagating on this geometry, we must understand more about how to study type IIB string theory propagating in backgrounds containing non–trivial R–R fields. This is at present an under–developed subject.

We might wonder if there might be some clues to be found to guide us in this endeavour by turning the calendar back to 1989, when the full renormalized $1/N$ expansion and indeed the whole non–perturbative story was uncovered. In particular, is there an analogous procedure to tuning couplings in the matrix

ⁱIt was pointed out⁵³ that the string propagating on AdS is similar to the non–critical string, where the radial coordinate in AdS is identified with the Liouville field, φ .

^jThe extra dimension in the count here is because the Liouville field φ , acts as an extra embedding coordinate.

model while taking $N \rightarrow \infty$ which will unlock a new box of tricks for studying non-perturbative string theory on AdS backgrounds?

Presumably we need either the analogue of the orthogonal polynomials (employed in the study of the zero dimensional matrix models) with which to re-express the Yang Mills partition function, or alternatively (and perhaps more realistically) a set of loop/Schwinger-Dyson equations for either macroscopic loops (*e.g.*, Wilson loops), or microscopic loops (*i.e.*, point-like operators). Once this rewriting of the content of the theory is done, perhaps in terms of a recursion relation in N , we might then be able to see how to tune the coupling constants with N in a manner which allows the resummation of the $1/N$ expansion, perhaps leading to a closed form analogous to the elegant “string equations” of old.

Is it at all reasonable to expect that such a procedure might be possible? Perhaps there is some hope. Notice that the AdS/CFT correspondence is not really founded upon supersymmetry or string duality in any essential way, but is of course consistent with it. So one expects that the fact that the type IIB superstring theory has the additional features of being highly supersymmetric and in possession of a large $SL(2, Z)$ strong-weak coupling self duality (as of course is the $\mathcal{N}=4$ supersymmetric 4D Yang Mills theory), will be an additional bonus feature which allows an elegant mathematical solution analogous to the case of ten years ago. Given also that the Coulomb branch of many $\mathcal{N}=2$ four dimensional theories also has an elegant description, as shown by the results of Seiberg and Witten⁵⁶ (and in work which followed) there is similar reason to expect that for cases with that amount of supersymmetry, there may also be a concise and beautiful non-perturbative dual string description waiting to be found.

If this is a complete analogue of the non-critical case, then the following question arises: Is it possible that in some sense we are already frozen at criticality? In analogy with the matrix models of ten years ago, we have solved tree level in the continuum limit, and hence our “matrix model” couplings $\{g_i\}$ are already tuned to their critical values, $\{g_i^c\}$. In order to achieve the double scaling limit which gave the complete resummed $1/N$ expansion, the couplings had to be temporarily taken off-criticality and tuned to their critical values at the correct rate while taking $N \rightarrow \infty$.

This is something that we have not yet done in this context: The fact that we are already doing correct continuum tree level string physics with the AdS/CFT correspondence translates into the fact that we are on shell, doing conformal field theory. The analogue of taking the defining couplings off-criticality therefore might necessitate finding a new off-shell definition of string theory in order to move away from the already fixed relationships between the

couplings (here, these are the Yang–Mills coupling, the string coupling, the D–brane charge, *etc.*) related by conformal invariance. We need to find a more general relationship between these couplings which represents the analogue of the “off–critical” situation. Then we could perform the double scaling limit in a systematic way.

Such an off–shell approach (perhaps *e.g.*, “string field theory”) has not been able to help us much in general, but it is worth noting that we may only need only understand it at tree level in open string theory to define the off–critical couplings on the D–brane world volume theory. For open string field theory at tree level, a fairly complete off–shell definition is understood⁶¹, and this might provide important clues.

A last observation is of course the strength of the effects which are non–perturbative in the $1/N$ expansion. Ten years ago, we learned²³ that they were essentially e^{-1/g_s} , and this is one of the features which survives in critical string theory today. In that context, this form arose as the action of the “instanton” representing tunneling of a matrix eigenvalue out of the potential well containing the “Dyson gas” of N eigenvalues. In critical string theory, we know that the D–branes produce these effects because their tension (mass per unit volume) goes like $1/g_s$, and so their appearance in loops will produce such non–perturbative effects.

Whatever the description of the full $1/N$ expansion of the AdS/CFT correspondence turns out to be, it is clear that these non–perturbative contributions will arise again, and this time clearly in terms of D–branes: Once we allow the full string theory to come into play in this AdS×sphere background with non–trivial R–R charge turned on, the e^{-1/g_s} contributions of D–branes will naturally appear in virtual loop “instanton” corrections due to *e.g.*, pair creation in the R–R background field. The analogy is very complete in this case: in the maximal Abelian subgroup of the $SU(N)$ gauge theory, the D–branes are in one–to–one correspondence with the eigenvalues of the matrices.

2 *Fugue: SO(32) String Duality and the Role of Extended Objects*

Let us discuss one of the string duality pairs discovered in 1995, the system of the $SO(32)$ Type I/Heterotic strings⁶³.

2.1 *Dual Actions*

Consider the low energy ($\alpha' \rightarrow 0$) limit, where we study only the massless space–time fields produced by the string, (the others of course having infinite mass in this limit). The effective action for the fields in both cases is a ten dimensional $\mathcal{N}=1$ supersymmetric gravity theory coupled to Yang Mills.

In the case of the heterotic string, the low energy fields are the metric g_{MN} , the dilaton Φ , the antisymmetric tensor field B_{MN} , and the gauge field A_M . At tree level in string loop perturbation theory, the bosonic part of the effective action is (in the “string frame”^{86,74}):

$$S_H = \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 - \frac{\alpha'}{4}\text{Tr}F^2 \right), \quad (10)$$

where $\kappa_0^2 = 64\pi^7(\alpha')^4$, and

$$H = dB + \alpha' \left(\omega_3^I(\Omega_-) - \frac{1}{4}\omega_3^{YM}(A) \right) + \dots \quad (11)$$

and ω_3 is the Chern–Simons three form with normalization such that⁸⁷

$$dH = \alpha'(\text{Tr}R \wedge R - \frac{1}{4}\text{Tr}F \wedge F). \quad (12)$$

Ω_- is the usual spin connection modified additively by the three form H and the dilaton Φ :

$$\Omega_{\pm M}^{ab} = \omega_M^{ab} \pm H_M^{ab} + \frac{1}{2} \left(e^{Na} e_M^b \partial_N \Phi - e^{Nb} e_M^a \partial_N \Phi \right), \quad (13)$$

where the indices (a, b) are tangent space indices, to which we refer spacetime indices using the standard vielbien e_M^a .

At this level the duality to the $SO(32)$ type I string is carried out by the following transformation:

$$g_{MN} \rightarrow e^\Phi g_{MN}, \quad \Phi \rightarrow -\Phi, \quad H_{LMN} \rightarrow H_{LMN}, \quad A_M \rightarrow A_M. \quad (14)$$

As the string coupling in each case is $g_s = e^\phi$, where ϕ is the expectation value of the dilaton Φ , this is a duality relating the strong and weak coupling limits of each theory. The resulting action after transforming the heterotic action (10) with (14) is:

$$S_I = \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-g} \left[e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 \right) - \frac{1}{12}H^2 - \frac{\alpha'}{4\pi} e^{-\Phi} \text{Tr}F^2 \right]. \quad (15)$$

This is the type I string effective low energy tree level action. The fields g_{MN} and Φ are still from the NS–NS sector of the theory, while H is the field strength of the R–R antisymmetric tensor field A_{MN} , up to important α' corrections which may be deduced from eqn.(11).

The Yang–Mills term for the gauge field A_M is multiplied by $e^{-\Phi}$, instead of $e^{-2\Phi}$ as was the case for the heterotic action (10). This is a reflection of the fact that the gauge fields arise at closed string tree level (sphere) in the heterotic string theory, while they arise at open string tree level (disk) in type I string theory. (String perturbative diagrams are two dimensional world sheets with h handles, b boundaries and c crosscaps, and come with a factor $g_s^{2h-2+b+c}$.)

The H^2 term for the R–R sector has no factor of e^{Φ} at all. This reflects the fact that this term does not arise in (NS–NS) closed or open string world sheet perturbation theory, a fact which is both a blessing (for string duality) and a curse (for computing perturbatively effectively in R–R backgrounds). It is crucial to notice that the NS–NS two form field $B_H^{(2)}$ of the heterotic string has transformed into the R–R two form field $B_I^{(2)}$ of the type I string theory. This means that objects which charged under these fields are exchanged under duality.

2.2 The Logic of Duality

One of the first checks of duality statements (as far as they were understood) was to examine the behaviour of the carriers of the basic degrees of freedom of each theory.

The reasoning is quite general and is as follows: Imagine that it is suggested that two theories, theory A and theory B, are dual to each other. To be precise let us imagine that the duality is organized by a single parameter λ such that as it gets small, the physics is best described by A, and if λ is large, theory B. Let us start by considering theory A, and so λ is small. Then the defining carriers of the basic degrees of freedom of theory A should be the lightest (easiest to excite) states in the theory. The carriers of the dual degrees of freedom (those of theory B) should be present in theory A somewhere, but infinitely heavy, and hence not elementary excitations of the vacuum. Now if the duality is true, then the masses of both the A carriers and the B carriers should be a function of the parameter λ in such a way that when λ is small the A carriers are light and the B carriers are heavy, and *vice-versa* when λ is large.

A duality which was already well known in closed string theory for some time before strong/weak coupling duality is T–duality⁴¹. In that case, one can compute quite simply in string perturbation theory that the situation is exactly as we described in the preceding paragraph. Theory A is a string propagating on a circle of radius R . The spectrum in one dimension lower has two types of states whose masses are functions of R , the “momentum states” whose masses go like n/R , and the “winding states”, whose masses go like wR/α' , where

n, w are integers. The former come from the usual quantization of a particle (string centre of mass) in a box of size R , while the latter come from the fact that the string can wrap around the box (wind around the circle), because it is an extended object. For large R , the description is best done in terms of string theory A: the light degrees of freedom are simply the center of mass degrees of freedom of the string itself, the momentum modes. For small R , those states become heavy, while instead the winding modes become light. These should now be thought of as the basic degrees of freedom, realizing a new theory, B. Theory B turns out not to be a mysterious theory, but simply a closed string propagating on a circle of radius $R'=\alpha'/R$, as can be seen in the complete spectrum by making the substitution^{41,33,39}.

In that example of duality, the parameter λ was the inverse radius of a circle, and the whole discussion can be carried out and checked in string perturbation theory (weak coupling). What happens if the conjectured duality is a strong/weak coupling duality? In other words, what if λ is the strength of the coupling of the theory? In that case, if our techniques for computation are limited to perturbation theory about $\lambda=0$ (which they often are) then we have a problem. How can we check that the spectrum at strong coupling is such that the dual degrees of freedom become the lightest?

In general, with no direct means of computing at strong coupling, we have no way of actually testing a strong/weak duality statement, as all of our perturbative tools are useless. However, when we have extended supersymmetry present, we have an additional lever to pull. The extended supersymmetry algebra admits central extensions, which are terms which commute with all of the elements of the algebra. Such extensions are equal to a charge that objects in the theory can carry. As a direct consequence of the algebra, there is a lower bound on the mass of states in terms of this charge. This is the BPS bound^{43,65}. States which are annihilated by some linear combination of the supersymmetries saturate this bound and are called “BPS states”. The formula for the mass and charge of these states is not subject to quantum corrections⁶⁵ and so if a weak coupling (*i.e.*, perturbative) computation of the BPS spectrum can be carried out, the results may be extended to arbitrary coupling.

So in our duality discussion, if we have extended supersymmetry, we have a hope of computing the masses for arbitrary coupling of at least the BPS part of the spectrum, secure in the knowledge that this part of the spectrum is true for all values of the coupling. Generically, there should be states in the theory which have mass which behaves inversely with the coupling ($M\sim 1/\lambda^x, x>0$), such that at strong coupling, they become the light stable states in the theory, defining the “dual” degrees of freedom. Such a behaviour for the mass of

objects in a field theory context is found in the spectrum of solitons⁶⁶.

2.3 Dual Strings

The technology for carrying out this computation in the string theory context to test duality centered around a similar expectation^{67,68,69,70}. The appropriate dual objects will be solitons⁷⁵ of the theory and should be extended solutions in the form of strings, for reasons which will be clear shortly.

One way to start is to notice that the heterotic supergravity has the following solution⁹³:

$$ds^2 = V(r)^{-1}(-dt^2 + dx_1^2) + dr^2 + r^2 d\Omega_7^2 \quad (16)$$

$$e^{-2\Phi} = 1 + \frac{M}{r^6} \equiv V(r), \quad B_{tx^1} = -V(r) \quad (17)$$

where $d\Omega_7^2$ is the metric on a unit round S^7 , with volume ω_7 , and

$$M = \frac{N\kappa_0^2}{3(2\pi\alpha')\omega_7}. \quad (18)$$

Meanwhile the type I supergravity has the following solution:

$$ds^2 = V(r)^{-\frac{1}{2}}(-dt^2 + dx_1^2) + V(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_7^2) \quad (19)$$

$$e^{2\Phi} = 1 + \frac{M}{r^6} \equiv V(r), \quad B_{tx^1} = -V(r) \quad (20)$$

In both cases, the solution represents a one dimensional extended object stretched along the x_1 direction. Taking into account an expectation value ϕ for the dilaton, a calculation of the tension (ADM mass per unit length) in each case gives, for the heterotic supergravity's solution and the type I supergravity's solution respectively:

$$T_1^F = \frac{N}{2\pi\alpha'} = \frac{\mu_1^F}{\sqrt{2}\kappa_0}, \quad T_1^D = \frac{N}{2\pi\alpha'g_s} = \frac{\mu_1^D}{\sqrt{2}\kappa_0g_s}, \quad (21)$$

in string frame, and

$$T_1^F = \frac{N}{2\pi\alpha'}g_s^{\frac{1}{2}}, \quad T_1^D = \frac{N}{2\pi\alpha'}g_s^{-\frac{1}{2}} \quad (22)$$

in Einstein frame. Here, $\mu_1^{F,D}$ are the NS-NS and R-R H-charges, respectively.

Obbligato: The ADM mass per unit length is computed in the Einstein frame metric, obtained from the string frame metric by multiplying it by $e^{\Phi/2}$. An expectation value ϕ for the dilaton rescales the constant κ_0 to $\kappa_0 e^{\phi} = \kappa_0 g_s$.

The quantities on the left hand side are the smallest masses allowed by the ten dimensional supersymmetry algebra, where $Z_1 = \mu_1 / (\sqrt{2}\kappa_0)$ is the central charge. Furthermore, the solutions are annihilated by half of the spacetime supersymmetries⁹³ (only half the number of Killing spinors can be defined in the presence of these solutions) in each case. They are BPS vacuum states.

The interpretation of these solutions is as follows: In the heterotic supergravity, solution (16) represents the fields around an infinite fundamental heterotic string. The existence of this solution ensures the self-consistency of the string theory: The fundamental string generates the quanta of the fields g_{MN} , Φ and B_{MN} , and therefore the effective low energy action for those fields should admit a solution representing the string. Due to the mass formulae (21–22) we see that at weak string coupling the string is light. It should also be pointed out that the surface $r=0$, representing the core of the string, is actually singular. This was thought to mean that one simply has to add a delta function source (the string itself) to the solution in order to make it a complete solution of the equations of motion, leaving conformal field theory to supply the missing description there. The form of the dilaton shows that the theory is weakly coupled at the core, and so this is consistent. (See later.)

In the type I supergravity, the solution (19) represents a special type of soliton. Its (string frame) mass goes like the inverse of a single power of the string coupling, and not the inverse squared, as is familiar from solitons in many other contexts. However, it is infinitely heavy at weak type I coupling, and therefore does not contribute to the perturbative type I string spectrum. As $g_s \rightarrow \infty$ however, the solution becomes lighter than the type I string. This can be seen because the duality transformation (which exchanges the two solutions) tells us that in the effective heterotic theory, it is a light object.

This suggests that the solitonic solution is actually the heterotic string, hiding in the the type I theory, waiting to come down to zero mass at infinite coupling and take over the job of dominating the spectrum. Of course, for consistency, what should actually be checked is that the putative dual string actually generates the massless spectrum of the heterotic string as it vibrates. Hence we look for the “collective motions” of the soliton *i.e.*, those deformations of that vacuum solution which have zero cost. We will do that in the next subsection.

It is worth pausing here to see what other consequences of the duality map we can explore in the light of this discussion. What we have done is found a pair of solutions which are mapped to each other under the duality map, and we have a satisfactory interpretation of the physics that they represent.

Are there other duality pairs which have interpretations in terms of duality? The answer is yes. There exist other solutions to the type I (and more

generally, type II) equations of motion with the interesting property that they carry R–R charges. Some of them are generalizations of black holes, and have been called “ p -branes” in general, being p -dimensional objects¹⁰⁵. They couple to R–R forms of rank $p+1$. One of the features of those solutions is that they often contain singularities at their core, and are hence generically not smooth soliton solutions. So their full interpretation was not fully understood for some time.

In fact, their singular nature might have been used as an argument for discarding them as pathological solutions of the equations of motion. As many of these solutions are mapped⁴⁴ by strong/weak coupling duality to smooth NS–NS charged soliton solutions of the heterotic string theory, such a neglect would be hard to justify: One does not throw out smooth soliton solutions, but considers them as additional sectors of the theory. Therefore duality implies that the “ p -brane” extended solutions are necessarily included in the type I and type II spectra, and must have some interpretation⁴⁴. At the time, this duality argument supplied complementary evidence⁴⁶ that R–R extended solitons were needed to understand type II spectra.

Duality suggests that these extended objects were an intrinsic part of the theory at least as important as the other extended solutions, including strings. The full spectrum of R–R charged extended objects must play an important role because of their relation to NS–NS objects.

2.4 *Collective Motions and World Volume Theories*

It is now a familiar story that the collective motions of an extended vacuum solution localized in some dimensions are described by an effective theory on its world–volume¹⁰⁴.

One way of seeing this is as follows: By placing the $p+1$ dimensional extended object at a position x^m ($m=p+1, \dots, D$) in the $D-p$ dimensional transverse space, one breaks the translational symmetry of that space. The Lorentz group decomposes as: $SO(1, D-p) \supset SO(1, p) \times SO(D-p-1)$, leaving a Lorentz symmetry in the space filled by the world–volume of the brane.

However, there is still the freedom to redefine the position, x^m , by shifting to another equivalent point. This freedom to “move” the object should cost no energy, (it is a trivial redefinition of the vacuum) and is therefore a simple “collective motion” of the configuration. This is summarized in terms of a collection of $D-p$ scalars $\phi^m(y)$, which are functions of the position, y^μ in the remaining $p+1$ dimensional spacetime in which $SO(1, p)$ invariance is still preserved. In the full theory, these are scalar fields —Goldstone bosons— living on the world–volume. Being functions of the position y^μ means that we

can reconstruct the shape of the object in spacetime, as should be clear from figure (3), where the shape of the D2-brane shown is described by the field configuration $Z(x, y) = (x-y)e^{-(x^2+y^2)}$.

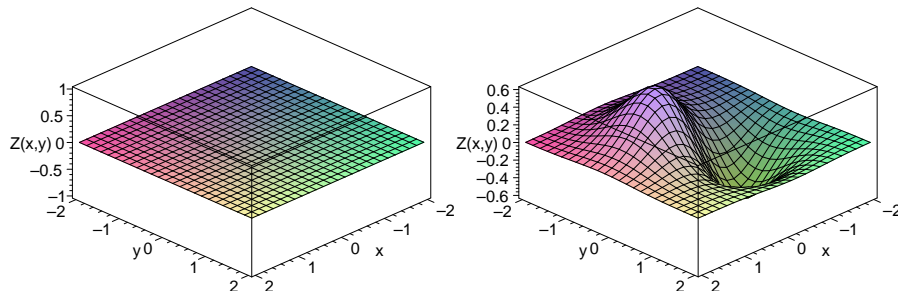


Figure 3: D2-brane: $Z(x, y) = 0$; Notes on D2-brane: $Z(x, y) = (x-y)e^{-(x^2+y^2)}$

The story extends to the case of supersymmetry. The supertranslation generators are “broken”, and a family of fermions $\psi(y)$ propagate on the world volume in exchange for the broken symmetry. The fermions and bosons form supermultiplets according to the amount of supersymmetry which was preserved, now reduced to the $p+1$ dimensional spacetime.

Now, let us be a bit more careful. The supergenerators in ten dimensional spacetime without the brane present close on momentum P^M . For example, for the case $\mathcal{N}=1$ we have:

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = (\mathcal{P}\Gamma_M\mathcal{C})_{\alpha\beta}P^M \quad (23)$$

Here, Γ_M are the gamma matrices satisfying the Clifford algebra, α, β are spinor indices, \mathcal{C} is the charge conjugation operator and the projector $\mathcal{P} \equiv (1+\Gamma_{11})/2$. $\Gamma_{11} \equiv \Gamma_0\Gamma_1 \cdots \Gamma_9$. In the presence of a p -brane, the supergenerators no longer close on the momentum, and have to be modified^{65,99,101}. The possible modifications of the algebra are by “central terms” like

$$Z^{M_1 \cdots M_p} = Q_p \int_{\mathcal{M}_p} dX^{M_1} \wedge \cdots \wedge dX^{M_p} \quad (24)$$

where \mathcal{M}_p is the space over which the p -brane is extended and Q_p is its charge, defined by integrating on the asymptotic S^{D-p-2} -sphere surrounding the p -brane at infinity:

$$Q_p = \frac{1}{\text{Vol}[S^{D-p-2}]} \int_{S^{D-p-2}} e^{-2\Phi} H^{p+2}. \quad (25)$$

It is worth noting that these terms have dimensions of mass, and so cannot be carried by massless particles in the spectrum.

The modified algebra in the case of $\mathcal{N}=1$ is:

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = (\mathcal{P}\Gamma_M\mathcal{C})_{\alpha\beta} (P^M + Z^M) + (\mathcal{P}\Gamma_{M_1\dots M_5}\mathcal{C})_{\alpha\beta} Z_+^{M_1\dots M_5} \quad (26)$$

and p is *only* either 1 or 5. The subscript $+$ means the restriction to the self-dual part. The $\Gamma_{M_1\dots M_p}$ are the antisymmetrized products of p Γ -matrices. That (26) is the appropriate form for the $\mathcal{N}=1$ case follows from the fact that¹⁰⁰ the supercharges \mathcal{Q}_α are 16 component spinors, and so the left hand side of the equation has $16 \times 17/2 = 136$ real components. The right hand side has $10 + 10!/(5!5!2) = 136$ also, from Z^M and $Z_+^{M_1\dots M_5}$. (Classically we can absorb Z^M into P^M , and so do not count their components separately. Quantum mechanically, however, they are distinct objects, and we must account for the possibility of objects which carry charges under them separately, by winding, Z^M , or center of mass momentum, P^M .)

Consider a static p -brane, *i.e.*, $P^M=0$ for all M except $M=0$. Align the brane along the directions x^1, \dots, x^p , and x^0 is time. Then, as the left hand side of (26) is positive, we have that

$$P^0 \geq |Z^{01\dots p}| \quad \Rightarrow \quad T_p \geq |Q_p|, \quad (27)$$

which is the Bogomol 'nyi-Prasad-Sommerfeld bound⁴³ on massive states in this supersymmetric theory. For branes which saturate the bound, we see that the brane must be annihilated by supersymmetries built out of spinors ϵ satisfying

$$\Gamma_{01\dots p}\epsilon = \pm\epsilon. \quad (28)$$

The number of solutions to this equation is 8, (the sign is correlated with the orientation of the brane) and we see that the BPS state is therefore annihilated by half of the 16 supersymmetries of the vacuum containing no branes. The other half do not annihilate the vacuum, but instead act on the solution, appearing as massless fermions propagating on the world volume⁹⁹. In this way we get a family of fields which propagate on the world-volume of the brane.

Now in the case of the heterotic supergravity, the solution (16) is the relevant object which carries the Z^M charge Q_1 . For the type I supergravity, solution (19) is the appropriate object carrying the charge Q_1 . The Lorentz algebra decomposes into $SO(1,1) \times SO(8)$ in the presence of this string. The ten dimensional 16 component spinor decomposes as $\mathbf{16} \rightarrow \mathbf{8}_+^s \oplus \mathbf{8}_-^c$, where the $\mathbf{8}^{s,c}$ are the spinor and conjugate spinor representations of $SO(8)$, which have opposite chirality. The \pm subscript denotes the $SO(1,1)$ chirality. As stated before, half of the supersymmetries (let's say the $\mathbf{8}^s$) annihilate the solution,

while the others do not. They instead give right moving zero modes S_-^α on the world volume. These are the superpartners of the eight bosonic scalars ϕ^i propagating on the world volume representing position in the transverse eight dimensions. The supersymmetry which relates them on the world-volume is the surviving half generated by the chiral $\mathbf{8}_-$.

The action for these modes is the Green-Schwarz free action:

$$S = \int d^2\sigma [T_1 \partial_\mu \phi^i \partial^\mu \phi^i - i S_-^\alpha \rho^\mu \partial_\mu S_-^\alpha] \quad (29)$$

That we have found with this simple analysis an oriented supersymmetric string in ten dimension with $(0, 8)$ world-sheet supersymmetry is already a big clue that this soliton is indeed the dual heterotic string, given the known short-list of superstrings in ten dimensions. Indeed, we know that we must have in addition the equivalent of a family of 32 left-moving fermions λ on the world-volume as well for consistency and that they must be coupled to a current-algebra of a 496 dimensional Lie Group. How can we see this? Well^{67,70}, we wrote a neutral solution (19) for the string, and therefore we may write new solutions equivalent to it by adding gauge fields which are pure gauge at infinity. As the gauge group of the low energy supergravity is $SO(32)$, we have a 496 parameter family of equivalent solutions of the equations of motion. On the world volume, this corresponds to a global $SO(32)$ symmetry of the couplings of the effective theory. That the action for λ is a *chiral* $SO(32)$ WZNW model⁹⁰ requires a touch more work. This may be analyzed in some more detail by studying^{67,69} the possible fluctuations about the background (19), but we will postpone a detailed analysis until later.

2.5 Dual Five-Branes

Now it is clear that there should also be five dimensional extended object solutions of the equations of motion for at least two reasons. The first is from examination of the low energy actions. The three form field strength H can in principle have magnetic charges as well. Its electric charge (coming from “electric type” non-zero components such as H_{rtx^1}) is found by integrating its Hodge dual on a seven sphere at infinity. This S^7 surrounds the stringy objects (lying along the x^1 direction; r is a radial coordinate in the transverse directions) which we studied last sub-section. We can instead have non-zero components of H (coming from “magnetic type” components $H_{\theta\phi\psi}$) which give a non-zero answer when we integrate it on an S^3 . This S^3 (with Euler coordinates θ, ϕ and ψ) would surround a five dimensional extended object in ten dimensions, a “five-brane”.

Such a solution was written down in the case of the heterotic string some time ago^{73,74}, and is:

$$\begin{aligned}
ds^2 &= -dt^2 + \sum_{i=1}^5 dx_i^2 + V(r)(dr^2 + r^2 d\Omega_3^2) \\
e^{2\Phi} &= 1 + N\alpha' \frac{(r^2 + 2\rho^2)}{(r^2 + \rho^2)^2} + O(\alpha'^2) \equiv V(r), \quad H_{\mu\nu\lambda} = -\epsilon_{\mu\nu\lambda}{}^\sigma \partial_\sigma \Phi \\
A_\mu &= \left(\frac{r^2}{r^2 + \rho^2} \right) g^{-1} \partial_\mu g, \quad g = \frac{1}{r} \begin{pmatrix} x_6 + ix_7 & x_8 + ix_9 \\ x_8 - ix_9 & x_6 - ix_7 \end{pmatrix} \quad (30)
\end{aligned}$$

where $r^2 = \sum_{i=6}^9 x_i^2$.

One striking thing about this solution is that it is essentially an instanton (localized in the x^6 - x^9 directions) dressed with some stringy fields. Indeed, an evaluation of the instanton number, using the metric in (30) gives instanton number N . The instanton has scale size ρ , and gives a non-zero contribution to dH via the equation (12).

This is truly a smooth soliton solution of the heterotic string equations of motion. One may compute the ADM tension to be

$$T_5^F = \frac{N}{(2\pi)^5 (\alpha')^3 g_s^2} = \frac{\mu_5^F}{\sqrt{2} \kappa_0 g_s^2}, \quad (31)$$

where μ_5^F is the NS-NS H-charge. The product of this charge with that (21) of the fundamental string is $\mu_5^F \mu_1^F = 2\pi n$, (n integer) as required for quantum consistency⁴². This follows for the other 1-5 soliton pair by duality. The fact that we can get the minimum allowed by setting $N=1$ for both the five and one-brane solutions is crucial. (Note that as a check that we have correctly modified the conventions of various papers to match ours, the numbers multiplying the $1/r^2$ part of the metric should be:

$$M = \frac{\kappa^2 T_5}{\omega_3} = \alpha' N, \quad (32)$$

which is what we have. (This is the analogue of equation (18), with $\omega_3=2\pi^2$ the volume of a round unit S^3 .) The value of the tension given in equation (31) is the minimum allowed by the supergravity algebra for the corresponding H-charge. The g_s^{-2} behaviour is just the type that we are accustomed to for ordinary solitons. An analysis of the available Killing spinors of the solution shows that it is indeed annihilated by half of the supersymmetries, verifying that it is a BPS state.

We can find a dual solution of the type I equations of motion by using the duality transformation (14). It is:

$$\begin{aligned}
ds^2 &= V(r)^{\frac{1}{2}}(-dt^2 + \sum_{i=1}^5 dx_i^2) + V(r)^{\frac{3}{2}}(dr^2 + r^2 d\Omega_3^2) \\
e^{-2\Phi} &= 1 + N\alpha' \frac{(r^2 + 2\rho^2)}{(r^2 + \rho^2)^2} + O(\alpha'^2) \equiv V(r), \quad H_{\mu\nu\lambda} = -\epsilon_{\mu\nu\lambda}{}^\sigma \partial_\sigma \Phi \\
A_\mu &= \left(\frac{r^2}{r^2 + \rho^2} \right) g^{-1} \partial_\mu g, \quad g = \frac{1}{r} \begin{pmatrix} x_6 + ix_7 & x_8 + ix_9 \\ x_8 - ix_9 & x_6 - ix_7 \end{pmatrix}. \quad (33)
\end{aligned}$$

This solution is also an instanton of scale size ρ . A computation of its tension gives:

$$T_5^D = \frac{N}{(2\pi)^5 (\alpha')^3 g_s}, \quad (34)$$

and we see that it has the g_s^{-1} behaviour that we saw for the tension of the stringy soliton solution of the type I theory.

Obbligato: It is worth noting that in order to get the correct value of the dual tensions by transforming with (14), we must remember that the tensions and charges have dimensions of $(\text{length})^{-(p+1)}$, and therefore an extra factor of $g_s^{\frac{p+1}{2}}$ must be inserted, given that the duality transformation involves a rescaling of the metric, which is used to measure $(\text{length})^2$.

This is an example of a phenomenon we anticipated earlier. A well-behaved smooth NS–NS charged soliton solution of the heterotic string gets mapped to a singular p -brane solution of the type I with R–R charges. As stressed before, this strongly suggests⁴⁴ that these singular R–R p -brane solutions are important. Now, we know that they are all understood within the framework of D-branes, a fact which is consistent with the fact that their tensions are proportional to g_s^{-1} , signaling that they have a description in tree level open string theory.

2.6 More Branes From The Other Extended Algebras

We may consider^k the extended algebra (26) as descending from the type IIB extended $\mathcal{N}=2$ algebra by projecting out the structures which are odd under Ω .

$$\{\mathcal{Q}_{i\alpha}, \mathcal{Q}_{j\beta}\} = \delta_{ij} (\mathcal{P}\Gamma_M \mathcal{C})_{\alpha\beta} P^M + (\mathcal{P}\Gamma_M \mathcal{C})_{\alpha\beta} \tilde{Z}_{ij}^M$$

^kThe reader ought to consult ref.¹⁰⁰ for more on these types of deductions, and how one may also deduce various brane intersections from closely related algebras.

$$\begin{aligned}
& + \varepsilon_{ij}(\mathcal{P}\Gamma_{M_1 M_2 M_3} \mathcal{C})_{\alpha\beta} Z^{M_1 M_2 M_3} + \delta_{ij}(\mathcal{P}\Gamma_{M_1 \dots M_5} \mathcal{C})_{\alpha\beta} Z_+^{M_1 \dots M_5} \\
& + (\mathcal{P}\Gamma_{M_1 \dots M_5} \mathcal{C})_{\alpha\beta} \tilde{Z}_{ij+}^{M_1 \dots M_5}
\end{aligned} \tag{35}$$

The two supercharges $\mathcal{Q}_{i\alpha}$, ($i=1, 2$) are of the same chirality: There is therefore an $SO(2)$ action on them which can mix them. The \tilde{Z} are traceless symmetric tensors of that $SO(2)$ and are therefore doublets. This is the full spectrum of charges allowed, given that the left hand side has $32 \times 33/2 = 528$ components, and the left has $10 + 2 \times 10 + 120 + 126 + 2 \times 126 = 528$. The $\mathcal{N}=1$ algebra (26) descends from this by Ω -projection because one linear combination of the \tilde{Z} 's (for $p=1, 5$) is odd under Ω , as is $Z_{M_1 M_2 M_3}$ and the five legged Z_+ .

For completeness, we list here also the type IIA extended superalgebra. The supercharges here are of opposite chirality, and so there can be no $SO(2)$ rotating them into each other.

$$\begin{aligned}
\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} & = (\Gamma_M)_{\alpha\beta} P^M + (\Gamma_{11})_{\alpha\beta} Z + (\Gamma_M \Gamma_{11})_{\alpha\beta} Z^M + (\Gamma_{M_1 M_2})_{\alpha\beta} Z^{M_1 M_2} \\
& + (\Gamma_{M_1 \dots M_4} \Gamma_{11})_{\alpha\beta} Z^{M_1 \dots M_4} + (\Gamma_{M_1 \dots M_5})_{\alpha\beta} Z^{M_1 \dots M_5}
\end{aligned} \tag{36}$$

and we see that $10 + 1 + 10 + 45 + 210 + 252 = 528$, as it should be.

This algebra descends from the extended $\mathcal{N}=1$ superalgebra in eleven dimensions:

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = (\mathcal{P}\Gamma_M \mathcal{C})_{\alpha\beta} P^M + (\mathcal{P}\Gamma_{M_1 M_2} \mathcal{C})_{\alpha\beta} Z^{M_1 M_2} + (\mathcal{P}\Gamma_{M_1 \dots M_5} \mathcal{C})_{\alpha\beta} Z^{M_1 \dots M_5} \tag{37}$$

The terms with Γ_{11} in the algebra (36) descend from terms in (37) with one more index by winding, while the others descend directly by dimensional reduction. The Γ matrices and associated projectors in (37) are eleven dimensional quantities. The supercharges are 32 component spinors and so the left hand side still has 528 components, while the right hand side has $11 + 55 + 462 = 528$.

We now understand that these algebras inform us about certain branes which can exist in the various theories. The type IIB has a doublet of strings, a doublet of five-branes, and a three-brane. Type IIA has a zero-brane, a string, a two-brane, a four-brane and five-brane, which in turn descend from eleven dimensional momentum, wrapping and direct reduction of the two-brane, and wrapping and direct reduction of the five-brane.

3 Trio: From p -Branes to Dp -Branes

3.1 Trouble at the Core?

Notice that something interesting happens when the scale size ρ of the instanton goes to zero^{74,79}. If this were purely gauge theory, the instanton would

simply be trivial, but here, there is still a lot of content. Examining the equations with $\rho=0$, we see immediately, that there is potentially a problem as $r=0$. There are at least two interesting things to say:

- The solution appears to diverge as $r \rightarrow 0$, but this is not the case. In this limit, we can neglect the additive 1 in the expression for $V(r)$ giving $V(r) \sim N\alpha'/r^2$ and we can change coordinates near the core of the configuration to a new radial coordinate⁷⁴ $\sigma = \sqrt{N\alpha'} \log(x/\sqrt{N\alpha'})$. This gives

$$ds^2 = -dt^2 + \sum_{i=1}^5 dy_i^2 + (d\sigma^2 + N\alpha' d\Omega_3^2) \quad (38)$$

$$\Phi = -\frac{\sigma}{\sqrt{N\alpha'}} + \text{constant}, \quad H = -N\alpha' \epsilon_3, \quad (39)$$

the transverse part of which has the topology $\mathbb{R}^1 \times S^3$, where ϵ_3 is the S^3 volume form. We see that the solution is smooth everywhere.

This product form of the solution is called the “throat” of the solution^l, because if S^3 was a circle, the geometry would be that of a cylinder, as shown in figure 4. More generally, what has happened is that instead of the size of the S^3 's of \mathbb{R}^4 increasing with the radial coordinate, it has frozen to constant value instead. This frozen value, R , is set by N and the string length: $R^2 = N\alpha'$. If we keep the product $N\alpha'$ large as $\alpha' \rightarrow 0$ and $N \rightarrow \infty$, we can keep the curvatures all small in the solution, making sure that at least the curvatures are under control. (Later we will see that this is apparently not necessary.)

- Unfortunately, while the solution is smooth everywhere, we see that the string coupling g_s diverges as we approach the core, because e^Φ does. In the new coordinates, this is of course still true, ($r \rightarrow 0 \equiv \sigma \rightarrow -\infty$), and we see that the string coupling gets arbitrarily large as we go down the throat. This “linear dilaton”⁷² type behaviour leads us to wonder whether we have a good description of the solution in this limit at all (but see later).

Placing the difficulties of interpretation of the strong coupling behaviour aside for a while, the form of the solution reminds us of an exact conformal field theory. In this limit, the fact that the solution has decoupled into such

^lActually, a more careful analysis⁷⁶ can show the “mouth” region where the throat opens up to connect to flat \mathbb{R}^4 as well.

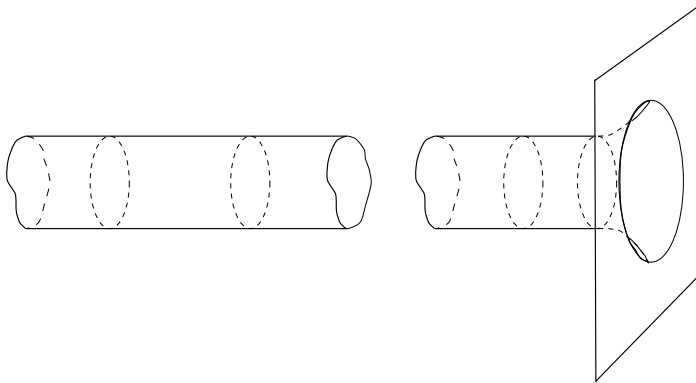


Figure 4: The infinite “throat”, to the left, is the product $\mathbb{R} \times S^3$ (see text). To the right is the “mouth” region, connecting the throat onto flat \mathbb{R}^4 .

a simple product form can be exploited. It turns out that there is a description^{77,74} (modulo the strong coupling) of a string propagating in this target space with that particular dilaton behaviour. It is the product of an $SU(2)$ WZNW model⁹¹ at level N (the S^3) with a Feigin–Fuchs⁹¹ field of background charge N (for σ). To complete the description, we take the product of this with a family of six free fields to represent the flat spacetime along the brane. This simple description is intriguing, and deserves better understanding, especially since the conformal field theory seems to contain a feature with a problematic interpretation: the string coupling becomes strong. We will return to this later.

Since these properties were noticed, we have a new handle on the whole problem: heterotic/type I duality. The fact that the string coupling is diverging near the core of our zero-size instanton should lead us to wonder if there is a better description in terms of the dual solution in the type I theory. Examining the form of the dual solution (33), we see that in the limit where $\rho=0$:

- We cannot find a change of variables which will remove the singularity at $r=0$. The solution is singular there.
- The string coupling gets arbitrarily weak as we approach the core of the solution.

So there is a duality between the good and the bad news in each case. Here we have a singularity (signaling that the $\alpha'=0$ limit has failed to capture some of the physics), but the string loop expansion in g_s is under control, while on

the heterotic side, the solution is smooth, and hence the α' expansion is under control, but we have no control over the string coupling.

Is this progress? At $\rho=0$, both sides of the discussion are apparently deficient in some way. It seems that there is still a component of the discussion missing.

3.2 Clues From Anomalies

Let us try to anticipate how new physics might arise to supply the missing sector. An important constraint on the allowed sectors in the theory is supplied by the anomaly. In the effective six dimensional theory on the world volume of the five-branes, we have $\mathcal{N}=1$ supersymmetry. To constrain our dynamics on the world-volume with the anomaly properly, we should actually make the transverse space into a compact manifold which preserves the amount of supersymmetry which we require. The only manifold with this property is called $K3$. It is a ‘‘Calabi–Yau’’ manifold, which for our purposes is simply a manifold with a Kähler structure (which is an extra-special form of a complex structure) with $SU(2)$ holonomy. This merely means that it preserves half of the spacetime supersymmetries¹¹³.

The allowed multiplets which can appear in $\mathcal{N}=1$ $D=6$ are called ‘‘vector multiplets’’, ‘‘hypermultiplets’’ and ‘‘tensor multiplets’’. The bosonic parts of these multiplets all have four field theoretic degrees of freedom. Their transformation properties under the $SU(2)\times SU(2)$ covering group of the $SO(4)$ little group of $SO(5,1)$ are $(\mathbf{2}, \mathbf{2})$, $4(\mathbf{1}, \mathbf{1})$ and $(\mathbf{3}, \mathbf{1})+(\mathbf{1}, \mathbf{1})$ respectively. So the hypermultiplet is simply four scalars, while the tensor is an antiself-dual antisymmetric field $B_{\mu\nu}^-$ plus a scalar.

The gravity supermultiplet has bosonic part $(\mathbf{3}, \mathbf{3})+(\mathbf{1}, \mathbf{1})$, the graviton $G_{\mu\nu}$ and dilaton Φ , and is completed by a self-dual antisymmetric tensor $B_{\mu\nu}^+$ $(\mathbf{1}, \mathbf{3})$, and three other scalars to contain the dilaton in a hypermultiplet. The familiar $B_{\mu\nu}$ field is assembled from the $B_{\mu\nu}^+$ $(\mathbf{1}, \mathbf{3})$ and a $B_{\mu\nu}^-$ $(\mathbf{3}, \mathbf{1})$ from an ordinary tensor multiplet. We will therefore take it as given that this tensor multiplet is always in the theory. Any tensor multiplets mentioned in a spectrum hereafter are understood to be in addition to this one.

Gauge and gravitational anomalies¹¹⁹ constrain the allowed content of the theory. Let us denote the number of vectors, hypers and (extra) tensors by n_V , n_H and n_T , respectively. A necessary (but far from sufficient!) condition that the anomalies vanish is that

$$n_H - n_V = 244 - 29n_T \tag{40}$$

(this is actually the coefficient of the irreducible $\text{Tr}R^4$ term in the gravitational anomaly.)

How might we make this work for us? Well, away from the small instanton limit, we may construct (for example) a consistent heterotic or type I vacuum using the fivebrane as follows: They are instantons and hence sources of $F \wedge F$. We saw from the supergravity (equations (10) and (15) with (12)) that this means that they are magnetic sources for H -charge (NS-NS sector in heterotic and R-R in type I) supported in x^0-x^5 . Now that we have a compact transverse space x^6-x^9 , the field equations for H become a Gauss-Law type condition on its field lines, requiring its sources to be accompanied by sinks within the compact spacetime. Fortunately, the same equation (12) which told us that instantons are a source also tells us that non-trivial $R \wedge R$ is a sink. $K3$ has 24 units of this (its Euler number) and so we can satisfy the equations of motion by having 24 fivebranes present at arbitrary positions in the $K3$. As $SO(32)$ instantons, they break the gauge group. We can choose how they are embedded into $SO(32)$ (choose the “gauge bundle”) in many ways. The minimal way is to embed them all into the same $SU(2)$ subgroup, breaking $SO(32) \supset SO(28) \times SO(4)$ to $SO(28) \times SU(2)$. An index theorem¹²⁰ tells us how many ways there are of doing this, which is 615. (This number includes the 24 positions of the instantons in $K3$ and their $SU(2)$ orientations.) In the six dimensional model, this translates into a number of extra hypermultiplets which parameterize these distinct choices. Their transformation properties under $SO(28) \times SU(2)$ are $10(\mathbf{28}, \mathbf{2}) + 45(\mathbf{1}, \mathbf{1})$. It is important to note that $K3$ also comes with 80 numbers (“moduli”) which specify its shape, which translates into 20 hypermultiplets. (The fact that they are naturally in groups of four is a consequence of the hyperKähler structure of $K3$.)

So we have $n_V = \dim[SO(28)] + \dim[SU(2)] = 381$, $n_T = 0$ and $n_H = 625$, which satisfies (40). Of course, we should also check that the other anomaly polynomials vanish —and they do— but we will not do that here¹¹⁸.

It should be immediately apparent from the form of equation (40), and the fact that vectors and hypers have the same number of bosonic components, that starting from a theory with some content allowed by the anomaly, there is the possibility to move to a new theory (or, more properly, a new *branch* of the theory) where we have either

- increased the number of vectors by the same amount by which we have decreased the number of hypers (or *vice-versa*), or similarly
- exchanged some number, n , of hypers with $29n$ tensors.

Both mechanisms, especially the first, should remind us of the Higgs mechanism, and that is the key.

3.3 An Economical Resolution

As the anomaly is a constraint on the *full* quantum theory and not just our classical analysis, whatever new physics might occur should give a spectrum consistent with the anomaly. So we can conclude the following: In either the heterotic or the type I picture, as we approach the limit of small instantons, we seem to get a singularity in the supergravity description. We must recall, however, that the supergravity description is an *effective* description of the massless degrees of freedom of the theory, and we have implicitly integrated out all of the massive degrees of freedom.

We therefore must consider the possibility⁴⁶ that the singularity we are encountering is *simply a result of having unintentionally integrated out fields which are becoming massless in the small instanton limit*. Put another way, the scale size of the instanton might correspond to the vacuum expectation value (“vev”) of a scalar in a hyper multiplet, or the mass of a vector or tensor multiplet. The anomaly (equation (40) and the other polynomials) allows a Higgs mechanism to take place and permit such new massless fields to appear.

We shall see that this is precisely what happens⁷⁸. The small instantons are D5–branes, for which an enhanced gauge symmetry lives naturally on their world–volume, carried by vectors. D5–branes are the description of the small $SO(32)$ instantons we seek^m.

So we see that we can complete the description of the fivebranes by adding a gauge theory when the type I supergravity description loses its predictability. On the other hand, the heterotic description apparently failed us in a different way: we simply did not know how to make an honest interpretation of the infinitely long throat because the linear dilaton placed us at arbitrarily strong coupling down there (see later, however).

This is an example of an effect which can take place in the heterotic string theory for any value of the coupling. Traditional perturbative heterotic methods cannot see this new massless sector at all. Indeed, this forces us to re–examine many of the conclusions made about the particle physics phenomenology of the heterotic string, as not only do new gauge groups appear, but new types of matter representations also. This is both a blessing and a curse for phenomenology as we understand it, as explained in the lectures of J. Louis in this school. (See also the lectures of B. Greene on F–theory, where the possible non–perturbative gauge groups and matter representations due to small heterotic instantons are described in a larger framework.)

^mActually, the possibility of extra *tensors* appearing to resolve the singularity also occurs, but for the small $E_8 \times E_8$ instantons⁸⁸. That interesting story will have to wait for another time, due to lack of space.

4 Type I String Theory Under the Microscope I: Dual Strings

4.1 D9-Branes

In modern parlance⁴⁵, we look at the type I string action (15) as follows: We started with the type IIB string and did a very simple orientifold, dividing by the \mathbb{Z}_2 action of world sheet parity Ω . As Ω acts on the type IIB string everywhere in spacetime we say that we have an “O9-plane” filling space, which has -16 units of charge of the R-R ten form potential, $A^{(10)}$. The form has no contribution to the action (15) in terms of its field strength. Instead, the “Gauss’ law” equation of motion simply requires us to cancel this charge by introducing 16 pairs of D9-branes (open string sectors with 32×32 Chan-Paton matrices), which each have $+1$ unit of charge.

The first three terms in the action reflect the equation of motion for the type IIB fields which survived the projection. The last term is the leading term in the expansion in α' of the world volume action of the D9-branes. The full action is the Born-Infeld action, and one can therefore read off the value of the D9-brane tension as $\tau_9 = (2\pi)^{-9} (\alpha')^{-5} g_s^{-1}$. The gauge symmetry is $SO(32)$ because 16 pairs of D9-branes give such a gauge symmetry, because their $U(32)$ gauge symmetry gets projected to $SO(32)$ by the action of Ω .

The D9-branes are introduced into the string theory as Neumann boundary conditions on the strings in all spatial directions x^1-x^9 . The world-volumes of the D9-branes fill the whole of spacetime. On the world-volume lives a gauge theory $SO(32)$.

4.2 D9-Branes and D1-Branes: The Dual Heterotic String

A D1-brane is placed into the theory extended along the x^1 direction by adding Dirichlet boundary conditions. We can ask for strings to end at a point in the x^2-x^9 directions, leaving them free (Neumann) in the x^1 direction. The collective coordinates of the soliton thus described are in the zero modes of the strings connecting the D1-brane to itself (“1-1 strings”), and those connecting it to the D9-branes (“1-9 strings”), as shown in figure 5. These zero modes are fields propagating on the (1+1)-dimensional world-volume of the D1-brane.

In the presence of the brane, the spacetime Lorentz group breaks as

$$SO(1,9) \rightarrow SO(1,1) \times SO(8), \quad (41)$$

where the $SO(1,1)$ refers to the world-volume of the D1-brane. The spacetime supercharges decompose as $\mathbf{16} \rightarrow \mathbf{8}_+^s \oplus \mathbf{8}_-^c$, where the $8^{s,c}$ are the spinor and conjugate spinor representations of the $SO(8)$ and the subscripts denote $SO(1,1)$ charge (chirality). The D1-brane is annihilated by one of these (choose the

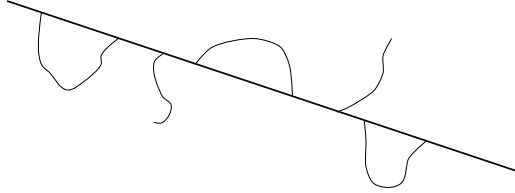


Figure 5: The type I D1–Brane and some of the strings that define it.

first), and the other remains as a supercharge of the (0,8) supersymmetric model in 1+1 dimensions representing the collective fluctuations of the brane.

Let us be fairly general^{83,114,115}, and add N D1–branes to the theory at one time. The 1–1 strings break up into two classes. Those with components in the directions transverse to the D1–branes, and those with parallel components. The massless excitations form vectors and scalars in 2D, and are created as follows. The latter class represents the collective motions parallel to the brane and are vectors of the $SO(1,1)$ Lorentz symmetry on the brane. As there are 8 Dirichlet–Dirichlet (DD) directions, the Neveu–Schwarz (NS) sector has zero point energy $-\frac{8}{24}-\frac{8}{48}=-\frac{1}{2}$. The vectors $A^\mu(x^0, x^1)$, ($\mu=0, 1$) come from the excitations of the Neumann–Neumann (NN) directions:

$$\lambda_V \psi_{-\frac{1}{2}}^\mu |0\rangle \quad \text{with} \quad \lambda_V = -\gamma_\Omega \lambda_V^T \gamma_\Omega^{-1}, \quad \mu = 0, 1. \quad (42)$$

Here, λ_V is an $N \times N$ Chan–Paton matrix, which must satisfy the conditions shown. The γ_Ω matrices are $N \times N$ matrices chosen to represent the action of Ω on λ_V . As a result the vectors carry an $O(N)$ gauge symmetry.

The transverse fluctuations are a family of eight scalars $\phi^i(x^0, x^1)$, ($i=2, \dots, 8$) on the world volume. These come from the 8 DD directions

$$\lambda_\phi \psi_{-\frac{1}{2}}^i |0\rangle \quad \text{with} \quad \lambda_\phi \rightarrow \gamma_\Omega \lambda_\phi^T \gamma_\Omega^{-1}, \quad i = 2, \dots, 8. \quad (43)$$

The scalars therefore transform in the $N(N+1)/2$ dimensional symmetric tensor representation of the gauge group. The $SO(8)$ global symmetry which rotates them into one another is a simple consequence of the symmetry the brane configuration and corresponds to the R–symmetry of the chiral $\mathcal{N}=8$ model we are studying.

The fermionic states ξ from the Ramond (R) sector (with zero point energy 0, by definition) are built on the vacua formed by the zero modes ψ_0^i , $i=0, \dots, 9$. This gives the initial **16** supercharges mentioned earlier. The GSO projection acts on the vacuum in this sector as:

$$(-1)^F = \Gamma^0 \Gamma^1 \dots \Gamma^9, \quad (44)$$

while as Ω acts as -1 on NN strings (*i.e.*, in the (x^0, x^1) directions), it is:

$$\Omega = \Gamma^2 \dots \Gamma^9. \quad (45)$$

So we have $(-1)^F \xi = \xi$ from the GSO projection, and with Ω , it simply correlates world-sheet chirality with spacetime chirality: $\Gamma^0 \Gamma^1 \xi_{\pm} = \pm \xi_{\pm}$, where ξ_- is in the $\mathbf{8}_c$ of $SO(8)$ and ξ_+ is in the $\mathbf{8}_s$. They are the superpartners of ϕ^i and A^μ , respectively, carrying the same $O(N)$ charges as their bosonic superpartners, ensuring that gauge symmetry respects supersymmetry.

The 1–9 strings will also form a family of fields on the world-volume. There are 8 Dirichlet–Neumann (DN) coordinates, giving ground state energy $\frac{1}{2}$, and so there are no massless states arising in the NS sector. The R sector excitations come from the NN (x^0, x^1) system giving just two ground states. In this sector, the GSO projection is simply $(-1)^F = \Gamma^0 \Gamma^1$, which picks the left-moving field⁸³. As we have gauge group $SO(32)$ from the D9-branes, we have left-moving fermions λ_+^M in the $(\mathbf{N}, \mathbf{32})$, where M is an $SO(32)$ (D9-brane) index.

Consider the case of one D1-brane for a moment. Then we have no gauge group on the world-volume, as the vectors are projected out by Ω , and the remaining fields are simply the eight scalars ϕ^i , their right moving superpartners ξ_- in the $\mathbf{8}_c$ and the 32 left moving fermions λ_+^M . This is simply the content of the $SO(32)$ heterotic string in static gauge where the λ_+^M are the current algebra fermions. The action for this theory is simply the light-cone gauge Green–Schwarz action (29) for the heterotic string with a current algebra term added.

In the case of the multiple D1-branes, we have a non-abelian generalisation of that model:

$$S = \int d^2\sigma \text{Tr} \left[T_2 \partial_\mu \phi^i \partial^\mu \phi^i - i S_\alpha^T \mathcal{D} S^\alpha - i \lambda_+^M \mathcal{D}_- \lambda_+^M + \frac{1}{g^2} F_{\mu\nu} F^{\mu\nu} + \text{extra} \right] \quad (46)$$

Here, $g^2 \sim g_s / \alpha'$ is the effective gauge coupling of the (1+1)–dimensional theory, and $S_\alpha^T \equiv (\xi_-^\alpha, \xi_+^\alpha / g)$, with $\mathcal{D} \equiv (\mathcal{D}_+, \mathcal{D}_-)$.

In models such as this, the “extra” terms may be written as a combination of commutators between the various fields, their precise form determined by gauge invariance and supersymmetry. In this example, one such term is $g^2 [\phi^i, \phi^j]^2$ and a similar term for the S_α , and a Yukawa term coupling the ξ_{\pm} and ϕ^i . Such terms constitute the “scalar potential” of the model.

Obbligato: The supersymmetric vacua of such a gauge theory are those for which the “scalar potential” is identically zero. We can immediately study the classical solutions of this condition by just treating the vanishing of those terms as an algebra problem. The space of *gauge inequivalent* solutions of this condition is grandly termed the “moduli space of classical vacua”. In general, quantum corrections can modify our answer, but with the right type or amount of supersymmetry (for example), the classical analysis is equivalent to the quantum analysis. This moduli space is the space of allowed values that the fields can take. Given that we have already realized that the fields on the world-volume of the branes are in one-to-one correspondence with the geometry that the brane encounters—both the embedding space and the shape that it can take in that embedding space—evaluating the moduli space of vacua is equivalent to discovering this geometry. This is the key to many relationships between geometry and field theory.

Turning to the moduli space of vacua of this model, we see that the point with gauge symmetry $O(N)$ is a special point of enhanced gauge symmetry. All of the scalar fields have zero vevs, and so the commutators (and hence the scalar potential) vanishes identically. This corresponds to all of the D1-branes being at the same point, which we have taken to be the origin of the eight-dimensional space \mathbb{R}^8 parameterized by ϕ^i , $i=2, \dots, 9$. Generically, the fields can have non-zero vevs, but we wish to still consider supersymmetric solutions, which is to say we want the potential still to vanish. A solution is to make the ϕ^i 's non-zero, but all commute. In this way, we break the gauge symmetry down to the “maximal torus” (the largest Abelian subgroup) of $SO(2N)$, which is $U(1)^{\frac{N}{2}}$. This corresponds to separating out N D1-branes pairs. This situation is further reducible however (in contrast to a similar situation for D5-branes^{78,89}), and we may split the D1-brane pairs. The resulting gauge group is $\{0\}$ as we have seen, and there is one eight component scalar ϕ^i left for each of the N D1-branes, representing their transverse positions. (Indeed, that we can Higgs the gauge group away leaving N scalars follows from the fact that the difference between the dimensions $N(N-1)/2$ of the adjoint and the $N(N+1)/2$ of the symmetric is N .)

As they all commute, we may find a basis where we simultaneously diagonalize the ϕ^i matrices, putting their eigenvalues down the diagonal: each eigenvalue represents an individual D1-brane. Notice that the Weyl group is still a gauge symmetry here, acting to permute the eigenvalues. This translates into the fact that the theory does not care if we rearrange the N D1-branes, as they are identical. Therefore the classical moduli space of vacua is not $(\mathbb{R}^8)^N$, but $(\mathbb{R}^8)^N/S_N$. The action of this S_N will have important conse-

quences shortly. Notice that we can get special points of $O(n)$ enhanced gauge symmetry whenever n D1-branes coincide, which corresponds to having n simultaneous eigenvalues in the eight ϕ^i 's.

We have not quite finished the job yet, as we have not discussed the allowed vacua of the superpartners ξ at all. However, this is not necessary, as we have sought supersymmetric solutions here. Therefore, their allowed values are determined by the unbroken supersymmetries. There remains to be determined the allowed values of the left-moving current algebra fermions λ_+^M . Up to subtleties we will mention later, this is simply parameterized by the fact that they are fundamentals of the D9-brane gauge group $SO(32)$, and hence parameterize the vector space $V_{32} \approx \mathbb{R}^{32}$ that it acts on. So the full moduli space is schematically $(\mathbb{R}^8 \times V_{32})^N / S_N$.

4.3 From D1-branes to Fundamental Strings

So we understand now that the type I supergravity model of the solitonic heterotic string that we were studying previously represents the fields around N coincident D1-branes. The world-volume theory of that soliton has been found more precisely to be our 1+1 dimensional gauge theory.

Notice that the coupling of the gauge theory is a function of the type I string coupling. We had promised that as g_s goes to infinity, we would arrive at the heterotic string theory. What does this mean for the 1+1 dimensional model? The 1+1 dimensional coupling gets strong too, and so as a 1+1 dimensional gauge field theory, it should flow to the infra-red, presumably to a fixed point. In the special case of one D1-brane, the conformal field theory that we flow to is clearly the $(0, 8)$ supersymmetric $(c_L, c_R) = (24, 12)$ conformal field theory of the free $SO(32)$ heterotic string, but what of other N ?

For general N , the model is a non-Abelian gauge theory, and therefore there are potential terms like $g^2[\phi^i, \phi^j]^2$. As the string coupling goes large, so does g , and this term becomes very important. Indeed, at infinite g , the only way to find supersymmetric vacua is to force this term to zero by demanding that the ϕ^i all commute, generically (to set them all to zero is highly non-generic). So in effect, strong coupling forces us out onto the Abelian (Coulomb) branch again, and the allowed values of the ϕ^i 's are in $(\mathbb{R}^8)^N / S_N$. What is the interpretation of this?

Given that we identify configurations related by the action of S_N , the permutation of the eigenvalues, it is useful to think of S_N as a sort of discrete gauge symmetry. The usefulness of this comes when we recall that our world-volume theory arose in representing the dynamics of a stable closed string made by winding it about a circle (x^1) with a very large radius R^1 . We have

not discussed that feature much so far, but it is crucial. Indeed, this model of N D1-branes is indistinguishable from a model of one D1-brane wound N times around the large circle, or any number of D1-branes with individual windings distributed among them to make total winding N . The moduli space $(\mathbb{R}^8)^N/S_N$ encodes precisely that. The interpretation as N D-branes that can be permuted by S_N is therefore a small part of the story. The ϕ^i are *matrix-valued* functions $\phi^i(x^1)$ which can go around the circle (with coordinate x^1 , recall) and return to their original value *up to an action of an S_N permutation of their eigenvalues*.

Imagine a particular field configuration $\phi_n^i(x)$ with such a “twisting” of its boundary conditions by $S_n \subset S_N$: $\phi_n^i(x^1) = S_n \cdot \phi_n^i(x^1 + 2\pi R^1)$. If the permutation S_n is irreducible, then we can return to the original matrix $\phi_n^i(x^1)$ only by going around the circle n times. In other words, this multi-valued configuration may be written as a single-valued one by using a circle of radius n times that of the basic circle: $\phi_n^i(x^1) = \phi_n^i(x^1 + 2n\pi R^1)$. We have just described a configuration representing the winding of a single D1-string n times around the circle. Figure 6 shows the situation for $n=3$, covering the circle three times. We have focussed on a 3×3 submatrix of ϕ^i , with entries $x_1(\sigma)$, $x_2(\sigma)$ and $x_3(\sigma)$, functions which get permuted every time we go around the circle. Here, σ is the world-sheet spatial coordinate of the string which gets identified with the circle x^1 in this “static gauge”. The physical string is the solid line, made by gluing the three functions together.

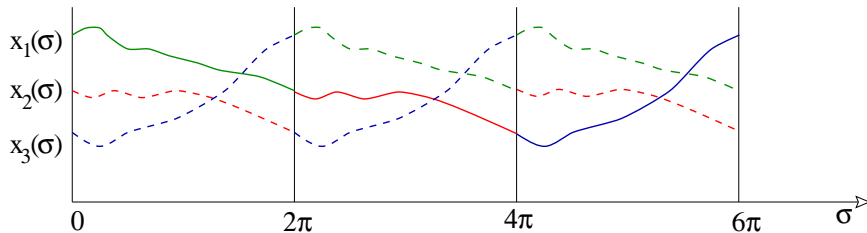


Figure 6: A winding D-string of length three, made by acting with an S_3 twist. The physical string is the solid line, made by gluing the three functions together. (Green, then red, then blue, for viewers in colour.) At strong coupling, this string becomes the fundamental heterotic string with three units of momentum in the discrete light-cone gauge. (See text.)

The beauty of this description is that it is a very economical way of describing strings of arbitrary shape n in the embedding space \mathbb{R}^8 (wound along a direction x^1 , with any amount of winding up to a total of N). The shortest

ⁿRecall from section 2.4 that the shape of the brane is encoded in the actual functional form of the eigenvalues of the matrix $\phi^i(x^1)$.

length of string is $2\pi R_1$ and the action of the discrete gauge group S_N glues together these short strings to make longer strings. We have another language for this: This (1+1)-dimensional field theory *is actually a light-cone string field theory*, as it has fields $\phi(x)$, which create and destroy strings of arbitrary shape in the transverse \mathbb{R}^8 . Notice that in this light-cone interpretation, the static gauge string with winding N is exchanged for a light-cone gauge string with momentum N/R , as is consistent with T-duality.

Now all of this could have been said before about the Coulomb branch, without recourse to the large g limit, but what makes this very relevant is the fact that at strong coupling, each of the short strings, which has tension $\sim 1/g$, becomes a light heterotic string. Furthermore, as N becomes large it can be shown¹¹¹ that the long strings made by gluing the short strings together have the same world-volume dynamics as the short strings, and that the interactions between the strings are generated by the field theory interactions. Unfortunately, lack of space does not permit us to describe the full story here, but luckily H. Verlinde will describe this subject of “Matrix String Theory” in his lectures¹¹².

Notice that in taking N large, but staying in the $\alpha' \rightarrow 0$ limit, we have taken ourselves back to the realm of validity of the original supergravity discussion of the previous discussion. The description of the N D-brane soliton conglomerate is well approximated by the supergravity solution (19). However, we have taken the string coupling to be very large, and therefore (because of the dilaton’s behaviour) we are studying the physics closer and closer to the core of the solution. In infinite coupling, we should simply exchange this solution for the heterotic one given in equation (16), arriving at the neighbourhood of the core of that solution.

As we noticed before, however, the fundamental string core is singular as a supergravity description. We were supposed to think of this as a result of the natural breakdown of the description of the self-consistent solution at weak coupling, and we can replace this singular core with a delta-function source, leaving conformal field theory vertex operators to take over the description⁹³. Now we see that we do have a complete description of the missing physics in the $SO(32)$ heterotic supergravity description, it is the $(\mathbb{R}^8 \times V_{32})^N / S_N$ orbifold conformal field theory^{95,52}.

There is an important subtlety here. We have seen that the orbifold conformal field theory describes heterotic strings, but we have not been careful to check which heterotic string. Recall that they are indistinguishable at tree level, and therefore the naive large N and g analysis above is good for either string. Similarly, the “duality” to the supergravity solution can be to either heterotic supergravity fundamental string solution if we are not careful.

It turns out that a more careful treatment of the model¹¹⁶ for arbitrary g and radius R requires that a Wilson line be turned on in the x^1 direction breaking the $SO(32)$ gauge symmetry to $G=SO(16)\times SO(16)$ and thus dividing the current–algebra fermions λ_+^M into two classes: periodic or antiperiodic as they wind around the circle x^1 . The full moduli space of the model is then $(\mathbb{R}^8 \times (V_{16} \oplus V_{16}))^N / (S_N \times \mathbb{Z}_2)$, where the \mathbb{Z}_2 exchanges fermions in either V_{16} factor. In the large N, R and g limit, it turns out that the long heterotic strings which are recovered are $E_8 \times E_8$ heterotic strings! This turns out to be consistent with the fact that the model can be obtained by compactifying Matrix theory on a line interval: Matrix theory is the light–cone representation of M–theory and M–theory on a line interval gives the $E_8 \times E_8$ heterotic string^o

To get the $SO(32)$ heterotic string in this way is more subtle. It has been shown⁹⁵ that the tree level description of the heterotic string is indeed identical to the above (including the “dual” supergravity description in terms of the core of a fundamental string), but the situation is different when the string coupling is turned on: The interacting physics is described by an exotic 2+1 dimensional model. This turns out to be consistent with the fact that the $SO(32)$ heterotic string arises as M–theory compactified on a cylinder.

4.4 Cadenza: Where Is The Fundamental Type I String?

Going back to the discussion of the previous section, it is reasonable to ask after the whereabouts of the solution representing the fundamental type I string. Surely, for consistency, there should also be a solution of the type I supergravity (15) representing the fields created by the string itself?

It is important to note that when we did this for the heterotic string, we constructed a stable solution. It was a BPS state, stable because it represented a closed string wrapped on a circle of large radius. As it is a closed string, it cannot shrink away to minimize its energy. By contrast, a type I string cannot be made stable by the same procedure because it can reduce its energy by snapping, (still ending on space–filling D9–branes) making shorter strings in this way. So we cannot make stable strings in this way, and therefore cannot expect to find a BPS state in the spectrum corresponding to such a

^oThe line interval is explicit in the description if one T–dualizes on x^1 and works in the dual theory. The action of T–duality combined with the orientifolding Ω makes the dual circle into a line interval by orbifolding. There is an O8–plane at each end of the interval. The D9–branes become point–like along the dual circle and turn into D8–branes. This is called “type IA” string theory, to contrast it with the ordinary type I theory, which might be called the “type IB” theory. (The terminology is good, as they each descend from each of the similarly named type II theories by simple orientifolds.) Local considerations^{83,107} require that there be 8 D8–branes at each end of the circle, breaking $SO(32)$ to $SO(16)\times SO(16)$.

configuration. One does not expect to find it in the type IB supergravity as a fundamental string solution, and not in the heterotic supergravity as a stable soliton solution (although one might be able to make metastable solutions).

One should not be discouraged, however. A failure to find a stable solution does not mean that the dual string (which we want to become light at strong coupling and take over the spectrum) does not exist. It simply means that things are a little more interesting.

It turns out that there is a way⁹⁴ of seeing the fundamental string, but only in a particular frame, and the winding circle is a clue as to how. Imagine that we place the type I string theory on a circle in the x^1 direction of radius R . We give it N units of momentum in that direction. Consider the limit where we take N and R to infinity, to define the string in the “Infinite Momentum Frame”. In that frame, the degrees of freedom which survive are those which have some finite fraction of positive momentum in that direction. So in the limit, we are probing arbitrarily small distances along that direction.

In string theory in this frame, there is a smallest possible “string bit” length¹⁰⁸ that strings can have. We are therefore able to stretch stable strings of that length, as they cannot break. Ultimately, therefore, we see a sub-structure in that direction corresponding to being able to resolve a transverse distance “between” the individual D9-branes. The D9-branes are point-like along this special transverse coordinate in this limit.

This is a “matrix string” representation of the type I string in the infinite momentum frame. The 1+1 dimensional lagrangian for this model (in the free string limit) is the large N limit of a matrix-valued Green–Schwarz type IIB light-cone action with type I boundary conditions as one goes around the spatial direction. The size of the matrices is N . Sectors of momentum n are represented by matrices which have their boundary conditions twisted by a non-trivial permutation of n eigenvalues, where n is tuned to be a finite fraction of N as we take the limit.

There is another language to describe this: to probe those very short distances, we have T-dualized to a type IA system, with N units of winding along the T-dual direction \hat{x}^1 . The 16 D8-branes are the places where the winding type IA strings can end. They are stable, as they cannot break unless another D8-brane is located at that point. The 1+1 dimensional Green–Schwarz action above controls the world-volume dynamics of the strings stretching along the light-cone direction, which is also a static gauge action for the type IA string.

This type IA description of the light-cone matrix type IB string^{95,94} can be used to show that the matrix string theory thus defined has $SO(32)$ gauge symmetry, and that the closed string sector is unoriented. This latter follows from the crucial fact that one must flip the copy of the line interval each time

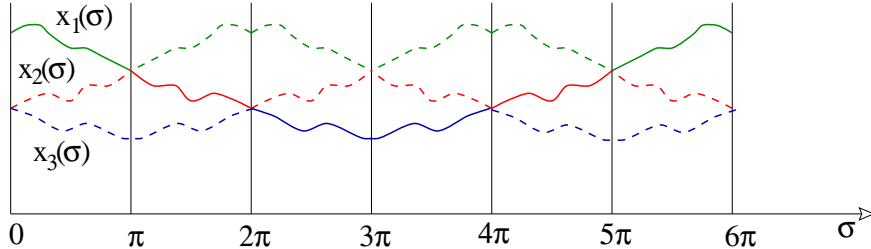


Figure 7: A winding closed unoriented type I string of length six, made by acting with an S_3 twist with an \mathbb{Z}_2 folded in. This can also be viewed as an open string of length three and its mirror. The physical string is the solid line, made by gluing the three functions together. (Green, then red, then blue, for viewers in colour.) The strings can end at any multiple of π , where there are $SO(16)$ Chan–Paton factors. Odd and even open strings fill out the full $SO(32)$ gauge multiplet. (See text.)

one extends to make a multiple cover of the line interval. See figure 7.

We take the $N, R \rightarrow \infty$ limit and $\alpha' \rightarrow 0$. In supergravity terms, the resulting limit is the core geometry of a fundamental string solution of the type IA (massive^{45,109} type IIA) supergravity^{95,94}. The string ends on D8–branes at infinity, which in this limit are domain walls. This solution represents the fields produced by a type I string in the limit where we probe to the substructure showing it stretched between D9–branes.

5 Type I String Theory Under the Microscope II: Instanton Redux

5.1 D9–Branes and D5–Branes

Instead of adding D1–branes to the type I theory, we can add D5–branes. Let us add N of them, all initially coincident. We do this by adding Dirichlet conditions to the theory requiring that strings can end at points in the subspace x^6-x^9 , thus defining a 5+1 dimensional world–volume of the D5–branes in x^0-x^5 .

In the presence of the brane, the spacetime Lorentz group breaks as

$$SO(1,9) \supset SO(1,5) \times SO(4), \quad (47)$$

where the $SO(1,5)$ refers to the world–volume for the D5–brane. The spacetime supercharges decompose as $\mathbf{16} \rightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2})$. Here the $\mathbf{4}$ ’s are vector representations of the $SO(4)$. The $SO(1,5)$ representations are given in terms of the $SU(2) \times SU(2)$ little group of the Lorentz group.

The D5–branes are annihilated by one of these (choose the first), and the other generates superpartners in the $\mathcal{N}=1$ supersymmetric model in 5+1

dimensions. This amount of supersymmetry has an $SU(2)$ R-symmetry which we shall denote as $SU(2)_R$.

We can study the content of the theory in a similar way to the above discussion for D1-branes, and the following content results⁷⁸:

The 5–5 strings with coordinates along the world-volume transform as vectors and give a $USp(2N)$ gauge theory. The 5–5 strings transverse give a family of 4 scalars transforming in the antisymmetric representation of $USp(2N)$. Ω acts on the fermions as $\Gamma^1 \dots \Gamma^5$, and correlates their $USp(2N)$ transformation properties with six dimensional chirality⁸².

There are four DN coordinates (x^1-x^5), and four DD coordinates (x^6-x^9) giving the NS sector a zero point energy of $-\frac{1}{2}+\frac{4}{8}=0$, with excitations coming from integer modes in the 1234 directions, giving initially a four component boson $h_{\alpha m}^M$ where M, m are $SO(32)$ and $USp(2N)$ indices respectively. As it acts by exchanging the ends, the Ω projection relates the 9–5 strings to the 5–9 strings: $(h_{\alpha m}^M)^* = \epsilon_{mn} \epsilon_{\alpha\beta} h_{\beta n}^M$, where the ϵ 's are the antisymmetric tensors of the $SU(2)_R$ and the $USp(2N)$. The 5–9 strings therefore give a half-hyper transforming in the $(2N, 32)$ of $USp(2N) \times SO(32)$. The R sector also has (as always) zero point energy 0, with excitations coming from the 6789 directions, giving a initially four component fermion χ , halved to two components by Ω , as before.

Returning to our six-dimensional $K3$ compactification for a moment, we can see¹⁰⁶ how this fits into the anomaly equation (40). There, as the transverse part of the space was the compact $K3$, we had 24 branes and therefore gauge group $SO(32) \times USp(48)$ in this point-instanton limit. So we have $n_V=496+1176=1672$, the dimension of the gauge group. The 9–5 sector supplies a half-hyper in the **(32, 48)** and the 5–5 sector has a set of hypers in the antisymmetric of $USp(48)$ which is therefore the **(1, 1128)**=**(1, 1127)**+**(1, 1)** with the reduction showing the center of mass position of the multi-instanton. Together with the 80 moduli of $K3$ (equivalent to 20 hypers), this gives $n_H=1916$, as required by the anomaly equation. Once again, further checking of the anomaly polynomials will reveal that all anomalies cancel.

This is of course a special point in the allowed space of vacua, and we can characterize the classical moduli space as we did previously for the D1-branes.

• *The Coulomb Branch*

There is a Coulomb branch analogous to that which we found for the D1-branes, simply corresponding to moving the D5-branes apart. This is done by giving vevs to the 5–5 hypermultiplets, breaking the gauge group by the Higgs mechanism. Done in the most complete way, we get $SU(2)^N$, corresponding

to N separated D5-branes ^{*p*}. The remaining 5–5 hypermultiplet is a singlet, whose four real components correspond to the position of the D5-brane in x^6-x^9 . The 9–5 fields on the other hand are N half-hypers in the $(\mathbf{2}, \mathbf{32})$. (It is interesting to verify that this spectrum is also anomaly free ¹⁰⁶.) The four components of the 9–5 half-hypers correspond to some additional data concerning the D5-brane, in the $SO(32)$ background which characterizes the “Higgs branch”.

• *The Higgs Branch*

At any stage, we can also give vevs to the 9–5 strings. Let us consider first the simplest case of a single unit, with gauge group $SU(2)$. The $(\mathbf{2}, \mathbf{32})$ half-hyper allows us to completely Higgs away the $SU(2)$ fivebrane group and break the $SO(32)$ to $SO(28)\times SU(2)$. Let us consider the details of this ⁷⁸.

Denote the half-hypermultiplet as $h_{\alpha m}^M$ where M is an $SO(32)$ index, m is an $SU(2)$ index and α is a doublet index of the the $SU(2)_R$ symmetry, labelling the components of the half-hyper. We wish to discover what allowed values of $h_{\alpha m}^M$ preserve the vanishing of the scalar potential.

The scalar potential can be written as a sum of “D-terms”

$$D_{\alpha\beta, mn} = \sum_M (h_{\alpha m}^M h_{\beta n}^M + h_{\beta m}^M h_{\alpha n}^M). \quad (48)$$

now we may alternatively consider the product of the gauge $SU(2)$ and the R-symmetry $SU(2)_R$ as $SO(4)$, in which case we may rewrite $h_{\alpha m}^M$ as a family of 32 $SO(4)$ vectors h_i^M where i is an $SO(4)$ vector index. Alternatively, h_i^M may be thought of as the components of four vectors, h_i , in a 32 dimensional vector space $V \approx \mathbb{R}^{32}$ on which $SO(32)$ acts.

So the “D-flatness condition” (vanishing of the D-terms) is equivalent to

$$(h_i, h_j) = \rho^2 \delta_{ij} \quad (49)$$

where $(\ , \)$ is the inner product on V . So the vectors $e^i = h^i / \rho$ are orthonormal vectors. Choosing the four orthonormal e^i in V breaks the $SO(32)$ down to $SO(28)\times SU(2)$, where we get the extra $SU(2)$ by treating equivalent h^i 's which are gauge related by $SU(2)$. Therefore that $SU(2)$ arises as a result of dividing by the original fivebrane gauge group to get the correct moduli space of vacua.

So what do we have? The allowed values of h leave us with gauge group $SO(28)\times SU(2)$. The subgroup of $SO(32)$ which commuted with this gauge

^{*p*}This is a sort of “non-abelian Coulomb branch”, given that the gauge group is not some power of $U(1)$.

group is an $SU(2)$, which is fully specified by choosing four parameters, the scale ρ and the orientation of the e^i basis. This is exactly the data needed to specify an $SO(32)$ instanton gauge bundle with structure group $SU(2)$. The scale size of the instanton (“thickness” of the fivebrane) is ρ .

What we have done is described the one-instanton version of the “hyperKähler quotient” technique⁸⁰ of constructing instantons. In the general case of $USp(2N)$ with half hypers in the $(2N, 32)$, we get the full ADHM construction of N $SO(32)$ instantons^{78,71}. So we discover that the Higgs branch of our gauge theory is parameterized by the 9–5 fields and is isomorphic to the moduli space of $SO(32)$ instantons⁹.

This harmonizes perfectly with our discussion of section 3. The supergravity and anomaly analysis led us to anticipate a new massless sector of the theory, arising as the vev of a hypermultiplet goes to zero. We see that this massless sector is in the form of the gauge theory on the world-volume of type I D5-branes. Let us call this D5-brane gauge theory with this particular content the “ADHM gauge theory” for short, after the structure of its Higgs branch analyzed above.

5.2 Cadenza: ADHM Gauge Theory as String Theory on a Throat

The 5+1 dimensional ADHM gauge theory has a well defined Coulomb and Higgs branch. At weak coupling (the infra-red (IR)), its Coulomb branch supplies the missing massless degrees of freedom when type IB the supergravity description of instanton fivebranes break down, as we saw.

It is interesting to speculate that the heterotic throat description which we were led to earlier did not break down so much as simply take us to a realm where we were unsure of the interpretation of some of our tools. There is the possibility that the D5-brane description and the throat description might be complementary⁹⁶, which is the oft-repeated lesson of duality. The

⁹This immediately generalizes. The important features in the above discussion was the type of half-hypermultiplet which appeared, in a bi-fundamental representation of a product gauge group where the products came from one brane inside another brane. such a multiplet will always appear when there are four DN direction which happens when a Dp -brane is inside a $D(p+4)$ -brane. Therefore, a D1-brane is an instanton of a D5-brane, a D0-brane is and instanton of a D4-brane, and so on⁸².

Yet another way to see this is from the world-volume action of the $D(p+4)$ -brane. It contains Chern–Simons couplings of the form $\int A_{p+1} \wedge F \wedge F$ where A_{p+1} is a R–R field, and the integral is over the $(p+5)$ -dimensional worldvolume. Therefore, a Dp -brane acts as a source of $F \wedge F$ in the world volume by virtue of being a source of A_{p+1} . It is also worth noting that there is also a term of the form $\int A_{p+1} \wedge R \wedge R$, which means that wrapping the $D(p+4)$ -brane on a space of non-zero $R \wedge R$, like $K3$, will result in it appearing as a p -dimensional source of A_{p+1} R–R charge.

linear dilaton description, with its exact (WZNW+FF) conformal field theory (CFT) representation, might still capture the physics of the gauge theory after all. Previously, however, we said that the heterotic string perturbation theory—including conformal field theory—cannot describe the D5-brane physics, not the least because the allowed gauge groups (*e.g.* as large as $USp(48)$) in our $K3$ example above) would give a central charge much greater than 24, so surely this is a contradiction?

It is not a contradiction⁹⁶ for at least two reasons:

- The conformal field theory of the throat is *completely disconnected* from the theory outside the throat. Indeed, it has been long known that throat conformal field theories are notoriously difficult to connect to the asymptotically flat region: The operators which describe the widening of the throat are singular⁷⁶. Now we know why. The correct interpretation of the string theory down the throat, as captured for example by the exact (WZW+FF) CFT is that it is a dual description of the theory on the world volume of the D5-branes.
- The gauge theory does not need to be explicit. This certainly has to be true, for the reasons stated above, and so it must be encoded in a different way. Given that the CFT naively seems to be describing a strongly coupled heterotic background, which invalidates many of the standard interpretations, is clear that there is some room for a new interpretation.

In capturing the physics of the ADHM gauge theory in a dual model, it is crucial to realize that we only need find the *gauge invariant* physics. This is why we need not find the gauge particles explicitly down the throat. An example of the gauge invariant information which the dual representation should capture is the moduli space of vacua. Part of this is the Coulomb branch representing the patterns of Higgsing which can occur as the D5-branes are moved around in x^6-x^9 directions, including possible coincidences. This is something which should be captured in the throat limit. Indeed, there are arguments to support this, with the $USp(2N)$ structure showing up in the modular invariant used to build the CFT partition function. This controls the moduli space of marginal deformations of the CFT which is isomorphic to the Coulomb branch of the ADHM gauge theory. This can be thought of as another generalization of the AdS/CFT correspondence, where gauge theory and geometry complement one another, this time with a non-trivial dilaton playing a crucial role^r instead of negative cosmological constant¹⁰².

^rThis relationship was pointed out in paper⁹⁶, but the author carelessly did not put the

Notice that the string coupling is $g_s=e^{-\sigma/R}$. The region where the throat conformal field theory does have the traditional interpretation is when $\sigma\rightarrow+\infty$, which is a *continuation* of the throat (continue the left-hand part of the diagram 4 infinitely to the right). While staying down the throat, we have continued to a region where the heterotic string is weakly coupled. What does this limit correspond to for the putative dual gauge theory?

As a six dimensional gauge theory, heading towards the ultra-violet (UV), the theory should break down at (mass)² scale $1/g_{YM}^2=1/(\alpha'g_s)$, where g_s is the type I string coupling, which is going strong if we keep α' fixed for a moment. As $g_s\rightarrow\infty$, we can ask what this physics looks like in the heterotic picture. We see that the (mass)² scale is simply $1/\alpha'$ in heterotic string units — the heterotic string coupling has gone from the formula. The independence of the gauge coupling of the string coupling in heterotic variables is a clue that there is a sensible theory¹¹⁰ ^s living on the world-volume of the NS-NS brane in the limit of vanishing heterotic string coupling. In other words, the heterotic string is telling us that there is no real problem with the ADHM theory in the UV, and we have found a description in heterotic variables in terms of the throat CFT! Indeed, we can control the approach to the UV in terms of the usual small α' expansion.

So we indeed have a new gauge theory/geometry correspondence. The theories are complementary, as the weakly coupled (IR) limit of the ADHM gauge theory is best described in terms of the defining lagrangian, because the throat theory is strongly coupled, while the UV of the ADHM theory is best described in terms of the weakly coupled string theory propagating on the throat, described by the WZNW+FF conformal field theory.

It is worth noting another attractive feature of this possibility. To properly describe string theory propagating on the throat in the weakly coupled supergravity limit, we should really take N large as $\alpha'\rightarrow 0$, in order to keep the radii ($R=\sqrt{N\alpha'}$) of the S^3 's large enough to keep the curvature corrections down. We actually seem to be able to describe much more than this restriction would suggest. The exact CFT representation means that we have not only the leading order in α' description of the geometry, but the full α' description. Furthermore note⁷⁴ that the precise combination of the $SU(2)_N$

word “holography” anywhere in the title, abstract or body of the paper. Fashion-conscious readers should therefore instead see ref.¹⁰³ for later (independent) work on holography and linear dilatons, in the context of the type IIA NS-NS branes. That work also discusses in detail how the holographic correspondence should work, given the peculiar properties of the throat theory.

^sThis type of theory, associated with the four types of NS-NS brane in the $SO(32)$, $E_8\times E_8$, type IIA and type IIB string theories, is sometimes called a “little string” theory, or a “micro-string” theory¹¹². See the lectures of H. Verlinde at this school.

WZNW model and the Feigin–Fuchs theory is such that the value of N is cancelled exactly in the central charge formula: The WZNW model and the three free fermions needed for supersymmetry ^t give $c=3-\frac{6}{N}+\frac{3}{2}$, while the Feigin–Fuchs scalar plus its fermion gives $c=1+\frac{6}{N}+\frac{1}{2}$. Their total central charge is exactly 6, which together with the $c=9$ from the superstring propagating on the flat x^0-x^5 gives the required $c_{\text{tot}}=15$ ^u This exact formula for the central charge should be reflected in the properties of the fields and vertex operators in the full conformal field theory. Therefore, when endowed with the correct interpretation, the exact CFT should contain the complete story for all N and all α' .

Therefore the full “holographic” statement is that the ADHM gauge theory at any N is described by the heterotic string propagating on the throat geometry, which is succinctly given by the exact WZWN+FF conformal field theory.

It is also natural to expect that a gauge theory interpretation of this type will exist for the many other types of exact throat CFT’s in existence in the literature.

6 Recapitulation

Duality and D–branes have taught us a number of lessons about non–perturbative string theory. Let us list a few of them. First, duality says that:

- The low–energy effective actions of the massless fields in the string spectra map into one another under the duality transformation.
- The solutions of the corresponding equations of motion also map into each other. In particular:
 - The NS–NS charged fundamental string solution, light at weak coupling and representing the perturbative string, maps into a R–R charged solitonic string, heavy at weak coupling (tension $\sim 1/g_s$) in the dual string theory.
 - The NS–NS charged fivebrane soliton solution (tension $\sim 1/g_s^2$) maps into a special type of R–R charged fivebrane soliton, (tension $\sim 1/g_s$). Both fivebranes are $SO(32)$ instantons. The scale size of the instanton is equivalent to the thickness of the fivebrane.

^tNote that chiral rotation ⁸⁴ needed to make the supersymmetry fermions free ⁸⁵ has the effect of shifting the level N to $N-2$.

^uThis is for the right hand, supersymmetric, side of the theory. The left hand side has $c_{\text{tot}}=26$, which can be arranged ⁹⁶.

- As supergravity solutions, the various fivebrane solitons have special properties as the instanton size vanishes:
 - The NS–NS fivebrane has an infinite throat at its core, down which the string coupling grows exponentially. Meanwhile, the R–R fivebrane is singular at the core, although the string coupling is weak there.

So the supergravity description evidently has problems when the fivebranes are thin.

We need another description of the physics in the small instanton/thin five-brane limit. This is where D–branes come in:

- The type I supergravity is singular because we missed some massless states. There are massless vectors associated with an enhanced gauge symmetry which appears in the small instanton limit.
- The small instanton (thin fivebrane) has a description as a D5–brane, introduced into type I conformal field theory by adding Dirichlet boundary conditions. N D5–branes in type I have a $USp(2N)$ gauge theory with two classes of hypermultiplets possessing transformation properties and couplings which constitute what we called an “ADHM gauge theory”. The space of allowed vevs of some of the hypermultiplets is isomorphic to the moduli space of instantons. The hypermultiplet vev which controls the thickening of the fivebrane also gives masses to the vectors, taking us back to the supergravity description.

So we saw that gauge theory supplements the type I supergravity description. It is hasty to throw out the heterotic supergravity throat description, however. It gives a dual representation of the Coulomb branch physics of the ADHM gauge theory. In particular, it has an exact conformal field theory description, with a strongly coupled regime (hard to interpret) and, by continuation, a weakly coupled regime, which supplies a complementary description of the UV of the ADHM theory. Indeed, the gauge invariant information — *e.g.*, the moduli space of deformations— can be encoded in the heterotic modular invariant of the conformal field theory⁹⁶.

This is another type of holography, this time with the linear dilaton playing the role that negative cosmological constant did in the AdS case^{102,103}. Notice that although the throat was properly a supergravity solution in the large N , small α' limit, the form of the exact conformal field theory description suggests that this is actually true for all N, α' which is at the least, very interesting.

We have therefore the complete holographic statement that the ADHM gauge theory is dual to heterotic string theory propagating on a throat, described by an exact conformal field theory.

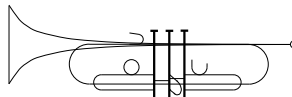
* * *

So we have come full circle over the last ten years. We started out with large N matrix model descriptions of very low dimensional string theory. The double scaling limit allowed for a complete description at all orders in the $1/N$ expansion (which is isomorphic to the string genus expansion), and non-perturbative information as well. The non-perturbative effects were associated with the tunneling of a matrix eigenvalue, giving $e^{-N} \sim e^{-1/g_s}$ effects.

D-branes are responsible for e^{-1/g_s} effects in critical string theory. They have a gauge theory on their world volume. These gauge theories are dual to string theories in a manner superficially similar to the simpler matrix models. It is hopefully clear, after the discussion in the five studies collected here, that these things are not coincidences: The gauge theories are also “matrix models” in an enlarged sense, and D-branes are eigenvalues in this framework. Clearly, the eigenvalues of the simpler matrix models of ten years ago correspond to D-branes of the low dimensional “non-critical” string theories, but whether this is a useful concept is not clear to the author.

A closer examination shows that the dual string theories to (at least) some of the gauge theories are string theories propagating on background spacetimes with unusual properties. In the simplest matrix models, we were able to solve the string theories exactly. In the case of AdS backgrounds, the correspondence is tested mainly at string tree level, which is supergravity. For the linear dilaton background however, it appears that the full stringy correspondence might be captured by an exact CFT.

Duality, in its various forms clearly has much to teach us about the nature of fundamental physics. It has pulled together a number apparently discordant approaches and hints over the years into single harmonious narrative. Matrix models, gauge theory, D-branes and other extended objects have been the chief means of instruction so far. With surprises happening with astonishing regularity in the field, the only safe prediction is that there is much more excitement in store for us in the next ten years.



Acknowledgments

This work was supported by an NSF CAREER grant, #9733173. I am grateful to the organisers of the 1998 Trieste Spring School for the invitation to give these lectures, and to them and the staff at the Abdus Salam Centre for Theoretical Physics for helping to make my stay there such a pleasant one. Many thanks to S. J. Butler for patience and hospitality while the crucial finishing touches were applied to this work.

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