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THE WHITNEY TECHNIQUE FOR POINCARÉ COMPLEX EMBEDDINGS

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ABSTRACT. The following result is proved.

THEOREM. *Let $f: K^k \to X^n$ be a map of a finite connected k-dimensional Poincaré complex K to a finite simply-connected n-dimensional Poincaré complex X, then if $n \geqq 2k+1$, $k>2$, f is homotopic to an embedding*. The proof involves a technique akin to the Whitney procedure for eliminating double points.

The object of this note is to prove the following result.

THEOREM 1. *Let $f: K^k \to X^n$ be a map between a finite connected k-dimensional Poincaré complex K and a simply-connected finite n-dimensional Poincaré complex X, then if $n \geqq 2k+1$ and $k>2$, f is homotopic to an embedding.*

This improves the result of Levitt [3] by one dimension but at the cost of requiring X to be simply-connected. The method of proof is to follow Levitt's construction, introducing a 'singularity' in K to kill the obstruction to engulfing. Then by a Whitney process we can remove the singularity.

1. **Construction of the stable model.** Theorem 1 is a corollary of the following result.

LEMMA 2. *Let (X^{2n+1}, Y^{2n}) be a finite Poincaré pair of dimension $2n+1$, with $\pi_1(Y)=0$, then any map $f: (D^n, S^{n-1}) \to (X, Y)$ is homotopic to an embedding.*

This gives Theorem 1, by embedding the $(n-1)$-skeleton of K in X using Levitt's theorem [3]; then by Wall [4] we have only the top cell left to embed. We note also that Levitt's theorem allows us to assume that $f|S^{n-1}$ is an embedding.

The proof of Lemma 2 is modelled on that of Lemma 5.1 in [3], since modifications are required at several points we recapitulate in some detail.

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First let $g:(X^{2n+1}, Y)\to(P^{2n+j+1}, Q^{2n+j})$ be a codimension j PL thickening, with $\partial P=Q\cup R$ and $\partial Q=\partial R$. Note that in homotopy terms we can think of (P, Q) as the mapping cone of a spherical fibration by [**1**, Chapter I, §4].

Now by general position we may homotope the composition

$$j\circ f:(D^n, S^{n-1})\to(P, Q)$$

to a PL embedding; further by the remarks above we have a map

$$F_0:(D^n\times D^j, S^{n-1}\times D^j; D^n\times S^{j-1}, S^{n-1}\times S^{j-1})\to(P, Q, R, \partial R)$$

obtained by considering the restriction of the spherical fibration to (D^n, S^{n-1}).

Let (M, N) be a regular neighborhood of (D^n, S^{n-1}) in (P, Q) and $(\mathscr{M}, \mathscr{N})$ the mapping cylinder of the composition

$$\bar{p}:(D^n\times S^{j-1}, S^{n-1}\times S^{j-1})\xrightarrow{p}(D^n, S^{n-1})\subset(M, N);$$

then F_0 can be extended to $F:(\mathscr{M}, \mathscr{N})\to(P, Q)$ so that $F|(M, N)$ is an inclusion. Further if $\bar{p}$ is altered by a homotopy in (M, N), then we can change F in such a way that $F|(D^n\times S^{j-1}, S^{n-1}\times S^{j-1})$ is unchanged and so that $F|(M, N)$ is still the inclusion. This follows from the fact that general position shows that a homotopy of $\text{Int}(D^n\times S^{j-1})$ (resp. $S^{n-1}\times S^{j-1}$) in M (resp. N) will miss Int D^n (resp. S^{n-1}) so that we can achieve the following.

(GP) $\qquad F(\mathscr{M}-M)\subseteq P-M, \qquad F(\mathscr{N}-N)\subset Q-N.$

From (GP) we obtain a map of pairs

$$\tilde{p}:(D^n\times S^{j-1}, S^{n-1}\times S^{j-1})\to(\partial M-\text{Int } N, \partial N)$$

and by the remarks following Lemma 2, we may assume $\tilde{p}|S^{n-1}\times S^{j-1}$ is an embedding. Let B be a regular neighborhood of its image in ∂N, then

$$\tilde{p}:(D^n\times S^{j-1}, S^{n-1}\times S^{j-1})\to(\partial M-\text{Int } N, B)$$

is a $(j-1)$-connected map. Thus by Theorem 2.3 of [**2**] we can find $(A^{2n+j}, B^{2n+j-1})\subset(\partial M-\text{Int } N, \partial N)$ and a simple homotopy equivalence

$$q:(D^n\times S^{j-1}, S^{n-1}\times S^{j-1})\to(A, B)$$

such that q followed by the inclusion $(A, B)\subset(\partial M-\text{Int } N, \partial N)$ is homotopic rel$(S^{n-1}\times S^{j-1})$ to $\tilde{p}$. Now we can apply Theorem 5.2 of [**3**] to the map q to change $\tilde{p}$ by a homotopy and get $\bar{p}(D^n\times S^{j-1}, S^{n-1}\times S^{j-1})\subseteq(W, V)$ where $(W, V)\subset(\partial M-\text{Int } N, \partial N)$ is a pair of dimension $(n+j-1, n+j-2)$ and the map

$$(D^n\times S^{j-1}, S^{n-1}\times S^{j-1})\to(W, V)$$

is a homotopy equivalence. Now let (W_0, V_0) be a disjoint copy of (W, V) and let $i:(W_0, V_0)\to(M, N)$ be the map corresponding to the inclusion $(W, V)\subset(M, N)$. Then the homotopy equivalence

$$(D^n \times S^{j-1}, S^{n-1} \times S^{j-1}) \to (W, V)$$

induces a homotopy equivalence

$$\alpha:(\mathscr{M}, \mathscr{N}, D^n \times S^{j-1}, S^{n-1} \times S^{j-1}) \to (\mathscr{M}_i, \mathscr{N}_i, W_0, V_0)$$

where $(\mathscr{M}_i, \mathscr{N}_i)$ is the mapping cylinder of $i:(W_0, V_0)\to(M, N)$. Thus we can get a map

$$\bar{F}:(\mathscr{M}_i, \mathscr{N}_i, W_0, V_0) \to (P, Q, R, \partial R)$$

such that $\bar{F}\circ\alpha$ is homotopic to F as a map of the 4-tuple $(\mathscr{M}, \mathscr{N}, D^n\times S^{j-1}, S^{n-1}\times S^{j-1})$ into $(P, Q, R, \partial R)$ and with $\bar{F}|(M, N)$ still the inclusion. Let (T, U) be the part of $(\mathscr{M}_i, \mathscr{N}_i)$ corresponding to $(W_0, V_0)\times I$; that is $T=\text{cl}(\mathscr{M}_i-M)$, $U=\text{cl}(\mathscr{N}_i-N)$.

Now by (GP) we can suppose

$$\bar{F}(T) \subseteq P - (\text{Int}\, M \cup \text{Int}\, N) \quad \text{and} \quad F(U) \subseteq Q - \text{Int}\, N.$$

Set $P_0=P-(\text{Int}\, M\cup\text{Int}\, N)$, $Q_0=Q-\text{Int}\, N$ so $\bar{F}(T, U)\subset(P_0, Q_0)$, and this implies that the pair $(W, V)\subseteq(\partial M-\text{Int}\, N, \partial N)$ can be deformed into $(R, \partial R)$ keeping V in Q_0 and W in P_0.

This completes the construction of the stable model.

2. **Engulfing.** In Levitt's original proof of the embedding theorem [**3**], the transition from the stable model to the required embedding was effected by engulfing (W, V) in a collar on $(R, \partial R)$, a procedure which corresponds to constructing a compressible embedding of $(D^n\times D^j, S^{n-1}\times D^j)$ in (P, Q). Unfortunately there are obstructions to doing the engulfing in our case (in fact the counterexample of Irwin shows that these will in general not vanish), so that we require a slightly different approach. We shall show how one can embed a complex (K^n, S^{n-1}) in (P, Q) where K is of the homotopy type of a wedge of S^1's, and then we will perform surgery to eliminate the S^1's. The reader may perhaps be helped by thinking of the S^1's as the union of the two paths joining the double points that one has in the Whitney procedure for elimination of double points; this is in fact the spirit of the construction of K, which involves attempting to engulf W in a collar on R.

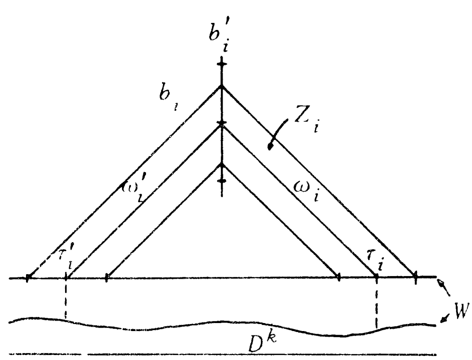
It follows from Lemma 5.3 of [**3**], that in Q_0 we can engulf V in a collar C_1 on ∂R in Q, whose inner boundary meets ∂N in a regular neighbourhood $\bar{V}$ of V with $C_1\cap\text{Int}\, N=\varnothing$. Further W is $R\cup C_1$-inessential, but as we remarked above the counterexample of Irwin shows that we cannot in general engulf W in a collar C on $R\cup C_1$; let us however recapitulate the proof of the engulfing theorem so as to see what modifications are possible.

We are given a homotopy of W to $R\cup C_1$ in P_0; choose a simplicial approximation and put this homotopy in general position; then its singular set is of dimension

$$\begin{aligned}&2(\dim \text{homotopy}) - \dim(\text{ambient space})\\ &\quad = 2(n+j)-(2n+j-1) = j-1.\end{aligned}$$

Thus the homotopy collapses onto a subcomplex of dimension j, but as (P_0, R) is only $(j-1)$-connected we cannot deform this subcomplex into R. The idea for continuing is to 'punch holes' in the $(j-1)$-simplexes of the singular set by adding 1-cells to D^k; (geometrically this one cell represents a double point) so that the singular set collapses to dimension $(j-2)$.

For each $(j-1)$-cell σ_i $(i=1, \cdots, l)$ of the singular set, let b_i be its barycentre; then there exist two disjoint paths ω_i, ω_i' from b_i to W (and thus to D^n) such that $\omega_i\cap W$ (resp. $\omega_i'\cap W$) is the barycentre of a simplex τ_i (resp. τ_i') of maximal dimension in W, and such that the track of τ_i (resp. τ_i') in the homotopy gives a neighbourhood of ω_i (resp. ω_i'), in the image of the homotopy (see diagram).



Now suppose we have a simplicial decomposition of P_0 so that ω_i and ω_i' are subcomplexes, and let (Z_i, Z_i') be disjoint relative regular-neighbourhoods of $(\omega_i\cup\omega_i')$.

Now $M'=M\cup\bigcup_{i=1}^l Z_i$ is homotopy equivalent to a complex $K=D^k\vee\bigvee_{i=1}^l S^1$ and we shall embed the pair (K, S^{k-1}) in (X, Y). To do this using Levitt's program requires us to embed a complex $W'\subset\partial M'$, with W' of the homotopy type of the normal Spivak fibration of X in P restricted to M', and then we must engulf W' in a collar on R. It is clear that W' is homotopy equivalent to $W\bigcup_{i=1}^l I\times S^{j-1}$ where the pairs $(I\times S^{j-1}, S^0\times S^{j-1})$ are to be embedded in $(\partial Z_i - Z_i', \partial Z_i')$. These embeddings exist by the

'Stalling's' theorem used in Levitt's paper [3, Theorem 5.2]. W' is obtained from this union by collapsing the I factors of the $S^{j-1}\times I$'s to $S^{j-1}\times 0$ in the Z_i's.

We claim that W' can be engulfed in a collar C on $R\cup C'$, so that the inner boundary of C meets M' in a regular neighbourhood of W'. This will follow if we can show that W' can be deformed into $R\cup C'$ in $P-\text{Int } M'$ by a homotopy whose singular set is of dimension $\leqq j-2$ (since once this is achieved the proof in [5] gives the required engulfing). To do this we note that we already have such a homotopy for $W'\cap W\,(=W-\bigcup_{i=1}^{l}\tau_i\cup\tau_i')$ so that it suffices to provide an extension over the $S^{j-1}\times I$. Now there is a homotopy of the $S^{j-1}\times I$ to R since the $S^{j-1}\times I$ represent part of the Spivak normal fibration of X in P, but now general position gives us the required condition on the dimension of the singular set. So the existence of C follows from engulfing this homotopy.

We thus obtain a splitting of (X, Y) mod boundary given by $(X, Y)\simeq (X_0, Y_0)\cup_{Y_2}(X_1, Y_1)$ where

$$Y_2 = \text{Closure}(\overline{P - C} - M') \cap M',$$

$$X_1 = M', \qquad Y_1 = Y_2 \cup N,$$

$$X_0 = \text{Closure}(\overline{P - C} - M'), \qquad Y_0 = Y_2 \cup \text{Closure}(\overline{Q - C} - N),$$

The verification that this is indeed a splitting is straightforward.

Surgery. Now choose $S_1^1, \cdots, S_l^1$ embedded in Y_2, generating $\pi_1(X_1)$ and corresponding to the $S_1^1, \cdots, S_l^1$ in $D^n\vee\bigvee_{i=1}^{l} S_i^1$. This is possible since Levitt's theorem tells us that a map $f: S^1\to Y_2$ is homotopic to an embedding since dim $Y_2\geqq 4$. But now S^1 is null homotopic in X_0 since X_0 is 1-connected by Van Kampen's theorem (this requires $k>2$). Hence, again by Levitt's theorem we can get disjoint embeddings $(D_i^2, S_i^1)\to (X_0, Y_2)$ representing these null homotopies. We show how to use each of these embeddings to do surgery on X_1 and X_0. Suppose $l=1$, and Y_2 is split as $Y_2\simeq(A, B)\cup_B(C, B)$. Where $C\sim S^1$, then (X_0, Y_0) is split as

$$(X_0, Y_0) = (D, E) \cup_F (G, H)$$

where $D\sim D^2$ and $\text{cl}(E-F)\sim C$ so we replace X_1 by $X_1'=X_1\cup_C D$. (X_0, Y_0) is replaced by (G, H) and Y_2 is replaced by $Y_2'=A\cup_B F$. $Y_1'=Y_2'\cup \text{cl}(Y_1-Y_2)$.

This surgery has the effect of constructing $X_1'\sim D^n$, and thus embedding (D^n, S^{n-1}) in (X, Y) in the required homotopy class, (X, Y) being split as $(X_1', Y_1')\cup_{Y_2'}(G, H)$, thus proving the lemma.

REFERENCES

1. W. F. Browder, *Surgery of simply connected manifolds*, Notes, Princeton University, Princeton, N.J., 1969.

2. J. P. E. Hodgson, *Obstructions to concordance for thickenings*, Invent. Math. **5** (1968), 292–316. MR **37** #5879.

3. N. Levitt, *On the structure of Poincaré duality spaces*, Topology **7** (1968), 369–388. MR **40** #2089.

4. C. T. C. Wall, *Poincaré complexes*. I, Ann. of Math. (2) **86** (1967), 213-245. MR **36** #880.

5. E. C. Zeeman, *Seminar on combinatorial topology*. Chap. 7, *Engulfing*, Inst. Haute Études Sci., Paris, 1963; revised 1965 (mimeographed).

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