Local Isometric Imbedding of Riemannian Manifolds with Indefinite Metrics

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Introduction. In 1873 SCHLÄFLI [9] conjectured that a Riemannian manifold R_m with positive definite and analytic metric can locally be imbedded isometrically as a submanifold of a euclidean space E_n, provided $n \geqq \frac{1}{2}m(m+1)$. In 1926 JANET [5] described a method of proof which however (as he himself observed) was incomplete. In 1927 E. CARTAN [3] gave a proof based on his theory of Pfaffian forms. BURSTIN [1] in 1931 completed the proof of JANET and also extended it to the case in which the enveloping space is a given Riemannian space R_n with positive definite and analytic metric. Recently LEICHTWEISS [6] gave a new proof of BURSTIN's extension based much more substantially (than [1]) on the GAUSS-CODAZZI equations of differential geometry. His proof is more involved than that of BURSTIN. Theorems on global isometric imbeddings of R_m into E_n which are of class C^k ($k \geqq 3$) provided the metric tensor of R_m is of class C^k have been established by NASH [8]. For compact R_m he assumes that $n \geqq \frac{1}{2}m(3m+11)$.

The first purpose of this paper is to extend the theorem of JANET-CARTAN-BURSTIN to Riemannian manifolds with indefinite metrics (such as the space of General Relativity). The metric tensors are still assumed to be analytic and non-degenerate (for semi-positive definite metric, see LENSE [7]). We prove the following theorem (Theorem 1): If the metric tensor g_{ab} of R_m has p positive and q negative eigenvalues ($p + q = m$) and if the metric tensor g_{ij} of R_n has at least p positive and at least q negative eigenvalues (no restriction being made on the signature of the remaining eigenvalues), then there exist local isometric and analytic imbeddings of R_m into R_n, provided $n \geqq \frac{1}{2}m(m+1)$. Our proof, like that of BURSTIN, is based on the general outline of JANET but otherwise it is a new proof even in the case of positive definite metrics.

We also consider in this paper the question of imbedding a given submanifold R_m of R_n in a family $R_m(t)$ of isometric submanifolds. We prove (Theorem 2) the existence of such a family of analytic submanifolds which varies analytically

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with t, $|t|$ small, and $R_m(0) = R_m$, provided $n \geqq \frac{1}{2}m\ (m + 1)$. The metrics of R_m and R_n are assumed to be analytic and R_m is assumed (roughly speaking) to be non-flat. The special case of R_n having a positive definite metric was proved by LEICHTWEISS [6] and, without establishing the analyticity of $R_m(t)$ in t, earlier by BURSTIN [2]. Our proof is based on the proof of Theorem 1, combined with a procedure used by LEICHTWEISS.

We finally consider the question of connecting two given submanifolds ${}_1R_m$ and ${}_2R_m$ of R_n which are isometric, by a family $R_m(t)$ of isometric submanifolds. This is solved (Theorems 3, 3′) under some assumptions (such as non-flatness of ${}_1R_m$, ${}_2R_m$) provided $n > \frac{1}{2}m\ (m + 1)$. Our result for positive definite metrics (Theorem 3) is slightly weaker than the analogous result of LEICHTWEISS [6].

Theorem 1 is proved in §1, Theorem 2 is proved in §2, and Theorems 3, 3′ are proved in §3. A few lemmas used in proving Theorems 1, 3′ are proved in an appendix.

1. Isometric imbedding of $\mathbf{R}_m$ into $\mathbf{R}_n$. All functions in this paper are real valued, m and n are positive integers. The indices a, b, c, d, e run from 1 to m, and the indices h, i, j, k, l run from 1 to n. Let R_m be a Riemannian space with indefinite non-degenerate metric tensor $g_{ab}(u^c)$ having p positive and q negative eigenvalues ($p + q = m$), and let R_n ($n \geqq m$) be a Riemannian space with indefinite non-degenerate metric tensor $g_{ij}(x^k)$ having at least p positive and at least q negative eigenvalues; no restriction being made on the signature of the remaining (non-vanishing) $n - p - q$ eigenvalues.

Theorem 1. *Let $g_{ab}(u^c)$ be analytic functions in a neighborhood of $u^c = 0$ and let $g_{ij}(x^k)$ be analytic functions in a neighborhood of $x^k = 0$. If $n \geqq \frac{1}{2}m\ (m + 1)$ then there exist analytic functions $x^k = x^k(u^c)$ in a neighborhood of $u^c = 0$ which satisfy the conditions*:

$$(1.1) \qquad x^k(0) = 0, \quad \operatorname{rank}\left(\frac{\partial x^k}{\partial u^c}(0)\right) = m, \qquad g_{ij}(x^k(u^c))\,\frac{\partial x^i(u^c)}{\partial u^a}\,\frac{\partial x^j(u^c)}{\partial u^b} = g_{ab}(u^c).$$

Thus, Theorem 1 asserts the existence of a local analytic and isometric imbedding of R_m into R_n.

Proof. By an analytic transformation ([4], pp. 52–56) we can obtain new g_{ij} and new x-coordinates which are euclidean at the origin, *i.e.*,

$$(1.2) \qquad g_{ij}(0) = \pm\,\delta_{ij}\,, \qquad \frac{\partial g_{ij}(0)}{\partial x^k} = 0.$$

Hence, without loss of generality we may assume that (1.2) is already satisfied for the original x-coordinates. Similarly we may assume that

$$(1.3) \qquad g_{ab}(0) = \pm\,\delta_{ab}\,, \qquad \frac{\partial g_{ab}(0)}{\partial u^c} = 0.$$

We may also assume that

$$(1.4) \qquad \operatorname{sgn} g_{ii}(0) = \operatorname{sgn} g_{aa}(0) \quad \text{if} \quad i = n - m + a.$$

The arrangement of the signs of the $g_{ii}(0)$ for $i \leqq n - m$ is immaterial. In what follows, m' will run from 2 to m, the indices a', b', c', d', e' will run from 1 to m', and the indices a'', b'', c'', d'', e'' will run from 1 to $m' - 1$. Further, we set $n' = n - m + m'$ and let the indices h', i', j', k', l' vary from 1 to n', and the indices h'', i'', j'', k'', l'' vary from 1 to $n' - 1$. We define $R_{m'}$ $(m' < m)$ by the equations $u^{m'+1} = \cdots = u^m = 0$ and $R_{n'}$ $(n' < n)$ by $x^{n'+1} = \cdots = x^n = 0$. We wish to obtain the imbedding of R_m into R_n by imbedding inductively $R_{m'}$ into $R_{n'}$, each step being an extension of the previous one. We need a few preliminaries.

Definition. Let $Y_1^i(x^k), \cdots, Y_\alpha^i(x^k)$ be n-dimensional vectors and set $b_{\lambda\mu}(x^k) = g_{ij}(x^k)\, Y_\lambda^i(x^k)\, Y_\mu^j(x^k)$. Consider the determinants

$$b_\nu(x^k) = \det(b_{\lambda\mu}(x^k)) \qquad (1 \leqq \lambda, \mu \leqq \nu) \quad \text{for} \quad 1 \leqq \nu \leqq \alpha.$$

If all the $b_\nu(x^k)$ are different from zero at a point x_0^k, then we say that the $Y_1^i, \cdots, Y_\alpha^i$ satisfy the *B-condition* at the point x_0^k. If $x^k = x^k(u^a)$ and $x_0^k = x^k(u_0^a)$, then we also say that the B-condition is satisfied at the point u_0^a.

It is well known ([4], pp. 103–104) that if the Y_λ^i satisfy the B-condition at a point x_0^k, then they can be orthonormalized by the process of E. SCHMIDT. Thus, there exists an orthonormal set E_λ^i (*i.e.*, $g_{ij}E_\lambda^iE_\mu^j = \pm\,\delta_{\lambda\mu}$) such that, at x_0^k,

$$Y_\lambda^i = \sum_{\gamma=1}^{\lambda} a_{\lambda\gamma}E_\gamma^i \quad \text{and} \quad a_{\lambda\lambda} \neq 0 \qquad (1 \leqq \lambda \leqq \alpha).$$

The converse of this theorem is also true (see Lemma 3 in §3) but it is not used in this section. We shall need the following result.

Lemma 1. *Let $E_1^i, \cdots, E_r^i$ be orthonormal vectors at a fixed point A in R_n. Then there exist vectors $E_{r+1}^i, \cdots, E_n^i$ at A which complete the given vectors to an orthonormal basis.*

The proof is given in the appendix at the end of this paper.

Clearly the B-condition depends on the order in which the vectors appear. In what follows we shall consider vectors indexed in the form

$$\xi_{\bar a}, \qquad \xi_{\bar a\bar b} \qquad (\bar a, \bar b = 1, \cdots, \beta),$$

and only those $\xi_{\bar a\bar b}$ appear for which $\bar a \geqq \bar b$. Unless the contrary is explicitly stated, it will be understood that these vectors are arranged in lexicographic order, that is, $\xi_{\bar a}$ precedes $\xi_{\bar b}$ if $\bar a < \bar b$; $\xi_{\bar a\bar b}$ precedes $\xi_{\bar c\bar d}$ if either $\bar a < \bar c$ or, in case $\bar a = \bar c$, if $\bar b < \bar d$; finally every $\xi_{\bar a}$ precedes every $\xi_{\bar b\bar c}$.

Consider the mapping $x^{\bar i} = x(u^{\bar i})$ in a neighborhood of $u^{\bar a} = 0$, where $\bar a$, $\bar b$, $\bar c$ run from 1 to $\bar m$, and $\bar i, \bar j, \bar k, \bar l$ run from 1 to $\bar n$. Consider the vectors

$$\frac{\partial x^{\bar i}(u^{\bar c})}{\partial u^{\bar a}}, \qquad \frac{\partial^2 x^{\bar i}(u^{\bar c})}{\partial u^{\bar a}\,\partial u^{\bar b}} + \Gamma_{\bar j\bar k}^{\bar i}(x^{\bar l}(u^{\bar c}))\,\frac{\partial x^{\bar j}(u^{\bar c})}{\partial u^{\bar a}}\,\frac{\partial x^{\bar k}(u^{\bar c})}{\partial u^{\bar b}},$$

where the $\Gamma_{\bar j\bar k}^{\bar l}$ are the CHRISTOFFEL symbols of the second kind in the $x^{\bar i}$-space.

If the first min $(\bar{n}, \frac{1}{2}\bar{m}\ (\bar{m} + 3))$ of these vectors satisfy the B-condition at $u^{\bar{c}} = 0$, then we say that *the mapping* $x^{\bar{i}} = x^{\bar{i}}(u^{\bar{c}})$ *satisfies the B-condition.*

We can now describe the inductive assumption. We assume that $R_{m'-1}$ has been imbedded in $R_{n'-1}$ and that the vectors

$$(1.5)\qquad \frac{\partial x^{i''}(u^{c''})}{\partial u^{a''}}, \qquad \frac{\partial^2 x^{i''}(u^{c''})}{\partial u^{a''}\,\partial u^{b''}} + \Gamma^{i''}_{j''k''}(x^{l''}(u^{c''}))\,\frac{\partial x^{j''}(u^{c''})}{\partial u^{a''}}\,\frac{\partial x^{k''}(u^{c''})}{\partial u^{b''}}$$

satisfy the B-condition. We shall prove that $R_{m'}$ can be imbedded in $R_{n'}$ in such a way that the imbedding satisfies the B-condition, that is, if $m' < m$ then the vectors analogous to (1.5) (with $''$ replaced by $'$) satisfy the B-condition, whereas if $m' = m$ then the first min $(n, \frac{1}{2}m\ (m + 3))$ of these vectors satisfy the B-condition.

We remark that in case of positive definite metric tensors, the B-condition is equivalent to linear independence whereas, in any case, the B-condition always implies linear independence. From the inductive procedure described above it follows that by way of proving Theorem 1 we shall also prove that the first min $(n, \frac{1}{2}m\ (m + 3))$ of the vectors

$$(1.6)\qquad \frac{\partial x^i(u^c)}{\partial u^a}, \qquad \frac{\partial^2 x^i(u^c)}{\partial u^a\,\partial u^b} + \Gamma^i_{jk}(x^l(u^c))\,\frac{\partial x^j(u^c)}{\partial u^a}\,\frac{\partial x^k(u^c)}{\partial u^b}$$

satisfy the B-condition; hence the rank of the matrix (1.6) is min $(n, \frac{1}{2}m\ (m + 3))$. A submanifold $x^i = x^i(u^c)$ with the last property is called *planar*. Geometrically this means that the vector space formed by the tangent vectors to the submanifold and the normal curvature vectors of the submanifold (with respect to R_n) is of maximal dimension. In the following we shall not make any use of this interpretation.

We start with the imbedding of R_1 in $R_{n'}$, $n' = n - m + 1$. The case $n = 1$ is immediate. If $n > 1$, we take $x^1 = x^2 = \cdots = x^{n'-2} = 0$ and try to find $x^{n'-1}$, $x^{n'}$ which satisfy

$$(1.7)\qquad \tilde{g}_{\lambda\mu}(x^\nu)\,\frac{dx^\lambda}{du^1}\,\frac{dx^\mu}{du^1} = \tilde{g}_{11}(u^1), \qquad (n' - 1 \leqq \lambda, \mu, \nu \leqq n'), \qquad x^\lambda(0) = 0,$$

where $\tilde{g}_{\lambda\mu}(x^\nu)$ is the function $g_{\lambda\mu}(x^k)$ when all but $x^{n'-1}$, $x^{n'}$ are identically zero, and similarly for $\tilde{g}_{11}(u^1)$. We only consider the case $g_{11}(0) = -1$ (the case of $+1$ can be handled similarly with minor changes). Then, by (1.4), $g_{n'n'}(0) = -1$. If a solution of (1.7) exists then, upon setting

$$\frac{dx^{n'-1}(0)}{du^1} = \alpha, \qquad \frac{dx^{n'}(0)}{du^1} = \beta, \qquad \frac{d^2x^{n'-1}(0)}{(du^1)^2} = \gamma, \qquad \frac{d^2x^{n'}(0)}{(du^1)^2} = \delta,$$

we find that (recall that $g_{n'-1,n'-1}(0) = \pm 1$)

$$(1.8)\qquad \pm\alpha^2 - \beta^2 = -1, \qquad \pm\alpha\gamma - \beta\,\delta = 0.$$

It is then easily seen that for the B-condition (of the imbedding) to hold it is enough to take $\beta \neq 0$, $\delta \neq 0$. We now take $x^{n'-1}(u^1)$ to be any analytic function of u^1 having numbers α and γ for its first and second derivatives at $u^1 = 0$. The numbers α, γ are taken in the range for which the solution β, δ of (1.8) exists and $\beta \neq 0$, $\delta \neq 0$. We next differentiate (1.7) with respect to u^1 and obtain a second order equation in $x^{n'}$. Taking the initial condition

$$x^{n'}(0) = 0 \qquad \frac{dx^{n'}(0)}{du^1} = \alpha,$$

the existence and uniqueness of an analytic solution follows by the CAUCHY-KOWALEWSKI theorem (note that the equation is of the normal form). By uniqueness, its second derivative at $u^1 = 0$ is equal to δ. Since the solution also satisfies (1.7) when $u^1 = 0$, we have actually obtained a solution $x^{n'}(u^1)$ of (1.7).

We have thus established the existence of an isometric and analytic imbedding of R_1 into $R_{n'}$, satisfying the B-condition.

We now assume that a local imbedding satisfying the B-condition has already been constructed for $R_{m'-1}$ (into $R_{n'-1}$), where $m' < m$, and we proceed to construct it for $R_{m'}$ (into $R_{n'}$). The case $m' = m$ will be considered later. We shall denote by $G_{m'-1}$ an appropriate neighborhood of the origin in $R_{m'-1}$. For the sake of definiteness we assume that $g_{m'm'}(0) = -1$, the case of $+1$ can be handled similarly with only minor changes.

We next introduce in $R_{m'}$ geodesically parallel coordinates $(p^1, \cdots, p^{m'})$ by constructing (analytic) functions $u^{a'}(p_b')$ which satisfy ([4], p. 57)

$$\frac{\partial^2 u^{a'}}{(\partial p^{m'})^2} = \Gamma^{a'}_{b'c'}(u^{d'}) \frac{\partial u^{b'}}{\partial p^{m'}} \frac{\partial u^{c'}}{\partial p^{m'}}, \tag{1.9}$$

$$u^{a''}(p^{b''}, 0) = p^{a''}, \qquad u^{m'}(p^{b''}, 0) = 0, \tag{1.10}$$

$$g_{a'b'}(u^{d'}(p^{e''}, 0)) \frac{\partial u^{a'}}{\partial p^{m'}}(p^{e''}, 0) \frac{\partial u^{b'}}{\partial p^{c'}}(p^{e''}, 0) = -\delta_{c'm'}. \tag{1.11}$$

Here the $\Gamma^{a'}_{b'c'}$ are the CHRISTOFFEL symbols of the second kind in $R_{m'}$.

We obtain a new metric tensor

$$\bar{g}_{a'b'}(p^{c'}) = g_{c'd'} \frac{\partial u^{c'}}{\partial p^{a'}} \frac{\partial u^{d'}}{\partial p^{b'}},$$

$$\bar{g}_{a''m'}(p^{b'}) \equiv 0, \qquad \bar{g}_{m'm'}(p^{b'}) \equiv -1. \tag{1.12}$$

Using (1.3), (1.10), (1.11) we get $\partial u^{a'}(0)/\partial p^{b'} = \delta^{a'}_{b'}$. By differentiating (1.11) and by (1.9) we also get $\partial^2 u^{a'}(0)/\partial p^{b'}\partial p^{c'} = 0$. Hence, using (1.3), (1.12) we conclude that

$$\bar{g}_{a'b'}(0) = g_{a'b'}(0) = \pm\delta_{a'b'}, \qquad \bar{g}_{m'm'}(0) = -1, \qquad \frac{\partial \bar{g}_{a'b'}}{\partial p^{c'}} = 0. \tag{1.13}$$

For brevity we omit, in the following, the bar above the $g_{a'b'}$. We observe, by

(1.10), that the imbedding $x^{i''} = x^{i''}(u^{a''}(p^{b''})) = x^{i''}(p^{b''})$, $x^{n'}(p^{b''}) \equiv 0$, satisfies the B-condition in the new coordinates $p^{b''}$.

We now introduce the expressions

$$E_{a'b'}(p^{c'}) = g_{i'j'}(x^{k'}(p^{c'})) \frac{\partial x^{i'}(p^{c'})}{\partial p^{a'}} \frac{\partial x^{j'}(p^{c'})}{\partial p^{b'}} - g_{a'b'}(p^{c'}), \tag{1.14}$$

where $g_{i'j'}(x^{k'})$, $g_{a'b'}(p^{c'})$ stand for

$$g_{i'j'}(x^1, \cdots, x^{n'}, 0, \cdots, 0), \qquad g_{a'b'}(p^1, \cdots, p^{m'}, 0, \cdots, 0)$$

respectively.

By the inductive assumption,

$$E_{a''b''}(p^{c''}, 0) = 0 \quad \text{on some} \quad G_{m'-1} . \tag{1.15}$$

We shall determine $\partial x^{i'}/\partial p^{m'}$ on $G_{m'-1}$ by the equations

$$E_{a''m'} = 0 \quad \text{on} \quad G_{m'-1} , \tag{1.16}$$

$$E_{m'm'} = 0 \quad \text{on} \quad G_{m'-1} , \tag{1.17}$$

$$\frac{\partial}{\partial p^{b''}} E_{a''m'} + \frac{\partial}{\partial p^{a''}} E_{b''m'} - \frac{\partial}{\partial p^{m'}} E_{a''b''} = 0 \quad \text{on} \quad G_{m'-1} . \tag{1.18}$$

Next we shall prove the existence of an analytic solution $x^{i'}(p^{c'})$ of the system

$$\frac{\partial}{\partial p^{m'}} E_{a''m'} = 0, \tag{1.19}$$

$$\frac{\partial}{\partial p^{m'}} E_{m'm'} = 0, \tag{1.20}$$

$$-\frac{\partial^2}{(\partial p^{m'})^2} E_{a''b''} + \frac{\partial^2}{\partial p^{b''}\, \partial p^{m'}} E_{a''m'} + \frac{\partial^2}{\partial p^{a''}\, \partial p^{m'}} E_{b''m'} - \frac{\partial^2}{\partial p^{a''}\, \partial p^{b''}} E_{m'm'} = 0 \tag{1.21}$$

in some neighborhood $G_{m'}$ of the origin in $R_{m'}$. If (1.16)-(1.21) are established, then it easily follows, as in [5], that

$$E_{a'b'}(p^{c'}) = 0 \quad \text{in} \quad G_{m'} , \tag{1.22}$$

which proves the isometry of the imbedding $x^{i'} = x^{i'}(p^{c'})$. By way of solving (1.16)-(1.21) we shall also prove that the imbedding satisfies the B-condition.

We first write the equations (1.16)-(1.18) more explicitly, namely,

$$g_{i'j'}(x^{h'}) \frac{\partial x^{i'}}{\partial p^{a''}} \frac{\partial x^{j'}}{\partial p^{m'}} = g_{a''m'} = 0 \quad \text{on} \quad G_{m'-1} , \tag{1.23}$$

$$g_{i'j'}(x^{h'}) \frac{\partial x_{j'}}{\partial p_{m'}} \frac{\partial x_{i'}}{\partial p_{m'}} = g_{m'm'} = -1 \quad \text{on} \quad G_{m'-1} , \tag{1.24}$$

$$g_{i'j'}(x^{h'})\left[\frac{\partial^2 x^{i'}}{\partial p^{a''}\,\partial p^{b''}} + \Gamma^{i'}_{k'l'}(x^{h'})\frac{\partial x^{k'}}{\partial p^{a''}}\frac{\partial x^{l'}}{\partial p^{b''}}\right]\frac{\partial x^{j'}}{\partial p^{m'}} = \Gamma_{a''b'',m'} = -\tfrac{1}{2}\frac{\partial g_{a''b''}}{\partial p^{m'}} \quad \text{on} \quad G_{m'-1}, \tag{1.25}$$

where the $\Gamma_{a''b'',m'}$ are CHRISTOFFEL's symbols of the first kind.

To solve these equations for $\partial x^{i'}/\partial p^{m'}$ we shall introduce (following LEICHTWEISS) an orthonormal basis of vectors at each point of $G_{m'-1}$. We first set

$$w^{i'}_{a''} = \frac{\partial x^{i'}}{\partial p^{a''}}, \quad w^{i'}_{a''b''} = w^{i'}_{\rho} = \frac{\partial^2 x^{i'}}{\partial p^{a''}\,\partial p^{b''}} + \Gamma^{i'}_{j'k'}\frac{\partial x^{j'}}{\partial p^{a''}}\frac{\partial x^{k'}}{\partial p^{b''}} \text{ on } G_{m'-1}, \tag{1.26}$$

where the set $\{a'', b''\}$ is indexed by ρ, $m' \leqq \rho \leqq m_0$, and $m_0 = \frac{1}{2}m'(m'+1) - 1$. By the inductive assumption it follows that the vectors (1.26) (as vectors in $R_{n'}$) satisfy the B-condition on $G_{m'-1}$ (recall that $x^{n'} = 0$ on $G_{m'-1}$). Hence, there exist vectors $e^{i'}_{d''}$, $e^{i'}_{\rho}$ such that

$$w^{i'}_{a''} = \sum_{d''=1}^{m'-1} a_{a''d''}e^{i'}_{d''}, \tag{1.27}$$

$$w^{i'}_{\rho} = \sum_{d''=1}^{m'-1} a_{\rho d''}e^{i'}_{d''} + \sum_{\sigma=m'}^{m_0} a_{\rho\sigma}e^{i'}_{\sigma}, \tag{1.28}$$

where $a_{a''d''} = 0$ if $d'' > a''$, $a_{\rho\sigma} = 0$ if $\sigma > \rho$, $a_{d''d''} \neq 0$, $a_{\rho\rho} \neq 0$, and the following relations hold:

$$g_{i'j'}(x^{k'})e^{i'}_{\lambda}e^{j'}_{\mu} = \pm\delta_{\lambda\mu} \qquad (\lambda, \mu = 1, 2, \cdots, m_0) \quad \text{on} \quad G_{m'-1}. \tag{1.29}$$

From the construction of the e's ([4], pp. 103–104) it follows that all the functions $a_{\lambda\mu}(p^{a''})$, $e^{i'}_{\nu}(p_c'')$ appearing above are analytic functions. Also, since $x^{n'}(p^{c''}) \equiv 0$,

$$e^{n'}_{a''}(p^{c''}) = 0, \qquad e^{n'}_{\rho}(p^{c''}) = 0 \quad \text{on} \quad G_{m'-1}. \tag{1.30}$$

We complete the vectors $e^{i'}_{a''}$, $e^{i'}_{\rho}$ to an orthonormal basis in $R_{n'}$ by adding $n' - m_0$ vectors $e^{i'}_{\omega}$. (The index ω runs from $m_0 + 1$ to n' and the index ω' runs from $m_0 + 1$ to $n' - 1$). The construction can be carried out in such a way that (in addition to the orthonormality relations) we have

$$e^{n'}_{\omega'}(p^{c''}) = 0, \qquad e^{i'}_{n'}(0) = \delta^{i'}_{n'}. \tag{1.31}$$

Indeed, we first take constant vectors $\tilde{e}^{i'}_{\omega}$ which satisfy together with $e^{i'}_{\lambda}(0)$ $(1 \leqq \lambda \leqq m_0)$ the B-condition and are such that

$$\tilde{e}^{n'}_{\omega'} = 0, \qquad \tilde{e}^{i'}_{n'} = \delta^{i'}_{n}.$$

(The existence of the $\tilde{e}^{i'}_{\omega'}$ follows by Lemma 1). We then have only to apply the orthonormalization method of SCHMIDT to the vectors $e^{i'}_{a''}(p^{c''})$, $e^{i'}_{\rho}(p^{c''})$,

$\bar{e}^{i'}_{\omega}$. We note that since $g_{n'n'}(0) = -1$, we have

$$g_{i'j'}(x^{k'}(p^{c''}))e^{i'}_{n'}(p^{c''})e^{j'}_{n'}(p^{c''}) = -1 \quad \text{on} \quad G_{m'-1} . \tag{1.32}$$

We proceed to construct $\partial x^{i'}/\partial p^{m'}$ on $G_{m'-1}$. We try to write it in the form

$$\frac{\partial x^{i'}}{\partial p^{m'}} = \sum_{d''=1}^{m'-1} b_{d''}e^{i'}_{d''} + \sum_{\sigma=m'}^{m_0} b_\sigma e^{i'}_\sigma + \sum_{\omega=m_0+1}^{n'} b_\omega e^{i'}_\omega \quad \text{on} \quad G_{m'-1} . \tag{1.33}$$

By (1.26), (1.27), the equations (1.23) are equivalent to

$$\sum_{d''=1}^{m'-1} \pm a_{a''d''}(p^{c''})b_{d''}(p^{c''}) = 0,$$

or to

$$b_{d''}(p^{c''}) = 0. \tag{1.34}$$

Similarly, the equations (1.25) are equivalent to

$$\sum_{\sigma=m'}^{m_0} \pm a_{\rho\sigma}b_\sigma(p^{c''}) = -\frac{1}{2}\frac{\partial g_\rho(p^{c''}, 0)}{\partial p^{m'}}. \tag{1.35}$$

This system of equations has a unique solution. Using (1.13) we conclude that, for $p^{c''} = 0$, equations (1.35) are equivalent to

$$b_\sigma(0) = 0. \tag{1.36}$$

Finally, equation (1.24) is equivalent to

$$\sum_{\sigma=m'}^{m_0} \pm (b_\sigma(p^{c''}))^2 + \sum_{\omega'=m_0+1}^{n'-1} \pm (b_{\omega'}(p^{c''}))^2 - (b_{n'}(p^{c''}))^2 = -1 \tag{1.37}$$

(here we made use also of (1.32)). Thus, we are free to choose the $b_{\omega'}(p^{c''})$, provided $b_{n'}(p^{c''})$ is taken such that (1.37) is satisfied. We define

$$b_{\omega'}(0) = 0. \tag{1.38}$$

Other restrictions on the $b_{\omega'}$ will be imposed later. If $p^{c''}$ is sufficiently close to the origin, as we may assume, then (1.37) can be solved by taking

$$b_{n'}(p^{c''}) = \{1 + \sum \pm (b_\sigma(p^{c''}))^2 + \sum \pm (b_{\omega'}(p^{c''}))^2\}^{\frac{1}{2}}. \tag{1.39}$$

We now differentiate (1.33) and obtain (using (1.34), (1.36), (1.38))

$$\begin{aligned} \frac{\partial^2 x^{i'}(0)}{\partial p^{m'}\,\partial p^{c''}} &= \sum A_{c''d''}e^{i'}_{d''}(0) \\ &\quad + \sum A_{c''\sigma}e^{i'}_\sigma(0) + \sum \gamma_{c''\omega'}e^{i'}_{\omega'}(0) + A_{c''n'}e^{i'}_{n'}(0). \end{aligned} \tag{1.40}$$

Since $\partial b_{n'}(0)/\partial p^{c''}$ does not depend on the choice of the $b_{\omega'}(p^{c''})$ (provided only (1.38) is satisfied), it is seen that all the A's in (1.40) are independent of the choice of $b_{\omega'}(p^{c''})$ (provided (1.38) holds). On the other hand the $\gamma_{c''\omega'}$ can be made arbitrary by arbitrary choice of $\partial b_{\omega'}(0)/\partial p^{c''}$. Indeed, we have

$$\gamma_{c''\omega'} = \frac{\partial b_{\omega'}(0)}{\partial p^{c''}} + A_{c''\omega'}\,, \tag{1.41}$$

where the $A_{c''\omega'}$ are independent of the $b_{\omega'}(p^{c''})$. Later on we shall make a special choice of the $\gamma_{c''\omega'}$.

We now turn to equations (1.19)–(1.21). We can write them in the form

$$g_{i'j'}(x^{h'}) \frac{\partial x^{i'}}{\partial p^{a''}} \left\{ \frac{\partial^2 x^{j'}}{(\partial p^{m'})^2} + \Gamma^{j'}_{k'l'}(x^{h'}) \frac{\partial x^{k'}}{\partial p^{m'}} \frac{\partial x^{l'}}{\partial p^{m'}} \right\} = F_{a''}\,, \tag{1.42}$$

$$g_{i'j'}(x^{h'}) \frac{\partial x^{i'}}{\partial p^{m'}} \left\{ \frac{\partial^2 x^{j'}}{(\partial p^{m'})^2} + \Gamma^{j'}_{k'l'}(x^{h'}) \frac{\partial x^{k'}}{\partial p^{m'}} \frac{\partial x^{l'}}{\partial p^{m'}} \right\} = F_{m'}\,, \tag{1.43}$$

$$\begin{aligned} g_{i'j'}(x^{h'}) &\left\{ \frac{\partial^2 x^{i'}}{\partial p^{a''}\,\partial p^{b''}} + \Gamma^{i'}_{k'l'}(x^{h'}) \frac{\partial x^{k'}}{\partial p^{a''}} \frac{\partial x^{l'}}{\partial p^{b''}} \right\} \cdot \\ &\qquad \left\{ \frac{\partial^2 x^{j'}}{(\partial p^{m'})^2} + \Gamma^{j'}_{k'l'}(x^{h'}) \frac{\partial x^{k'}}{\partial p^{m'}} \frac{\partial x^{l'}}{\partial p^{m'}} \right\} = F_{a''b''}\,, \end{aligned} \tag{1.44}$$

where $F_{a'}$, $F_{a''b''}$ are given analytic functions which involve $p^{a'}$, $x^{h'}$, $\partial x^{h'}/\partial p^{a'}$, $\partial^2 x^{h'}/\partial p^{a''}\partial p^{b'}$ (but not $\partial^2 x^{h'}/(\partial p^{m'})^2$). (1.42)–(1.44) is a differential system of $m_0 + 1$ equations in n' unknowns $x^{i'}$. Suppose that a solution exists which takes on $G_{m'-1}$ the initial values $x^{i'}$, $\partial x^{i'}/\partial p^{m'}$ already determined. Let us then calculate $\partial^2 x^{i'}(p^{c''})/(\partial p^{m'})^2$ at $p^{c''} = 0$.

We write

$$\begin{aligned} \frac{\partial^2 x^{i'}(0)}{(\partial p^{m'})^2} = \sum A_{m'd''} e^{i'}_{d''}(0) + \sum A_{m'\sigma} e^{i'}_{\sigma}(0) \\ + \sum \gamma_{m'\omega'} e^{i'}_{\omega'}(0) + A_{m'n'} e^{i'}_{n'}(0). \end{aligned} \tag{1.45}$$

By (1.42), (1.44) we get (using (1.27), (1.28) and noting that $\Gamma^{i'}_{k'l'}(0) = 0$ by (1.2)),

$$\sum \pm a_{a''d''} A_{m'd''} = F_{a''}\,, \tag{1.46}$$

$$\sum \pm a_{\rho d''} A_{m'd''} + \sum \pm a_{\rho\sigma} A_{m'\sigma} = F_{a''b''} \quad (\text{where } \rho = \{a'', b''\}). \tag{1.47}$$

By (1.43) we get (using (1.33))

$$\sum \pm b_{\sigma} A_{m'\sigma} + \sum \pm b_{\omega'} \gamma_{m'\omega'} - b_{n'} A_{m'n'} = F_{m'}\,. \tag{1.48}$$

Note that all the functions in (1.46)–(1.48) have to be taken at the point $p^{c''} = 0$. Equations (1.46)–(1.48) (at $p^{c''} = 0$) show that $A_{m'd''}$, $A_{m'\sigma}$ and $A_{m'n'}$ (for fixed $\gamma_{c''\omega'}$) are determined by the equations (1.42)–(1.44) in a unique manner, depending on the $\gamma_{c''\omega'}$, but independently of the $\gamma_{m'\omega'}$ (since $b_{\omega'}(0) = 0$) whereas the $\gamma_{m'\omega'}$ can be taken in an arbitrary manner.

We shall now determine the $\gamma_{a'\omega'}$ which appear in (1.40), (1.45). By analog with (1.26) we define the vectors

$$w_{a'}^{i'}, \qquad w_{a'b'}^{i'} \tag{1.49}$$

(taken in lexicographic order). We want to choose the $\gamma_{a'\omega'}$ in such a way that the vectors (1.49) satisfy the B-condition at $p^{c''} = 0$. The Grammian matrix $(g_{i'j'}(0)\, w_\lambda^{i'}(0)\, w_\mu^{j'}(0))$ $(\lambda, \mu = a', \{a', b'\})$ has the form

$$C = \left(\begin{array}{ccccc|cc}
 & & 0 & & & & \vdots \\
 & B_1 & 0 & B_2 & & B_4 & \tilde{A}_{m'c'} \\
 & & \vdots & & & & \vdots \\
0\;\;0\;\;\cdots & & -1 & & 0\;\;\cdots\;\;0 & \cdots\;\; -A_{c''m'} \;\;\cdots & \vdots \\
 & & 0 & & & & \\
 & B_2^* & \vdots & B_3 & & B_5 & \tilde{A}_{m'\rho} \\
 & & -0 & & & & \vdots \\
\hline
 & & \vdots & & & & \\
 & B_4^* & -A_{c''m'} & B_5^* & & P\Gamma\Gamma^* + B_6 & \\
 & & \vdots & & & & \\
\cdots & \tilde{A}_{m'c'} & \cdots & \tilde{A}_{m'\rho} & \cdots & &
\end{array}\right)$$

$$\equiv (c_{pq}), \qquad (p, q = 1, \cdots, \tfrac{1}{2}m'(m' + 3)),$$

where the $\tilde{A}_{m'c'}$, $\tilde{A}_{m'\rho}$ and the last row and column of B_6 depend on the $\gamma_{c''\omega'}$ but otherwise the B_i are independent of the $\gamma_{a'\omega'}$; D^* denotes the transpose of D, $\Gamma = (\gamma_{a'\omega'})$, and $P\Gamma\Gamma^* = (\sum \epsilon_{\omega'}\gamma_{a'\omega'}\gamma_{b'\omega'})$ where $\epsilon_{\omega'} = g_{ij}\, e_{\omega'}^i\, e_{\omega'}^j$. Γ is a matrix of m' rows and $n' - m_0 - 1$ columns. Since we are considering the case $m' < m$, $n' - m_0 - 1 \geqq m'$. From the form of C it is seen that if we take

$$\Gamma = \begin{bmatrix} \lambda + \delta_{11} & \delta_{12} & \cdots & \delta_{1m'} & \cdots & \delta_{1\beta} \\ \delta_{21} & \lambda + \delta_{22} & \cdots & \delta_{2m'} & \cdots & \delta_{2\beta} \\ & \cdots & & & & \\ \delta_{m'1} & \delta_{m'2} & \cdots & \hat{\lambda} + \delta_{m'm'} & \cdots & \delta_{m'\beta} \end{bmatrix}, \quad (\beta = n' - m_0 - 1), \tag{1.50}$$

where $|\delta_{\mu\nu}| \leqq$ const., and if λ is sufficiently large and $\hat{\lambda}$ is sufficiently large compared to λ, then $\det (c_{pq}) \neq 0$ $(p, q = 1, \cdots, r)$ for all $r \leqq \frac{1}{2}m'(m' + 3)$, that is, the vectors (1.49) satisfy the B-condition. We thus take in (1.50) λ, $\hat{\lambda}$ sufficiently large and $\lambda = o(\hat{\lambda})$, and from now on the $\gamma_{a'\omega'}$ are fixed. We also fix the $b_{\omega'}(p^{c''})$ such that they are analytic functions satisfying (1.38), (1.41).

We return to the equations (1.42)–(1.44). Since the number n' of the unknown functions $x^{i'}$ is larger than the number $m_0 + 1$ of the equations, and since the rank of the coefficients matrix is $m_0 + 1$ at $p^{c''} = 0$ (and hence in a neighborhood of $p^{c''} = 0$ which we may assume to be $G_{m'-1}$), we can determine $n' - m_0 - 1$ of the $x^{i'}$ in an arbitrary manner and then solve for the rest of the unknowns. For simplicity we may assume these unknowns to be x^ν, $1 \leqq \nu \leqq m_0 + 1$. We then take the x_μ ($m_0 + 2 \leqq \mu \leqq n'$) to be any analytic functions subjected only to the initial conditions (on $G_{m'-1}$) on the x^μ, $\partial x^\mu / \partial p^{m'}$ and to the condition that at $p^{c'} = 0$ (1.45) holds for $i' = \mu$ where the A's are determined by (1.46)–(1.48).

The system (1.42)–(1.44) is now reduced to a normal system in the x^ν, that is,

$$\text{(1.51)} \qquad \frac{\partial^2 x^\nu}{(\partial p^{m'})^2} = G^\nu\left(p^{a'}, x^\gamma, \frac{\partial x^\gamma}{\partial p^{a'}}, \frac{\partial^2 x}{\partial p^{a''} \partial p^{b'}}\right) \qquad (1 \leqq \nu, \gamma \leqq m_0 + 1).$$

Combining (1.51) with the initial conditions $x^\nu(p^{a''}) = 0$ and (1.33), we have a system to which the Cauchy-Kowalewski theorem can be applied. Thus there exists a unique analytic solution x^ν. Combined with the x^μ ($m_0 + 2 \leqq \mu \leqq n'$,) we obtain an analytic solution of (1.42)–(1.44). From the uniqueness of the solution x^ν and from the way the $\partial^2 x^\mu(0)/(\partial p_m{}')^2$ have been determined it follows that the $\partial^2 x^\nu(0)/(\partial p^{m'})^2$ (are uniquely determined and) satisfy (1.45). Thus the solution $x^{i'}$ which we have just constructed satisfies (1.45). Hence the vectors (1.49) satisfy the B-condition. We have thus completed the construction of an analytic isometric imbedding of $R_m{}'$ ($m' < m$) into $R_{n'}$ which satisfies the B-condition.

It remains to consider the case $m' = m$. In this case the number of the $b_{\omega'}$ is $n' - m_0 - 1 = n - \frac{1}{2}m(m + 1)$ (thus the $b_{\omega'}$ may not appear at all). The proof proceeds in a similar manner. By making a good choice of the $\gamma_{c'\omega'}$, which is a matrix of order $m \times (n - \frac{1}{2}m(m + 1))$, we can get the first

$$m_0 + 1 + \min(m, n - \tfrac{1}{2}m(m + 1)) = \min(n, \tfrac{1}{2}m(m + 3))$$

vectors of the set (1.6) to satisfy the B-condition. The proof is then completed as above. We have thus proved that there exists a local analytic and isometric imbedding of R_m into R_n which satisfies the B-condition.

Remark. The above proof shows that the imbedding is not uniquely determined and that the number of free parameters does not depend on the signatures of the metric tensors. The calculation of the number of these parameters for positive definite metrics is given in [6].

2. Isometric imbedding of a submanifold in a family. Let R_m be an analytic submanifold of R_n and let the metric tensor $g_{ij}(x^k)$ of R_n be analytic and non-degenerate, no restriction being made on its signature. We say that R_m is locally imbedded isometrically in a family of submanifolds $R_m(t)$ of R_n, if the local equations $x^i(u^a, t)$ of $R_m(t)$ satisfy

$$\text{(2.1)} \qquad x^i(u^a, 0) = x^i(u^a), \qquad \frac{\partial}{\partial t}\left[g_{ij}(x^k(u^c, t)) \frac{\partial x^i(u^c, t)}{\partial u^a} \frac{\partial x^j(u^c, t)}{\partial u^b}\right] = 0,$$

for all t sufficiently small, where $x^i = x^i(u^c)$ is the local representation of R_m. It is agreed once and for all that we do not consider here trivial imbeddings, that is, imbeddings induced by a continuous family of isometries of R_n. LEICHTWEISS [6] proved the existence of analytic submanifolds $R_m(t)$ satisfying (2.1) and varying analytically with t, and BURSTIN [2] proved earlier the same result but without the analyticity in t; in both papers R_n has a positive definite metric tensor. The only assumption made in [2], [6] is that the origin (we take $x^i(0) = 0$) is not a flat point of R_m. For the sake of completeness we define this concept here.

A twice continuously differentiable submanifold $\bar{R}_{m-1}$ of R_m, which contains the origin $u^a = 0$, is said to be *non-asymptotic* at the origin if the vector space generated by the tangent vectors of $\bar{R}_{m-1}$ (at $u^a = 0$) and by the normal curvature vectors (at $u^a = 0$) has a maximal dimension. Thus, if $u^a = u^a(v^{\bar{b}})$ are the equations of $\bar{R}_{m-1}$ and if the x^i-coordinates and u^a-coordinates are euclidean at the origin (*i.e.*, if (1.2), (1.3) hold), then

$$\operatorname{rank}\left\{\frac{\partial x^i(0)}{\partial u^a}, \frac{D}{Dv^{\bar{b}}}\left(\frac{\partial x^i}{\partial v^{\bar{c}}}\right)(0)\right\} \tag{2.2}$$

$$= \operatorname{rank}\left\{\frac{\partial x^i(0)}{\partial u^a}\frac{\partial u^b(0)}{\partial v^{\bar{b}}}\frac{\partial u^c(0)}{\partial v^{\bar{c}}}\frac{D}{Du^b}\left(\frac{\partial x^i}{\partial u^c}\right)(0)\right\} = \frac{m(m+1)}{2},$$

where $\bar{a}$, $\bar{b}$, $\bar{c}$ run from 1 to $m - 1$, $u^a(0) = 0$, and $D/Dv^{\bar{b}}$, D/Du^b are mixed covariant derivatives.

If there exists at least one non-asymptotic submanifold $\bar{R}_{m-1}$ at the origin, then the origin is not a *flat point*. Now, for positive definite $g_{ij}(x^k)$ it is easily shown [6] that by change of coordinates we may obtain, in case the origin is not a flat point, a non-asymptotic submanifold defined by $u^m = 0$, that is, R_{m-1} is non-asymptotic. For indefinite metric tensors, we shall need (below) to assume more than the condition (2.2), namely, we shall need the B-condition. Since the B-condition is not invariant under change of coordinates, we shall introduce the following condition:

(A) The x^i-coordinates and the u^a-coordinates are euclidean at the respective origins, $x^i(0) = 0$, and the vectors

$$\frac{\partial x^i(0)}{\partial u^{a'}}, \frac{\partial^2 x^i(0)}{\partial u^{b'}\,\partial u^{c'}} \qquad (a', b', c' = 1, \cdots, m'), \tag{2.3}$$

satisfy the B-condition for any $m' = 1, 2, \cdots, m - 1$.

Condition (A), for a positive definite metric $g_{ij}(x^k)$, merely states that R_{m-1} is non-asymptotic.

Theorem 2. *Let R_n be a Riemannian manifold with analytic and non-degenerate metric tensor $g_{ij}(x^k)$ in a neighborhood of the origin and let R_m be an analytic submanifold with local coordinates u^a, such that the condition* (A) *is satisfied. If $n \geq \frac{1}{2}m\,(m + 1)$ then there exists in a neighborhood of the origin a family $R_m(t)$*

of analytic submanifolds which is analytic also in t, for $|t|$ small, and which is isometric (for any fixed t) to R_m.

By the remarks made above it follows that the special case of Theorem 2 in which the metric $g_{ij}(x^k)$ is positive coincides with the analogous result of [6].

Proof. We shall construct the imbedding of R_m by induction: We suppose that we have already constructed the imbedding family $R_{m'-1}(t)$ for $R_{m'-1}$ and we shall extend it to a family $R_{m'}(t)$. We first consider the imbedding of R_1 in $R_1(t)$.

We have to solve the equation (for $x^k = x^k(u^1, t)$)

$$g_{ij}(x^k) \frac{\partial x^i}{\partial u^1} \frac{\partial x^j}{\partial u^1} = g_{11}(u^1, 0, \cdots, 0). \tag{2.4}$$

In order that the imbedding will not be trivial, the expression (compare [6], p. 468)

$$g_{ij}(x^k(0, t)) \frac{D}{Du^1}\left(\frac{\partial x^i}{\partial u^1}\right)(0, t) \frac{D}{Du^1}\left(\frac{\partial x^j}{\partial u^1}\right)(0, t) \tag{2.5}$$

must be a non-constant function of t. We now note that at least one of the $g_{ii}(0)$ must be equal to $g_{11}(0)$; for simplicity we take it to be $g_{nn}(0)$. We can clearly find a solution

$$x^i(0, t) = 0, \qquad \frac{\partial x^i(0, t)}{\partial u^1} = \bar{x}^i(t), \qquad \bar{x}^n(t) \neq 0 \tag{2.6}$$

of (2.4) at $u^1 = 0$. Indeed, the $\bar{x}^i$ have only to satisfy

$$\sum_{i=1}^{n} g_{ii}(0)(\bar{x}^i(t))^2 = g_{11}(0), \qquad \bar{x}^n(t) \neq 0, \tag{2.7}$$

(recall that $g_{nn}(0) = g_{11}(0)$). Next, differentiating (2.4) we get

$$g_{nn}(x^h) \frac{\partial^2 x^n}{(\partial u^1)^2} \frac{\partial x^n}{\partial u^1} + \cdots = \frac{\partial g_{11}}{\partial u^1}, \tag{2.8}$$

which is equivalent to (2.4), since (2.4) is already satisfied at $u^1 = 0$. We can choose x^α ($1 \leqq \alpha \leqq n - 1$) as arbitrary analytic functions satisfying only the restrictions of (2.6), and then solve (2.8) for x^n, by the CAUCHY-KOWALEWSKI theorem. The solution will then depend analytically also on t. We can clearly make use of the freedom of choosing $\partial^2 x^\alpha/(\partial u^1)^2$ and the $\bar{x}^\alpha$ to obtain a solution for which (2.5) varies with t.

We now proceed by induction. We do not have to take care any more of the non-triviality of the imbedding, since it is non-trivial already for R_1. To construct $R_{m'}(t)$, we first (as in §1) introduce in $R_{m'}$ geodesically parallel coordinates $p^{a'}$. Since the vectors $w^i_{a''}(p^{c''})$, $w^i_{a''b''}(p^{c''})$ corresponding to $R_{m'-1}$ satisfy the B-condition, we can represent them (as in §1) in terms of orthonormal vectors $e^i_{a''}(p^{a''})$, $e^i_\rho(p^{a''})$. The vector

$$e^i_n(0) = \frac{\partial x^i(0)}{\partial p^{m'}} \tag{2.9}$$

is orthogonal to $e^i_{d''}(0)$, $e^i_\rho(0)$, as follows by the analogs of (1.23), (1.25) which clearly hold for $R_{m'}$. We now construct by Lemma 1 vectors $e^i_{\omega'}(0)$ which form together with $e^i_{d''}(0)$, $e^i_\rho(0)$, $e^i_n(0)$ an orthonormal basis. Next, by the SCHMIDT process we obtain an orthonormal basis $e^i_{d''}(p^{c''})$, $e^i_\rho(p^{c''})$, $e^i_n(p^{c''})$, $e^i_{\omega'}(p^{c''})$ for $p^{c''}$ in some neighborhood of $p^{c''} = 0$. In the following we shall assume that $g_{m'm'}(0) = -1$ (the case of $+1$ is treated in the same way).

We now try to construct (on $p^{m'} = 0$)

$$\frac{\partial x^i(p^{c''}, 0, t)}{\partial p^{m'}} = \sum b_{d''}(p^{c''}, t)e^i_{d''}(p^{c''}) + \sum b_\rho(p^{c''}, t)e^i_\rho(p^{c''}) + \sum b_\omega(p^{c''}, t)e^i_\omega(p^{c''}) \tag{2.10}$$

such that it agrees for $t = 0$ with $\partial x^i(p^{c''}, 0)/\partial p^{m'}$ which is known and whose coefficients in the representation analogous to (2.10) are $b_{d''}(p^{c''}) \equiv 0$, $b_\rho(p^{c''})$ and $b_\omega(p^{c''})$. In the following, whenever we refer to equations (1.23)–(1.25), (1.43)–(1.45) it is to be understood that the sign $'$ is to be omitted from the the quantities of $R_{n'}$; thus $x^{i'}$, n', *etc.* are to be replaced by x^i, n, *etc.*

We shall determine the right side of (2.10) in such a manner that equations (1.23)–(1.25) are satisfied for any small t; the right sides of these equations are independent of t.

Noting that

$$w^i_{a''} \equiv \frac{\partial x^i(p^{c''}, 0, t)}{\partial p^{a''}} = \frac{\partial x^i(p^{c''}, 0)}{\partial p^{a''}} + O(t) = w^i_{a''}(p^{c''}) + O(t),$$

and similarly for $w^i_{a''b''}$, we are led to the equations

$$\sum \pm a_{a''d''}(p^{c''})b_{d''}(p^{c''}, t) = \sum_{\lambda=1}^{n} b_\lambda(p^{c''}, t)O(t), \tag{2.11}$$

$$\sum \pm a_{\rho d''}(p^{c''})b_{d''}(p^{c''}, t) + \sum \pm a_{\rho\sigma}(p^{c''})b_\sigma(p^{c''}, t) = -\frac{1}{2}\frac{\partial g_\rho(p^{c''}, 0)}{\partial p^{m'}} + \sum_{\lambda=1}^{n} b_\lambda(p^{c''}, t)O(t), \tag{2.12}$$

$$\sum \pm (b_{d''}(p^{c''}, t))^2 + \sum \pm (b_\sigma(p^{c''}, t))^2 + \sum \pm (b_{\omega'}(p^{c''}, t))^2 + \epsilon(b_n(p^{c''}, t))^2 = -1, \tag{2.13}$$

where $\epsilon = g_{ij}(0)\, e^i_n(0)\, e^j_n(0)$. All the functions $O(t)$ are analytic functions in $(p^{c''}, t)$. Now, equations analogous to (2.11)–(2.13) with $O(t) \equiv 0$ are satisfied for the coefficients of $\partial x^i(p^{c''}, O)/\partial p^{m'}$. Making use of (2.9) we conclude that $b_{a''}(0) = b_\sigma(0) = b_{\omega'}(0) = 0$, $b_n(0) = 1$ and hence (by the analogue of (2.13)) $\epsilon = -1$. We set

$$b_{\omega'}(p^{c''}, t) = b_{\omega'}(p^{c''}). \tag{2.14}$$

Then the equations (2.11)–(2.13) are clearly solvable if $O(t) \equiv 0$, and if we make the restriction $b_n > 0$ then the solution is unique, provided we consider the $b_{\omega'}$ as given functions.

Hence, using the implicit-function theorem we conclude that if t is sufficiently small then equations (2.11)–(2.13) have an analytic solution which is uniquely determined if we take

$$b_n(p^{c''}, t) = \{1 + \sum \pm (b_{d''}(p^{c''}, t)^2 + \sum \pm (b_\sigma(p^{c''}, t))^2 + \sum \pm (b_{\omega'}(p^{c''}, t))^2\}^{\frac{1}{2}} \tag{2.15}$$

and if we set (2.14) from the beginning. Clearly, the solution coincides with the b's of $\partial x^i(p^{c''}, 0)/\partial p^{m'}$ when $t = 0$.

We turn to equations (1.42)–(1.44). We fix $n - m_0 - 1$ of the $x^i(p^{a'}, t)$ such that they are analytic functions in $(p^{a'}, t)$ which coincide with $x^i(p^{a'})$ when $t = 0$ and such that they and their first $p^{m'}$-derivatives at $p^{m'} = 0$ coincide with $x^i(p^{a''}, t)$ (as given by the inductive assumption) and with the functions (2.10) constructed above, respectively. We can fix $n - m_0 - 1$ of the x's such that the remaining x^i satisfy a normal CAUCHY-KOWALEWSKI system if $|t|$ is small. This system coincides, for $t = 0$, with a system which the $x^i(p^{a'})$ satisfy. Hence the solution $x^i(p^{a'}, t)$ together with the $n - m_0 - 1$ already prescribed x's form a solution $x^i(p^{a'}, t)$ of (1.42)–(1.44) which coincides with $x^i(p^{a'})$ when $t = 0$. We thus obtain an imbedding of $R_{m'}$ in $R_{m'}(t)$. Since the $x^i(p^{a'}, t)$ satisfy the equations (1.23)–(1.25), (1.42)–(1.44), or equivalently, (1.16)–(1.21), and since (1.15) is also satisfied (since $R_{m'-1}(t)$, determined by $x^i(p^{a''}, t)$, is isometric to $R_{m'-1}$), we conclude that $E_{a'b'} = 0$. Here, $E_{a'b'}$ is defined as in (1.14) but with $x^{k'}(p^{c'})$, *etc.* replaced by $x^k(p^{c'}, t)$, *etc.* $E_{a'b'} = 0$ means that $R_{m'}(t)$ is isometric to $R_{m'}$, hence the proof of Theorem 2 is completed.

Remark. We note in Theorem 1 we have imbedded R_m as a submanifold of R_n and that this submanifold satisfies the condition (A) of Theorem 2; hence we can imbed it in a family of isometric submanifolds of R_n.

3. Isometric imbedding of two isometric submanifolds in a family. Let ${}_1R_m$ and ${}_2R_m$ be two analytic submanifolds of R_n with local coordinates u^a and let ${}_1x^i(u^a)$ and ${}_2x^i(u^a)$ be their local representation in a neighborhood of $u^a = 0$. The metric tensor $g_{ij}(x^k)$ of R_n is assumed to be analytic and non-degenerate. We assume that the mapping ${}_1x^i(u^a) \to {}_2x^i(u^a)$ is isometric.

Theorem 3. *Let the metric tensor $g_{ij}(x^k)$ be positive definite, let ${}_\alpha x^i(0)$ ($\alpha = 1, 2$) be non-flat points of ${}_\alpha R_m$ with respect to R_n, let the ${}_\alpha x^i(0)$ belong to one x-coordinate patch N_x and let $n > \frac{1}{2}m\,(m + 1)$. Then there exists a family $R_m(t)$ ($0 \leqq t \leqq 1$) of analytic submanifolds isometric to ${}_1R_m$, which is piecewise analytic in t, and which connects ${}_1R_m$ to ${}_2R_m$. More precisely, there exist functions $x^i(u^a, t)$ analytic in u^a in a neighborhood of $u^a = 0$ and continuous and piecewise analytic in t, $0 \leqq t \leqq 1$, such that*

$$x^i(u^a, 0) = {}_1x^i(u^a), \qquad x^i(u^a, 1) = {}_2x^i(u^a), \tag{3.1}$$

$$\frac{\partial}{\partial t}\left[g_{ij}(x^k(u^c, t))\,\frac{\partial x^i(u^c, t)}{\partial u^a}\,\frac{\partial x^j(u^c, t)}{\partial u^b}\right] = 0, \qquad (0 \leqq t \leqq 1). \tag{3.2}$$

Remark 1. In [6] Theorem 3 is proved by a method different from ours and, actually, a slightly stronger result is established, namely, the $R_m(t)$ are analytic in t. Our proof however can easily be generalized to the case of indefinite metrics, under suitable assumptions. This is discussed following the proof of Theorem 3.

Remark 2. The restriction $n > \frac{1}{2}m\,(m+1)$ is essential. However, for euclidean spaces it is easily proved (see [6]) that if $n = \frac{1}{2}m\,(m+1)$ then either ${}_1R_m$ or its reflection, with respect to any given hyperplane in R_n, can be connected to ${}_2R_m$ by an analytic family $R_m(t)$ of isometric submanifolds.

Proof of Theorem 3. The B-condition is equivalent to linear independence (since (g_{ij}) is positive definite). However, in order to make clear the generalization of Theorem 3 to indefinite (g_{ij}), we prefer to use the term B-condition.

As is easily shown (see [6]), we can assume, by change of coordinates, that ${}_\alpha R_{m-1}$ (defined by $u^m = 0$) is a non-asymptotic submanifold of ${}_\alpha R_m$, $\alpha = 1, 2$. We next introduce, step by step, geodesically parallel coordinates p^a and denote by ${}_0R_m$ the space of p^a-coordinates with the metric tensor $g_{ab}(p^c)$ induced by ${}_1R_m$. We may also assume that the p^a-coordinates are euclidean at $p^a = 0$. The condition that ${}_\alpha R_{m-1}$ (defined by $p^m = 0$) is non-asymptotic can be formulated as follows: For every $m' = 1, 2, \cdots, m-1$, the vectors

$$(3.3)\quad \frac{\partial\, {}_\alpha x^i(0)}{\partial p^{a'}}, \frac{\partial^2\, {}_\alpha x^i(0)}{\partial p^{a'}\,\partial p^{b'}} + \Gamma^i_{kl}({}_\alpha x^h(0))\,\frac{\partial\, {}_\alpha x^k(0)}{\partial p^{a'}}\,\frac{\partial\, {}_\alpha x^l(0)}{\partial p^{b'}},\ (a', b' = 1, \cdots, m')$$

satisfy the B-condition.

We proceed to prove Theorem 3 by induction. We first construct a family $R_1(t)$ connecting ${}_1R_1$ to ${}_2R_1$. $R_1(t)$ will be represented by $x^i(p^1, t)$. We take $x^i(0, t)$ $(0 \leqq t \leqq 1)$ to be any analytic curve in N_x which satisfies

$$(3.4)\qquad x^i(0, 0) = {}_1x^i(0), \qquad x^i(0, 1) = {}_2x^i(0).$$

We next consider the equation

$$(3.5)\qquad g_{ij}(x^k(p^1, t))\,\frac{\partial x^i(p^1, t)}{\partial p^1}\,\frac{\partial x^j(p^1, t)}{\partial p^1} = g_{11}(p^1, 0, \cdots, 0),$$

which at $p^1 = 0$ becomes

$$(3.6)\qquad g_{ij}(x^k(0, t))\,\frac{\partial x^i(0, t)}{\partial p^1}\,\frac{\partial x^j(0, t)}{\partial p^1} = g_{11}(0).$$

For $t = 0, 1$ the equation (3.6) has a solution $\partial_\alpha x^i(0)/\partial p^1$ for $\alpha = 1, 2$. We connect these two unit vectors by unit vectors $e^i_1(t)$ $(0 \leqq t \leqq 1)$ analytic in t, and then define

$$(3.7)\qquad \frac{\partial x^i(0, t)}{\partial p^1} = e^i_1(t).$$

We next differentiate (3.5) with respect to p^1 and obtain

$$(3.8)\qquad g_{ij}(x^h)\,\frac{\partial x^i}{\partial p^1}\left\{\frac{\partial^2 x^j}{(\partial p^1)^2} + \Gamma^j_{kl}(x^h)\,\frac{\partial x^k}{\partial p^1}\,\frac{\partial x^l}{\partial p^1}\right\} = \frac{1}{2}\,\frac{\partial g_{11}(p^1, 0, \cdots, 0)}{\partial p^1}.$$

Since (3.6) is satisfied, (3.8) is equivalent to (3.5). We denote by $e^i_{11}(t)$ the braces in (3.8) when $p^1 = 0$. Then, for $p^1 = 0$, (3.8) is equivalent to the statement that $e^i_1(t)$ and $e^i_{11}(t)$ are orthogonal. Now $e^i_{11}(0)$ and $e^i_{11}(1)$ are given vectors which are orthogonal to $e^i_1(0)$ and $e^i_1(1)$ respectively and they are non-zero vectors (because the B-condition holds for the vectors (3.3) when $m' = 1$). Hence, we can connect $e^i_{11}(0)$ to $e^i_{11}(1)$ by a vector $e^i_{11}(t)$, analytic in t, which is orthogonal to $e^i_1(t)$. We then define

$$(3.9) \qquad \frac{\partial^2 x^i(0, t)}{(\partial p^1)^2} + \Gamma^i_{kl}(x^h(0, t)) \frac{\partial x^k(0, t)}{\partial p^1} \frac{\partial x^l(0, t)}{\partial p^1} = e^i_{11}(t).$$

We want to solve (3.8) with the conditions (3.4), (3.7), (3.9). An entirely similar problem arises in the inductive passage from $m' - 1$ to m'. Hence we shall omit details here. The solution $x^i(p^1, t)$ is analytic in p^1 in some neighborhood of $p^1 = 0$ and in t, $0 \leqq t \leqq 1$. Since the vectors (3.7), (3.9) are orthogonal, the mapping $x^i(p^1, t)$ satisfies (for any $0 \leqq t \leqq 1$) the B-condition provided p^1 is sufficiently close to the origin.

We assume that we have already connected ${}_1R_{m'-1}$ to ${}_2R_{m'-1}$ by $R_{m'-1}(t)$ having the asserted analytic and isometric properties and satisfying (for any $0 \leqq t \leqq 1$) the B-condition. We proceed to construct $R_{m'}(t)$. We first consider the case $m' < m$. In what follows, whenever we refer to equations (1.23)–(1.25), (1.42)–(1.44) it is to be understood that $x^{i'}$, n', *etc.* (but not a', b', a'', m', *etc.*) are to be replaced by x^i, n, *etc.* Furthermore, the coefficients as well as the solutions are usually understood to depend on t. We shall make free use of the notation of §1, and sometimes introduce the parameter t into some of the quantities defined in §1, without further explanation.

We define

$$(3.10) \qquad {}_\alpha e^i_n(0) = \frac{\partial\, {}_\alpha x^i(0)}{\partial p^{m'}}, \qquad (\alpha = 1, 2).$$

Since the B-condition is satisfied for $R_{m'-1}(t)$ (by the inductive assumption) we can apply the process of orthonormalization of Schmidt to the vectors $w^i_{a''}(p^{c''}, t)$, $w^i_\rho(p^{c''}, t)$ (where $\rho = \{a'', b''\}$) and obtain orthonormal vectors $e^i_{a''}(p^{c''}, t)$, $e^i_\rho(p^{c''}, t)$ analytic in $p^{c''}$ and piecewise analytic in t. By (1.23), (1.25) we find that the vectors (3.10) are orthogonal to the $e^i_\lambda(0, t)$ ($\lambda = a''$, ρ) for $t = 0, 1$ respectively. Since the number of the vectors e^i_λ is not more than (and actually less) than $n - 2$, we can construct a piecewise analytic vector $e^i_n(0, t)$ ($0 \leqq t \leqq 1$) which connects the vectors (3.10) and which is orthogonal to the vectors $e^i_\lambda(0, t)$. Indeed it is easy to construct such $e^i_n(0, t)$ which varies continuously with t. Then we approximate it by an analytic vector $\tilde{e}^i_n(t)$ which connects the vectors (3.10) such that the B-condition is still satisfied for the $e^i_\lambda(0, t)$ and $\tilde{e}^i_n(t)$. Finally, by using Schmidt process we obtain from $\tilde{e}^i_n(t)$ a new vector orthogonal to the $e^i_\lambda(0, t)$, which is piecewise analytic in t and of unit length, we again denote it (for simplicity) by $e^i_n(0, t)$.

We complete the vectors $e^i_\lambda(0, t)$, $e^i_n(0, t)$ to an orthonormal basis by adding vectors $e^i_{\omega'}(t)$. These vectors can be taken to be continuous in t, and by using analytic approximation and the SCHMIDT process we obtain new vectors which we denote by $e^i_{\omega'}(0, t)$. These vectors not only complete $e^i_\lambda(0, t)$, $e^i_n(0, t)$ to an orthonormal basis, but they are also piecewise analytic in t. Applying the method of SCHMIDT to the vectors $e^i_\lambda(p^{c''}, t)$ $e_{\omega'}(0, t)$, $e^i_n(0, t)$ we obtain an orthonormal basis $e^i_\lambda(p^{c''}, t)$, $e^i_{\omega'}(p^{c''}, t)$, $e^i_n(p^{c''}, t)$ at the points on $R_{m'-1}(t)$ (for $p^{c''}$ in some neighborhood of the origin).

We now try to construct

$$\frac{\partial x^i(p^{c''}, 0, t)}{\partial p^{m'}} = \sum b_{d''}(p^{c''}, t)e^i_{d''}(p^{c''}, t) + \sum b_\rho(p^{c''}, t)e^i_\rho(p^{c''}, t) + \sum b_\omega(p^{c''}, t)e^i_\omega(p^{c''}, t), \tag{3.11}$$

which coincides for $t = 0, 1$ with $\partial_\alpha x^i(p^{c''}, 0)/\partial p^{m'}$ when $\alpha = 1, 2$, and is such that (1.23)–(1.25) hold for $0 \leqq t \leqq 1$. We then find (by (1.23)) that $b_{d''}(p^{c''}, t) = 0$ and (by (1.25)) that the $b_\rho(p^{c''}, t)$ are uniquely determined extensions of the analogous coefficients ${}_\alpha b_\rho(p^{c''})$, and $b_\rho(0, t) = 0$. We next want to define $b_{\omega'}(p^{c''}, t)$ in such a way that

$$b_{\omega'}(p^{c''}, 0) = {}_1b_{\omega'}(p^{c''}), \qquad b_{\omega'}(p^{c''}, 1) = {}_2b_{\omega'}(p^{c''}), \qquad b_{\omega'}(0, t) = 0. \tag{3.12}$$

Then, since ${}_\alpha b_n(0) = 1$, the function

$$b_n(p^{c''}, t) = \{1 - \sum (b_\rho(p^{c''}, t))^2 - \sum (b_{\omega'}(p^{c''}, t))^2\}^{\frac{1}{2}} \tag{3.13}$$

both satisfies (1.24) and is an extension of ${}_\alpha b_n(p^{c''})$. Later on we shall show how to choose in a useful way the $b_{\omega'}(p^{c''}, t)$ which satisfy (3.12).

We next differentiate (3.11) with respect to $p^{a''}$ and obtain, upon substituting $p^{c''} = 0$,

$$\begin{aligned}\frac{\partial^2 x^i(0, 0, t)}{\partial p^{m'}\, \partial p^{a''}} + \Gamma^i_{kl}(x^h(0, 0, t)) \frac{\partial x^k(0, 0, t)}{\partial p^{m'}} \frac{\partial x^l(0, 0, t)}{\partial p^{a'}} &= \sum A_{a''d''}(t)e^i_{d''}(0, t) + \sum A_{a''\rho}(t)e^i_\rho(0, t) \\ &\quad + \sum \gamma_{a''\omega'}(t)e^i_{\omega'}(0, t) + A_{a''n}(t)e^i_n(0, t),\end{aligned} \tag{3.14}$$

where

$$\gamma_{a''\omega'}(t) = A_{a''\omega'}(t) + \frac{\partial b_{\omega'}(0, t)}{\partial p^{a''}}. \tag{3.15}$$

Here the A's are given functions, independent of the choice of the γ's. The A's, for $t = 0, 1$, are equal to the corresponding ${}_\alpha A$'s for $\alpha = 1, 2$ and

$$\gamma_{a''\omega'}(0) = {}_1\gamma_{a''\omega'}, \qquad \gamma_{a''\omega'}(1) = {}_2\gamma_{a''\omega'} \tag{3.16}$$

(the ${}_\alpha\gamma$'s are given).

We next write

$$(3.17)\qquad \begin{aligned}\frac{\partial^2 x^i(0,0,t)}{(\partial p^{m'})^2} + \Gamma^i_{kl}(x^h(0,0,t))\,\frac{\partial x^k(0,0,t)}{\partial p^{m'}}\,\frac{\partial x^l(0,0,t)}{\partial p^{m'}} &= \sum A_{m'd''}(t)e^i_{d''}(0,t) + \sum A_{m'\rho}(t)e^i_\rho(0,t)\\ &\quad + \sum \gamma_{m'\omega'}(t)e^i_{\omega'}(0,t) + A_{m'n}(t)e^i_n(0,t).\end{aligned}$$

The equations (1.42)–(1.44) determine the A's, depending on the $\gamma_{a''\omega'}$ but independently of the $\gamma_{m'\omega'}$, in a unique way, and when $t = 0, 1$ the A's coincide with the analogous coefficients ${}_\alpha A$ (corresponding to $\partial^2_\alpha/x^i/(\partial p^{m'})^2$) for $\alpha = 1, 2$. The γ's are given for $t = 0, 1$ but are otherwise free. Thus we have to choose the γ's such that

$$(3.18)\qquad \gamma_{m'\omega'}(0) = {}_1\gamma_{m'\omega'}\,, \qquad \gamma_{m'\omega'}(1) = {}_2\gamma_{m'\omega'}\,.$$

We shall later on show how to choose the $\gamma_{m'\omega'}(t)$, $b_{\omega'}(p^{c''}, t)$ such that the vectors $w_{a'}(0, t)$, $w_{a'b'}(0, t)$ satisfy the B-condition. Once this is shown, we can complete the proof of Theorem 3 as follows.

For every t_0 there is a t-neighborhood for which $m_0 + 1$ of the $x^i(p^{c'}, t)$ satisfy a normal system; the remaining $n - m_0 - 1$ of the $x^i(p^{c'}, t)$ can be taken as arbitrary analytic functions. We divide the interval [0, 1] into a finite number of such intervals $I_\mu = [t_\mu, t_{\mu+1}]$. For $t \in I_1$ we fix $n - m_0 - 1$ of the $x^i(p^{c'}, t)$ such that they are analytic functions, coinciding for $t = 0$ with ${}_1x^i(p^{c'})$, having at $p^{m'} = 0$ the values $x^i(p^{a''}, 0, t)$, for their first $p^{m'}$-derivatives at $p^{m'} = 0$ the values of (3.11), and for their second $p^{m'}$-derivative at $p^{a'} = 0$ the values of (3.17).

We then solve for the remaining $x^i(p^{c'}, t)$ with the initial data $x^i(p^{c''}, 0, t)$, $\partial x^i(p^{c''}, 0, t)/\partial p^{m'}$ (of (3.11)). The solution has $\partial^2 x^i(0, 0, t)/(\partial x^{p^{m'}})^2$ as determined above in 3.17 (by uniqueness) and, furthermore, it coincides for $t = 1$ with ${}_1x^i(p^{c'})$. We next turn to I_2 and proceed in a similar manner, then to I_3, *etc.* Finally, in the last interval I_ν we can proceed as before, provided we can determine $n - m_0 - 1$ of the $x^i(p^{c'}, t)$ in such a manner that they are analytic, connect ${}_1x^i(p^{c'}, t)$ to ${}_2x^i(p^{c'})$ and such that $x^i(p^{c'}, t)$, $\partial x^i(p^{c'}, t)/\partial p^{m'}$ on $p^{m'} = 0$ and $\partial^2 x^i(p^{c'}, t)/(\partial p^{m'})^2$ for $p^{c'} = 0$ have the values prescribed above. We can take

$$(3.19)\qquad \begin{aligned} x^i(p^{c'}, t) &= \frac{t - t_\nu}{1 - t_\nu}\,{}_2x^i(p^{c'}) + \frac{1 - t}{1 - t_\nu}\,x^i(p^{c'}, t_\nu)\\ &\quad + \left\{x^i(p^{c''}, 0, t) - \left[\frac{t - t_\nu}{1 - t_\nu}\,{}_2x^i(p^{c''}, 0) + \frac{1 - t}{1 - t_\nu}\,x^i(p^{c''}, 0, t_\nu)\right]\right\}\\ &\quad + \left\{\frac{\partial x^i(p^{c''}, 0, t)}{\partial p^{m'}} - \left[\frac{t - t_\nu}{1 - t_\nu}\,\frac{\partial\, {}_2x^i(p^{c''}, 0)}{\partial p^{m'}} + \frac{1 - t}{1 - t_\nu}\,\frac{\partial x^i(p^{c''}, 0, t_\nu)}{\partial p^{m'}}\right]\right\}p^{m'}\\ &\quad + \frac{1}{2}\left\{\frac{\partial^2 x^i(0, 0, t)}{(\partial p^{m'})^2} - \left[\frac{t - t_\nu}{1 - t_\nu}\,\frac{\partial^2\, {}_2x^i(0, 0)}{(\partial p^{m'})^2} + \frac{1 - t}{1 - t_\nu}\,\frac{\partial^2 x^i(0, 0, t_\nu)}{(\partial p^{m'})^2}\right]\right\}(p^{m'})^2. \end{aligned}$$

The reason that we can assert, in Theorem 3, only piecewise-analyticity in t is that the $n - m_0 - 1$ functions $x^i(p^{c'}, t)$ cannot, in general, be extended analytically in t at the points $t = t_\mu$.

The solution $x^k(p^{c'}, t)$ just found satisfies the B-condition and also defines $R_{m'}(t)$ with the required analyticity and isometry properties.

It remains to choose the $b_{\omega'}(p^{a''}, t)$, $\gamma_{m'\omega'}(t)$. We shall first determine $\gamma_{a'\omega'}(t)$. Since (by assumption) the vectors (3.3) satisfy the B-condition, we can apply the process of SCHMIDT and get

$$\frac{\partial\ {}_\alpha x^i(0)}{\partial p^{a''}} = \sum_{d''=1}^{a''} {}_\alpha a_{a''d''}\ {}_\alpha e^i_{d''}\ , \tag{3.20}$$

$$\frac{\partial^2{}_\alpha x^i(0)}{\partial p^{m'}} = {}_\alpha e^i_n\ , \tag{3.20'}$$

$$\frac{\partial^2\ {}_\alpha x^i(0)}{\partial p^{a''}\ \partial p^{b''}} + \Gamma^i_{kl}({}_\alpha x^h(0))\frac{\partial\ {}_\alpha x^k(0)}{\partial p^{a''}}\frac{\partial\ {}_\alpha x^l(0)}{\partial p^{b''}} = \sum_{d''=1}^{m'} {}_\alpha a_{\rho d''}\ {}_\alpha e^i_{d''} + \sum_{\sigma=m'+1}^{\rho} {}_\alpha a_{\rho\sigma}\ {}_\alpha e^i_\sigma\ , \qquad (\rho = \{a'', b''\}), \tag{3.21}$$

$$\frac{\partial^2\ {}_\alpha x^i(0)}{\partial p^{m'}\ \partial p^{a'}} + \Gamma^i_{kl}({}_\alpha x^h(0))\frac{\partial\ {}_\alpha x^k(0)}{\partial p^{m'}}\frac{\partial\ {}_\alpha x^l(0)}{\partial p^{a''}} = \sum {}_\alpha B_{a'd''}\ {}_\alpha e^i_{d''} + {}_\alpha B_{a'n}\ {}_\alpha e^i_n + \sum {}_\alpha B_{a'\rho}\ {}_\alpha e^i_\rho + \sum_{\tau=1}^{a'} {}_\alpha C_{a'\tau}\ {}_\alpha e^i_{m'\tau}\ , \tag{3.22}$$

for $\alpha = 1, 2$. Here ${}_\alpha a_{a'a'} \neq 0$, ${}_\alpha a_{\rho\rho} \neq 0$ and we may take ${}_\alpha C_{a'a'} > 0$. The ${}_\alpha a$'s and the ${}_\alpha e^i_\lambda$'s ($\lambda = a''$, ρ) are the values of the $a(0, t)$'s and the $e^i_\lambda(t)$'s corresponding to the $w^i_{a'}(0, t)$, $w^i_{a''b''}(0, t)$ when $t = 0, 1$. Since the number of the e^i_λ's is not more (and actually less) than $n - 3$, we can find orthogonal unit vectors $e^i_{m'\tau}(t)$ which coincide for $t = 0, 1$ with ${}_\alpha e^i_{m'\tau}$ for $\alpha = 1, 2$, which are orthogonal to $e^i_n(0, t)$ and to the $e^i_\lambda(0, t)$, and which are continuous in t, $0 \leqq t \leqq 1$.

By approximating them by analytic vectors and applying the method of SCHMIDT we obtain new vectors, again denoted by $e^i_{m'\tau}(t)$, which are also piecewise analytic in t.

We next try to write (3.14), (3.17) in the form

$$\frac{\partial^2 x^i(0, 0, t)}{\partial p^{m'}\ \partial p^{a'}} + \Gamma^i_{kl}(x^h(0, 0, t))\frac{\partial x^k(0, 0, t)}{\partial p^{m'}}\frac{\partial x^l(0, 0, t)}{\partial p^{a'}} = \sum B_{a'd''}(t)e^i_{d''}(0, t) + B_{a'n}(t)e^i_n(0, t) + \sum B_{a'\rho}(t)e^i_\rho(0, t) + \sum_{\tau=1}^{a'} C_{a'\tau}(t)e^i_{m'\tau}(t). \tag{3.23}$$

By comparison with (3.14), (3.17) we see that (3.23) is valid if $B_{a'\lambda}(t) = A_{a'\lambda}(t)$ and if

$$\sum \gamma_{a'\omega'}(t)e^i_{\omega'}(0, t) = \sum_{\tau=1}^{a'} C_{a'\tau}(t)e^i_{m'\tau}(t). \tag{3.24}$$

Since the $e^i_{\omega'}$ complete the e^i_λ ($\lambda = a''$, n, ρ) to a basis, and since the $e^i_{m'\tau}$ are orthogonal to the e^i_λ, we have

$$e^i_{m'\tau}(t) = \sum d_{\tau\omega'}(t) e^i_{\omega'}(0, t). \tag{3.25}$$

If we multiply (3.25) scalarly by $e^i_{\omega'}(0, t)$ we obtain an expression for $d_{\tau\omega'}(t)$ from which we conclude that it is piecewise analytic in t. Combining (3.24), (3.25) we get

$$\gamma_{a'\omega'}(t) = \sum_{\tau=1}^{a'} d_{\tau\omega'}(t) C_{a'\tau}(t). \tag{3.26}$$

We are now ready to show how to choose the $\gamma_{a'\omega'}(t)$. We take $C_{a'a'}(t)$ to be any positive analytic function which coincides for $t = 0, 1$ with ${}_\alpha C_{a'a'}$ for $\alpha = 1, 2$. We construct the remaining $C_{a'\tau}(t)$ as arbitrary analytic functions which connect the ${}_\alpha C_{a'\tau}$. Then the $\gamma_{a'\omega'}(t)$ are defined by (3.26). We also define $B_{a'\lambda}(t) = A_{a'\lambda}(t)$. Clearly the vectors $w_{a'}(0, t)$, $w_{a''b''}(0, t)$ and (3.23) satisfy the B-condition.

It remains to extend the $b_{\alpha'}(p^{c''}, t)$ such that (3.15) holds and such it is an extension of the ${}_\alpha b_{\omega'}(p^{c''})$. This can be done by the formula

$$\begin{aligned} b_{\omega'}(p^{c''}, t) = {} & t\, {}_2b_{\omega'}(p^{c''}) + (1 - t)\, {}_1b_{\omega'}(p^{c''}) \\ & + \sum \{\gamma_{a''\omega'}(t) - [t\, {}_2\gamma_{a''\omega'} + (1 - t)\, {}_1\gamma_{a''\omega'}]\} p^{a''} \\ & - \sum \{A_{a''\omega'}(t) - [t\, {}_2A_{a''\omega'} + (1 - t)\, {}_1A_{a''\omega'}]\} p^{a''}. \end{aligned} \tag{3.27}$$

We have thus completed the construction of $R_{m'}(t)$ for $m' < m$. If $m' = m$ the proof is similar and is obtained from the previous proof by some minor modifications (compare the proof of Theorem 1, for $m' = m$). We make use of the fact that $n > \frac{1}{2}m\,(m + 1)$ in establishing the existence of $e^i_n(0, t)$.

Remark 1. From the above proof it follows that $p^m = 0$ is a non-asymptotic submanifold (at $p^a = 0$) for any $R_m(t)$, $0 \leqq t \leqq 1$.

Remark 2. If instead of (3.19) in I_ν we use a more refined formula, and similarly in the other intervals I_μ, then we can prove that $R_m(t)$ is q-times continuously differentiable in t ($0 \leqq t \leqq 1$), as well as piecewise analytic in t, where q is an arbitrarily given positive integer.

We shall now extend Theorem 3 to R_n with an indefinite metric. We need some assumptions.

General assumptions. ${}_1R_m$ and ${}_2R_m$ are analytic submanifolds of R_n represented locally by ${}_1x^i(u^a)$ and ${}_2x^i(u^a)$, and ${}_1x^i(0)$, ${}_2x^i(0)$ belong to one x-coordinate patch of R_n which we denote by N_x. $g_{ij}(x^k)$ is an analytic, non-degenerate and indefinite metric in R_n. ${}_1x^i(u^a) \to {}_2x^i(u^a)$ is an isometry. We introduce (step by step) in the u^a-space, with the metric induced by ${}_1R_m$, geodesically parallel coordinates p^a and denote the space of p^a-coordinates with the metric $g_{ab}(p^a)$ (induced by ${}_1R_m$) by ${}_0R_m$. The isometry between ${}_1R_m$ and ${}_2R_m$ makes ${}_2x^i(p^a)$

correspond to ${}_1x^i(p^a)$. We take $x^i(0, t)$ to be an analytic curve (in N_x) for $0 \leqq t \leqq 1$ which satisfies: $x^i(0, 0) = {}_1x^i(0)$, $x^i(0, 1) = {}_2x^i(0)$. We denote $g_{ij}(x^k(0, t))$ simply by $g_{ij}(t)$.

Definition. For any t, $0 \leqq t \leqq 1$, a vector $\xi^i(t)$ is called *positive*, *negative* or *null* if $g_{ij}(t)\ \xi^i(t)\xi^j(t)$ is positive, negative or zero, respectively. We denote by $K^+(t)$, $K^-(t)$ and $K(t)$ the sets of positive, negative and null vectors, respectively.

We denote by $K^*(t)$ any component of either $K^+(t)$ or $K^-(t)$ which varies continuously with t and which is never empty. Incidentally, as is well known, for $n \geqq 3$ the number of components (for every fixed t) is two if there are at least two eigenvalues (of $g_{ij}(t)$) of each sign and is three if all the eigenvalues except one have the same signature. We now state further assumptions.

Assumption (A₁). The p^a are euclidean coordinates of ${}_0R_m$ at the origin $p^a = 0$, and the vectors (3.3) satisfy the B-condition for $m' = m - 1$ and for $\alpha = 1, 2$.

Noting that $\partial_\alpha x^i(0)/\partial p^{m'+1}$ is orthogonal to the vectors (3.3) (see (1.23), (1.25)) for any $m' < m$, it follows (by (A_1)) that we can write down the equations (3.20), (3.21) where the ${}_\alpha e^i_\lambda$ are orthonormal and where ${}_\alpha a_{a''d''} = 0$ if $a'' \neq d''$, $a_{d''d''} = 1$, and we also may assume that

$$(3.28) \qquad {}_\alpha a_{\rho\rho} > 0 \qquad (\rho = \{a'', b''\}) \quad \text{for} \quad m' = 1, 2, \cdots, m; \qquad \alpha = 1, 2.$$

Assumption (A₂). For $m' = m$ and $\alpha = 1, 2$ the inequalities (3.28) (in the representation (3.21)) hold and there exist vectors $e^i_a(0, t)$, $e^i_\rho(0, t)$ which are orthonormal and continuous in $0 \leqq t \leqq 1$, and which satisfy:

$$(3.29) \quad e^i_d(0, 0) = \frac{\partial\ {}_1x^i(0)}{\partial p^d}, \ e^i_d(0, 1) = \frac{\partial\ {}_2x^i(0)}{\partial p^d}, \ e^i_\rho(0, 0) = {}_1e^i_\rho, \ e^i_\rho(0, 1) = {}_2e^i_\rho .$$

If $n > \frac{1}{2}m\,(m + 1)$ and if for any λ the vectors ${}_\alpha e^i_\lambda$ for $\alpha = 1$ and $\alpha = 2$ lie is some $K^*(0)$ and $K^*(1)$ respectively, then conditions on the mutual positions of the ${}_\alpha e^i_\lambda$ can be given under which the assumption (A_2) follows.

We can now state a generalization of Theorem 3.

Theorem 3′. *Let the general assumptions and assumptions (A_1), (A_2) be satisfied and let $n \geqq \frac{1}{2}m\,(m + 1)$. Then there exists a family $R_m(t)$ $(0 \leqq t \leqq 1)$ of isometric cnalytic submanifolds, which is continuous and piecewise analytic in t, and which aonnects ${}_1R_m$ to ${}_2R_m$ in the sense of Theorem* 3.

Proof. The proof of Theorem 3′ proceeds very similarly to the proof of Theorem 3. A few general facts about vectors which are tacitly used in the previous proof are not self evident in the present case of indefinite metric. We state these facts in a few lemmas whose proofs are given in the appendix, and omit further details of the proof of Theorem 3′.

Lemma 2. *Let $E^i_1, \cdots, E^i_r$ be orthonormal vectors at a point A of R_n. Then* *the dimension of the vector space of solutions f^i (at A) of*

$$g_{ij}E^i_\beta f^j = 0, \qquad (1 \leqq \beta \leqq \tau),$$

is $n - \tau$.

Corollary. *If in Lemma* 1, F^i *is orthogonal to the vectors* $E^i_1, \cdots, E^i_\tau$ *at the point* A *of* R_n, *then*

$$F^i = \sum_{\beta=\tau+1}^{n} c_\beta E^i_\beta .$$

Lemma 3. *If the vectors* Y^i $(1 \leqq \lambda \leqq \beta)$, *at the point* A, *can be represented in the form*

$$Y^i_\lambda = \sum_{\gamma=1}^{\lambda} a_{\lambda\gamma} E^i_\gamma \quad \textit{where} \quad a_{\lambda\lambda} \neq 0, \qquad (1 \leqq \lambda \leqq \beta),$$

where the vectors E^i_γ *are orthonormal vectors at* A, *then the* Y^i_λ *satisfy the B-condition at* A.

The converse of this lemma was already used in §§1, 2.

Corollary. *If* Y^i_λ $(1 \leqq \lambda \leqq \beta)$ *satisfy the B-condition and if*

$$Z^i_\lambda = \sum_{\gamma=1}^{\lambda} b_{\lambda\gamma} Y^i_\gamma \quad \textit{where} \quad b_{\lambda\lambda} \neq 0, \qquad (1 \leqq \lambda \leqq \beta),$$

then the Z^i_λ *satisfy the B-condition.*

Lemma 4. *Let* $g_{ij}(t)$, $E^i_\lambda(t)$ $(1 \leqq \lambda \leqq \tau)$ *depend continuously on* t, $0 \leqq t \leqq 1$, *where* det $(g_{ij}(t)) \neq 0$ *and where the* $E^i_\lambda(t)$ *are orthonormal, for every* t. *Then we can complete the* $E^i_\lambda(t)$ *to an orthonormal basis by adding vectors* $E^i_\beta(t)$ $(0 \leqq t \leqq 1)$ *which depend continuously on* t.

4. Appendix.

Proof of Lemma 1. If the lemma is not true, then there exists k, $\tau \leqq k \leqq n - 1$, such that every solution f^i of

(4.1) $$g_{ij}E^i_\beta f^j = 0, \qquad (1 \leqq \beta \leqq k)$$

satisfies

(4.2) $$g_{ij}f^i f^j = 0,$$

where the E^i_β are orthonormal vectors. It is easily seen that the matrix $(g_{ij}E^i_\beta)$ has rank k. Hence the solutions f^i form a vector space V of dimension $n - k$. If f^i, h^i are vectors of V then

(4.3) $$g_{ij}f^i h^j = 0.$$

Indeed, this follows by noting that (4.2) holds also for h^i and for $f^i + h^i$. Let

$E^i_{k+1}, \cdots, E^i_n$ be linearly independent vectors in V. Then

$$g_{ij}E^i_\gamma E^j_n = 0, \qquad (1 \leqq \gamma \leqq n),$$

which implies that the matrix $(g_{ij}E^i_\gamma)$ is singular. Since det $(g_{ij}) \neq 0$, the vectors E^i_γ must be linearly dependent. Writing $\sum \mu_\gamma E^i_\gamma = 0$ and multiplying scalarly by E^i_β $(1 \leqq \beta \leqq k)$ we obtain $\mu_\beta = 0$. Hence, the vectors $E^i_{k+1}, \cdots, E^i_n$ are linearly dependent, which is a contradiction.

Lemma 2 follows immediately from the fact that the matrix $(g_{ij}E^i_\beta)$ has rank τ.

Proof of Lemma 3. Set $g_{ij}E^i_\lambda E^j_\mu = \epsilon_\lambda \delta_{\lambda\mu}$, so that $\epsilon_\lambda = \pm 1$. To establish the lemma it is enough to prove that the Grammian determinant

$$\begin{vmatrix} \epsilon_1 a_{11}a_{11} & \epsilon_1 a_{11}a_{21} & \epsilon_1 a_{11}a_{31} & \cdots & \epsilon_1 a_{11}a_{\beta 1} \\ \epsilon_1 a_{21}a_{11} & \epsilon_1 a_{21}a_{21} + \epsilon_2 a_{22}a_{22} & \epsilon_1 a_{21}a_{31} + \epsilon_2 a_{22}a_{32} & \cdots & \epsilon_1 a_{21}a_{\beta 1} + \epsilon_2 a_{22}a_{\beta 2} \\ \epsilon_1 a_{31}a_{11} & \epsilon_1 a_{31}a_{21} + \epsilon_2 a_{32}a_{22} & \epsilon_1 a_{31}a_{31} + \epsilon_2 a_{32}a_{32} + \epsilon_3 a_{33}a_{33} & \cdots & \epsilon_1 a_{31}a_{\beta 1} + \epsilon_2 a_{32}a_{\beta 2} + \epsilon_3 a_{33}a_{\beta 3} \\ \cdots & & & & \\ \epsilon_1 a_{\beta 1}a_{11} & \epsilon_1 a_{\beta 1}a_{21} + \epsilon_2 a_{\beta 2}a_{22} & \epsilon_1 a_{\beta 1}a_{31} + \epsilon_2 a_{\beta 2}a_{32} + \epsilon_3 a_{\beta 3}a_{33} & \cdots & \epsilon_1 a_{\beta 1}a_{\beta 1} + \cdots + \epsilon_\beta a_{\beta\beta}a_{\beta\beta} \end{vmatrix}$$

is different from zero. Extracting $-\epsilon_1 a_{11}$ from the first row, then multiplying the first row by $a_{\mu 1}$ and adding to the μ-th row, for $2 \leqq \mu \leqq \beta$, we obtain a new determinant which has zeros at each place $(\mu, 1)$ $(2 \leqq \mu \leqq \beta)$. We then see that the original Grammian determinant is equal to $\epsilon_1 a_{11} a_{11}$ times the Grammian determinant of $Y^i_2, \cdots, Y^i_\beta$. Proceeding inductively we conclude that the Grammian determinant of $Y^i_1, \cdots, Y^i_\beta$ is equal to

$$\prod_{\lambda=1}^{\beta} \epsilon_\lambda (a_{\lambda\lambda})^2 \neq 0$$

and the proof is completed.

Proof of Lemma 4. It is enough to construct $f^i(t) = E^i_{\tau+1}(t)$, since then we can apply the same construction $n - \tau - 1$ additional times. We have to find a continuous solution $f^i(t)$ $(0 \leqq t \leqq 1)$ of

$$g_{ij}(t)E^i_\alpha(t)f^j(t) = 0, \qquad (1 \leqq \alpha \leqq \tau), \tag{4.4}$$

$$g_{ij}(t)f^i(t)f^j(t) \neq 0. \tag{4.5}$$

Let $V(t)$ be the space of solutions of (4.4). We first consider the case $n - \tau \geqq 2$.

By Lemma 2, the dimension of $V(t)$ is $n - \tau \geqq 2$. It is also easy to see that $V(t)$ varies continuously with t. Hence we can construct two linearly independent vectors $f^i_1(t)$, $f^i_2(t)$ in $V(t)$, which vary continuously with t. The line $L(t)$ through these vectors cannot lie entirely on the cone $K(t)$, and we denote by $L^+(t)$ an unbounded component of $L(t)$ which varies continuously (and hence is never empty) with t. We can now take any continuous vector $f^i(t)$ $(0 \leqq t \leqq 1)$ which lies on $L^+(t)$.

If $n - \tau = 1$ then Lemma 1 implies that, for $0 \leqq t \leqq 1$, the line $V(t)$ has only the origin in common with $K(t)$. Hence, $V(t) - \{0\}$ lies either in $K^+(t)$ or in $K^-(t)$ for all t and it is clear how to construct $f^i(t)$.

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