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# ON THE UPWARD EMBEDDING ON THE TORUS

ARDESHIR DOLATI, S. MEHDI HASHEMI, AND MASOUD KHOSRAVANI

ABSTRACT. An upward embedding of a digraph on an embedded surface is an embedding of its underlying graph on that surface such that all arcs are represented by monotonic curves that point to a fixed direction. In this paper we study the concept of upward embedding on the torus. We shall introduce a partition of the arcs of a digraph and based on that we shall investigate some characteristics of digraphs that admit upward embedding on the horizontal torus. We also present a polynomial time algorithm for upward embedding testing of single source and single sink digraphs on the horizontal torus. We shall investigate the relation between the vertical and the horizontal tori with respect to the upward embedding.

**1. Introduction.** Graph embeddings and their generalization on surfaces have many applications, such as VLSI layout and graphical representations of a poset. In fact, it is customary and convenient to draw a diagram of an ordered set on the plane, whether or not edges cross. We may also wish to draw them on other surfaces, especially if this avoids the crossing of edges. In this paper we deal with the upward embedding of digraphs, which is defined as follows.

An upward embedding of a digraph $D$ on an embedded surface $S$ is an embedding of its underlying graph on the surface such that all arcs are represented by monotonic curves that point to a fixed direction.

The study of upward embedding on surfaces has been motivated by graph embedding, and topological graph theory, whose literature is extensive (cf., for example, [**8, 15**]). However, there are major differences between graph embedding and upward embedding of digraphs. For instance, all genus one orientable surfaces are topologically homeomorphic to a ring torus, which in turn, from the point of view of graph embedding is equivalent to horizontal and vertical tori. But in this paper we show that in upward embedding the critical points of these

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surfaces are important and a digraph with an underlying graph with genus one may have an upward embedding on the vertical torus, which has two saddle points and a minimum and a maximum, but may fail to have an upward embedding on the horizontal torus, which has infinitely many minimum and maximum points and has no saddle point.

While the question that whether an undirected graph has an embedding on a fixed surface has a polynomial time algorithm [**5, 13**], there exist polynomial time algorithms for upward embedding on the plane for some special cases such as triconnected [**1**], single source [**1, 12**], outerplanar [**16**] and bipartite digraphs [**4**]. However, in general, upward embedding testing of a digraph on the plane and on the sphere is NP-complete [**6, 9**].

Mohar and Rosenstiehl [**14**] presented an algorithm to orient the edges of a toroidal map to obtain an upward embedding of it on the torus by the aid of flow techniques. Their definition of upward embedding is based on the model of the torus as a parallelogram in the plane whose opposite sides are identified. They defined horizontal and vertical circuits as the lines parallel to the sides of the parallelogram due to their definition an arc is monotonic if it crosses horizontal circuits from bottom to top. Here we try to define the concept of upward embedding more naturally, and so we first consider a fixed embedded torus in $\mathbf{R}^3$ and then we say that a digraph has an upward embedding on it if there is an embedding of its underlying graph on the surface and its arcs are represented by monotonic curves that flow toward positive direction of the $z$ axis.

In this paper we study the problem of upward embeddings on the two different embeddings of the torus, namely horizontal and vertical tori. We refer to them by $\mathbf{T}_h$ and $\mathbf{T}_v$, respectively.

After some preliminaries in Section 2 we shall introduce a partition of the arcs of a digraph and then we shall present some characteristics of digraphs that admit upward embedding on $\mathbf{T}_h$ in Section 3. Then in Section 4 we present a polynomial time algorithm for upward embedding testing of single source and single sink digraphs on $\mathbf{T}_h$. In Section 5 we show that the class of digraphs that have upward embedding on $\mathbf{T}_h$ is a proper subset of the class of digraphs that have upward embedding on $\mathbf{T}_v$. In Section 6 we present conclusions and some related open problems that, from our point of view, are worth investigating.

**2. Preliminaries.** Here we introduce some definitions and notations which we use throughout the paper. By a digraph $D$ we mean a pair $D = (V, A)$ of vertices $V$, and arcs $A$. A *source* of $D$ is a vertex with no incoming arc. A *sink* of $D$ is a vertex with no outgoing arc. An *internal* vertex of $D$ has both incoming and outgoing arcs. Let $H_1 = (V_1, A_1)$ and $H_2 = (V_2, A_2)$ be subdigraphs of a digraph $D$. Their union is defined as $H_1 \cup H_2 = (V_1 \cup V_2, A_1 \cup A_2)$. An *st-digraph* is a digraph with exactly one source and exactly one sink with an arc connecting them. If we replace some edges of a graph $G$ with independent paths between their ends so that none of these paths has inner vertex on another path in $G$, the resulting graph is called a *subdivision* of $G$. Let $e = xy$ be an edge of a graph $G = (V, E)$. By $G/e$ we denote the graph obtained from $G$ by *contracting* the edges into a new vertex $v_e$, which becomes adjacent to all the former neighbors of $x$ and $y$. A *minor* of a graph $G$ can be obtained from it by first deleting some vertices and edges, and then contracting some further edges. Notice that any minor of a planar graph is planar.

By a surface we mean a two-dimensional connected compact manifold like sphere and torus.

Now we restate some well-known results for upward embedding on the plane and the sphere which play key roles in the proof of our main results.

**Theorem 2.1** [4]. *A digraph $D$ has an upward embedding on the plane if and only if it is a spanning subdigraph of an st-digraph whose underlying graph is planar.*

**Theorem 2.2** [10]. *A digraph $D$ has an upward embedding on the sphere if and only if it is a subdigraph of an acyclic single source and single sink digraph whose underlying graph is planar.*

We now introduce two different embeddings of the torus, which we use for the problem of upward embedding.

Our definition of horizontal and vertical tori is based on the concept of the surface of revolution [**7**]. We define the *horizontal torus* $\mathbf{T}_h$ as the surface obtained by revolution of the curve $c : (y-2)^2+(z-1)^2 = 1$ round the line $L : y = 0$ as its axis of revolution in the $yz$-plane. In this case we refer as *inner layer* to that part of $\mathbf{T}_h$ resulting from the revolving of the part of $c$ in which $y \leq 2$. The other part of $\mathbf{T}_h$ resulting from the revolving of that part of $c$ in which $y \geq 2$ is called *outer layer.* The curves generating from revolving points $(0, 2, 0)$ and $(0, 2, 2)$ round the axis of revolution are minimum and maximum of the torus and are denoted by $c_{\min}$ and $c_{\max}$, respectively. It is obvious that $c_{\min}$ and $c_{\max}$ are common between the inner layer and the outer layer.

We also define a *vertical torus* $\mathbf{T}_v$ as the surface of revolution that results from revolving the curve $c' : (x-1)^2 + (z-1)^2 = 1$ round the line $L' : z = 3$ in the $xz$-plane. In this case $b = (1, 0, 0)$ is the single minimum point of $\mathrm{T}_v$, $s_b = (1, 0, 2)$ and $s_t = (1, 0, 4)$ are its saddle pints, and $t = (1, 0, 6)$ is its single maximum. We also refer to that part of $\mathrm{T}_v$ with nonnegative $y$ coordinates as the *positive portion* and to the part of it with nonpositive $y$ coordinate as the *negative portion.* Note that $c'$ and $c'' : (x-1)^2+(z-5) = 1$ in the $xz$-plane are common between the positive portion and the negative portion.

A *parallel* on $\mathbf{T}_h$ or on $\mathbf{T}_v$ is the intersection of the surface with a plane orthogonal to its axis of revolution. A parallel may consist of two curves. In this case, for simplicity, we refer to each of them as a parallel.

Consider the orientations of these simple closed curves on $\mathbf{T}_v$ and $\mathbf{T}_h$, whose projections on the $xy$-plane are also simple closed curve. We denote the family of these curves by $\mathcal{C}$. An orientation on $c \in \mathcal{C}$ is called *clockwise* (*counterclockwise*) *orientation* if the inherited orientation of its projection on the $xy$-plane is clockwise (counterclockwise).

**3. Some characteristics of digraphs that have an upward embedding on $\mathbf{T}_h$.** In this section we concentrate on the upward embedding on $\mathbf{T}_h$. We first introduce a partition of the arcs of a digraph into equivalence classes with this special property that whenever an arc of a class is drawn on a layer (inner or outer) of $\mathbf{T}_h$, the other arcs in the same class must be drawn on the same layer; otherwise, the upward property would not be satisfied.

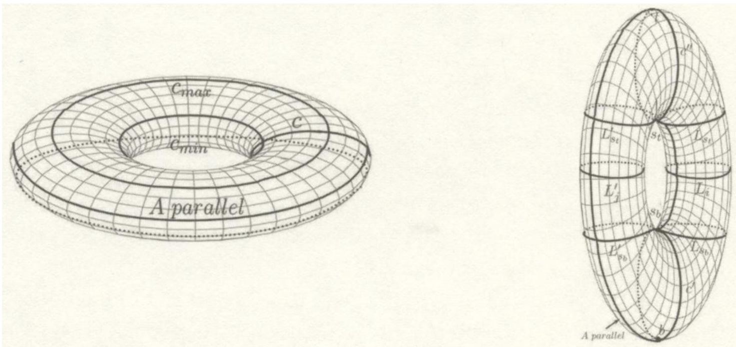


FIGURE 1. $\mathbf{T}_h$ and $\mathbf{T}_v$.

**Definition 3.1.** Given a digraph $D = (V, A)$, we say two arcs $a, a' \in A(D)$ are related by relation $\mathbf{R}$ denoted by $a\mathbf{R}a'$ if they belong to a directed path or there is a sequence $P_1, P_2, \ldots, P_k (k \geq 2)$ of directed paths with the following properties:

(i) $a \in P_1$ and $a' \in P_k$.

(ii) Every $P_i$, $i = 1, \ldots, k-1$, has at least one common vertex with $P_{i+1}$ which is an internal vertex.

Applying Definition 3.1 one can easily verify the following propositions.

**Proposition 3.2.** *Given a digraph $D = (V, A)$, the relation $\mathbf{R}$ is an equivalence relation on the set of arcs $A(D)$.*

**Proposition 3.3.** *Let $a = (u, v)$ be an arc of a digraph $D$ which belongs to class $C$. Then all incoming arcs to $u$ and all outgoing arcs from $v$ also belong to class $C$.*

**Proposition 3.4.** *In a digraph $D$, the elements of every two distinct classes $C$ and $C'$ are connected just via sources and sinks.*

**Theorem 3.5.** *Given a digraph $D$. In every upward embedding of $D$ on $\mathbf{T}_h$, all arcs that belong to the same class must be drawn on the same layer.*

*Proof.* Suppose that $D$ has an upward embedding on $\mathbf{T}_h$, and let $a$ and $a'$ be two arcs of it that belong to a class. We distinguish two cases: (i) $a$ and $a'$ belong to a directed path. In this case they must be in a layer. Otherwise, this path passes through a maximum point of $\mathbf{T}_h$ in an upward embedding that is impossible. (ii) Assume that they do not belong to any directed path and they are related via a sequence of length two of paths, say $(P_1, P_2)$. Suppose that $v$ is the common internal vertex of $P_1$ and $P_2$, in this case $a$ and $a'$ belong to two subpaths $P_1'$ and $P_2'$ (or $P_1''$ and $P_2''$) of $P_1$ and $P_2$, respectively, which are terminated at $v$ (or started from $v$). Let $a''$ be an outgoing arc from (or an incoming arc to) $v$. Since $a$ and $a''$ belong to the directed path $P_1' \cup a''$ (or $a'' \cup P_1''$), they are drawn on a same layer and since $a'$ and $a''$ belong to the directed path $P_2' \cup a''$ (or $a'' \cup P_2''$) they are drawn on the same layer. So $a$ and $a'$ are drawn on the same layer. The case in which the length of the sequence that relates $a$ and $a'$ is more than two can be easily verified by induction. □

**Corollary 3.6.** *If a digraph $D$ has an upward embedding on the horizontal torus, then the underlying graph of every class must be planar.*

It is not difficult to see that the reverse of the last corollary is not true. Moreover, it may be impossible to find an upward embedding of a digraph on the horizontal torus, even if the induced subdigraph of each equivalence class has an upward embedding on the plane. For an example, see Figure 2.

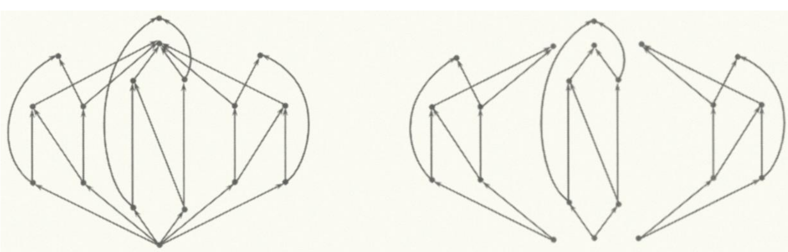

FIGURE 2. A digraph and the induced subdigarphs of its equivalence classes.

**4. Upward embedding testing on the horizontal torus for single source and single sink digraphs.** In this section we present an algorithm for testing whether an acyclic digraph $D$ with a single source $s$, and a single sink $t$, has an upward embedding on the horizontal torus. Let $C_1, C_2, \ldots, C_k$ be the equivalence classes of the arcs of $D$ with respect to the relation defined in Definition 3.1. We denote the induced subdigraphs of $C_i$ by $D_i$, for $i = 1, \ldots k$. Recall that $D_i \cup D_j$ where $1 \leq i < j \leq k$ is a subdigraph of $D$ resulting from identifying the common source and the common sink of $D_i$ and $D_j$. Since $D$ is acyclic, all $D_i$s are acyclic, too. Also, due to Proposition 3.4, we have $D_i \cap D_j = \{s, t\}$. In the rest of this section we freely use the same notation as above.

Since each $D_i$ is acyclic and has only one sink and one source, if it is planar, by Theorem 2.2 we can deduce that $D_i$ has an upward embedding on the sphere. The following lemma is a generalization of this fact.

**Lemma 4.1.** *If the underlying graph of $H = D_{i_1} \cup D_{i_2} \cup \ldots \cup D_{i_m}$, where $1 \leq i_1, i_2, \ldots, i_m \leq k$, is planar, then it has an upward embedding on the sphere.*

As a result we state that if two sets of indices, say $\{i_1, i_2, \ldots, i_p\}$ and $\{j_1, j_2, \ldots, j_q\}$, construct a partition on $\{1, 2, \ldots, k\}$, and the underlying graphs of $H = D_{i_1} \cup D_{i_2} \cup \ldots \cup D_{i_p}$ and $H' = D_{j_1} \cup D_{j_2} \ldots \cup D_{j_q}$ are both planar, then each of them has an upward embedding on the sphere and consequently has an upward embedding on a layer of the horizontal torus. Since $H \cap H' = \{s, t\}$, we can deduce that the digraph $D$ has an upward embedding on the horizontal torus.

The following lemma shows the necessary and sufficient condition for each of $D_i$s to have an upward embedding on the plane.

**Lemma 4.2.** *The underlying graph of $D_i + (s, t)$ is planar if and only if $D_i$ has an upward embedding on the plane.*

Let $D$ have an upward embedding on $\mathbf{T}_h$. Which pair of $D_i$s cannot be drawn simultaneously on a same layer? The following lemma answers this question.

**Lemma 4.3.** *Suppose that $D$ has an upward embedding on $T_h$. If $D_i \cup D_j$, $1 \leq i < j \leq k$, has no upward embedding on the sphere, then $D_i$ and $D_j$ are not in the same layer.*

The following lemma has an important role in the algorithm.

**Lemma 4.4.** *If $D_i$ does not have an upward embedding on the plane, then in any upward embedding of $D$ on $\mathbf{T}_h$, $D_i$ and $D_j$, $1 \leq j \leq k$, $i \neq j$, cannot be drawn on the same layer.*

*Proof.* The underlying graph of $D_i + (s, t)$ is the minor of the underlying graph of $D_i \cup D_j$ for each $1 \leq j \leq k$. On the other hand, by Lemma 4.2, the underlying graph of $D_i + (s, t)$ is not planar, so the underlying graph of the digraph $D_i \cup D_j$ is not planar. That means $D_i \cup D_j$ does not have an upward embedding on the sphere. Therefore, $D_i$ and $D_j$ cannot be in the same layer in any upward embedding on $\mathbf{T}_h$. □

Now we state a theorem that is the basis of the correctness of our algorithm.

**Theorem 4.5.** *Suppose that $D$ is a single source and single sink acyclic digraph, and let $C_1, C_2, \ldots, C_k$ be the equivalence classes of its arcs with respect to the relation $\mathbf{R}$. Also suppose that the digraphs $D_1, D_2, \ldots, D_k$ are the induced subdigraphs on $C_1, C_2, \ldots, C_k$, respectively. The digraph $D$ has an upward embedding on $\mathbf{T}_h$ if and only if the underlying graphs of $D_i$s are planar and either $k \leq 2$ or there is only one induced subdigraph that has no upward embedding on the plane.*

*Proof.* Suppose that the underlying graphs of $D_i$s are planar. If there are only two classes, since their induced subdigraphs are planar and acyclic and they have one source and one sink, by Theorem 2.2 each of them has an upward embedding on the sphere. So we embed upwardly one of them on the inner and the other on the outer layer of $\mathbf{T}_h$ separately, then attach them together via the common source on $c_{\min}$ and the common sink on $c_{\max}$. In the case that the induced subdigraphs of the equivalence classes are upward planar, by using Lemma 4.1 one can deduce that $D_1 \cup D_2 \cup \cdots \cup D_k$ has an upward embedding on the sphere, so it can be embedded on a layer of the horizontal torus. If just one of the induces subdigraphs is not upward planar, then in this case to gain an upward embedding, we embed it on a layer of $\mathbf{T}_h$ and embed the rest of them on the another layer and identify their sources on $c_{\min}$ and their sinks on $c_{\max}$.

To prove the converse, Suppose that $D$ has an upward embedding on $\mathbf{T}_h$. According to Corollary 3.6, the underlying graph of the digraph $D_i$, $i = 1, \ldots, k$, is planar. Consider the subdigraphs of $D$ that are embedded on the inner and the outer layers of $\mathbf{T}_h$. If the arcs of these subdigraphs consist of just two or less equivalence classes, by Theorem 3.5, nothing remains to prove. So suppose that $k$, the number of equivalence classes, is more than two. In that case we show that there is at most one of them that has no upward embedding on the plane. Suppose, for the sake of contradiction, that there are at least two subdigraphs $D_{i_1}$ and $D_{i_2}$, $1 \leq i_1 < i_2 \leq k$, that have no upward embeddings on the plane. By Lemma 4.4 each of them must be embedded, individually, on a layer of $\mathbf{T}_h$ in any upward embedding of $D$. On the other hand, since $k > 2$, there is a subdigraph $D_{i_3}$ distinct from $D_{i_1}$ and $D_{i_2}$. By Lemma 4.4, $D_{i_3}$ cannot be embedded on any layer of $\mathbf{T}_h$. That means $D$ has no upward embedding on $\mathbf{T}_h$, a contradiction. □

We use the last theorem in the following algorithm to decide whether a single source and single sink digraph has an upward embedding on the horizontal torus.

**Algorithm: Upward embedding testing**

**INPUT** An acyclic single source and single sink digraph $D = (V, A)$.

**OUTPUT** YES or NO, based on whether or not $D$ has an upward embedding on the horizontal torus.

1. Find all equivalence classes $C_1, \ldots, C_k$ of $A$ with respect to the relation $\mathbf{R}$ and construct their induced subdigraphs $D_1, \ldots, D_k$.

2. **if** there is any $D_i$, $i = 1, \ldots, k$, whose underlying graph is not planar, **then** answer:=NO

3. **else if** $k \leq 2$ **then** answer:=YES

4. **else if** $r \leq 1$ (2here $r$ is the number of $D_i$s for which the underlying graph of $D_i + (s,t)$ is not planar) **then** answer:=YES

5. **else** answer:=NO

6. **return** answer

The correctness of the algorithm is obvious with respect to the preceding theorem. Suppose that $n$ and $m$ are the numbers of vertices and arcs of the given acyclic digraph, respectively. To find the time complexity of the algorithm, note that the construction of $D_1, D_2, \ldots, D_k$ can be done in $O(m+n)$. Then, if we show the number of vertices in $D_i$ by $n_i$, planarity testing of the underlying graphs of every $D_i$ and $D_i + (s,t)$ can be done in $O(n_i)$ by Hopcroft and Tarpan algorithm [**11**], and since $\sum_{i=1}^{k} n_i = n + 2k - 2$ and $k \leq m$, we have $\sum_{i=1}^{k} n_i \leq n + 2m - 2$. So it takes $O(m+n)$ time and we have:

**Theorem 4.6.** *Let $D$ be an acyclic single source and single sink digraph with $n$ vertices and $m$ arcs. There is an algorithm that tests whether $D$ has an upward embedding on the horizontal torus in $O(m+n)$ time.*

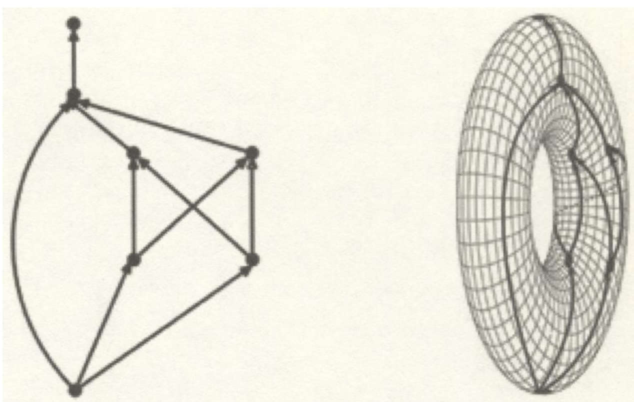

FIGURE 3. A digraph and its upward embedding on the vertical torus.

**5. The relation between horizontal and vertical tori.** Recall that, unlike graph embedding on a surface which completely relies on the topology of the surface, the problem of upward embedding of a digraph on a surface depends not only on the topological properties of the surface but also on the specified embedding of that surface in $\mathbf{R}^3$. In order to verify this fact, in this section we shall show that if a digraph has an upward embedding on the horizontal torus then it has an upward embedding on the vertical torus whereas the converse is not true.

For example, consider the digraph and an upward embedding of it on $\mathbf{T}_v$ that are shown in Figure 3. This digraph does not have an upward embedding on $\mathbf{T}_h$, because all its arcs are in the same class and the underlying graph of the induced subdigraph on this class is not planar. So, due to Corollary 3.6, it does not have an upward embedding on $\mathbf{T}_h$.

Now we prove one of our main results that shows the relation between $\mathbf{T}_h$ and $\mathbf{T}_v$ with respect to the upward embedding problem.

**Theorem 5.1.** *If a digraph $D$ has an upward embedding on the horizontal torus $\mathbf{T}_h$ then it has an upward embedding on the vertical torus $\mathbf{T}_v$.*

*Proof.* Suppose that $D$ has an upward embedding on $\mathbf{T}_h$. Let $D_{\text{in}}$ and $D_{\text{out}}$ be the induced directed subgraphs on the vertices on the inner and the outer layers, respectively. We shall construct an upward embedding of $D_{\text{in}}$ and $D_{\text{out}}$ on the positive and the negative portions of $\mathbf{T}_v$, respectively. Suppose that $D_{\text{in}}$ is not the empty graph. Let $P_1, P_2, \ldots, P_k$ be the parallels (except $c_{\min}$ and $c_{\max}$) that pass through at least one vertex of $D_{\text{in}}$. We suppose that they are indexed such that if $i < j$ then $\text{height}(P_i) < \text{height}(P_j)$, where *height* is a function that assigns to a parallel the common $z$-value of the coordinates of its points. If there is no parallel with the above conditions, we let $k = 0$. We let $P_0 = c_{\min}$ and $P_{k+1} = c_{\max}$. For any arc of $D_{\text{in}}$ whose endpoints are on two nonconsecutive parallels, we consider its intersection points with other parallels as dummy vertices. Then we assign a bottom to top direction to the new edges. We denote the resulting subdivision digraph of $D_{\text{in}}$ by $\widetilde{D}_{\text{in}}$ (If none of the arcs of $D_{\text{in}}$ has been divided, obviously $\widetilde{D}_{\text{in}} = D_{\text{in}}$). We intersect the positive portion of $\mathbf{T}_v$ with the planes $z_i = (2i/k+1)+2$, $i = 0, \dots, k+1$, and denote the resulting curves by $L_i$. Let $v_1^i, \dots, v_{m_i}^i$ be the vertices that appear in the cyclic order when we traverse $P_i$ on $\mathbf{T}_h$ in the clockwise direction. We put the vertices $\hat{v}_1^i, \dots, \hat{v}_{m_i}^i$ as the copies of them on $L_i$ such that the counterclockwise order of $\hat{v}_1^i, \dots, \hat{v}_{m_i}^i$ on $L_i$ is the same as the clockwise order of $v_1^i, \dots, v_{m_i}^i$ on $P_i$. Because of the roles of the curves that pass through the saddle points of $\mathbf{T}_v$ in the proof, to distinguish them we refer to $L_0$ and $L_{k+1}$ by $L_{s_b}$ and $L_{s_t}$, respectively. We put the copies of $v_1^0$ and $v_1^{k+1}$, if they exist, on saddle points $s_b$ and $s_t$, respectively. Moreover, we can assume that the copies of the vertices on $L_{s_b}$ and $L_{s_t}$ are distributed uniformly.

Similarly, we obtain $\widetilde{D}_{\text{out}}$ and then put copies of its vertices on the negative portion of $\mathbf{T}_v$ with slight changes in the above method. These changes correspond to the order of the distribution of the vertices such that the clockwise order of the vertices on each parallel is the same as the clockwise order of their copies on the corresponding curve.

Notice that any arc of $\widetilde{D}_{\text{in}}$ and $\widetilde{D}_{\text{out}}$ connects a pair of vertices on two consecutive parallels. Thus, the copies of its endpoints are on two consecutive $L_i$s in $\mathbf{T}_v$. On the other hand, we know that the order of appearance of vertices on any parallel is the same as the order of appearance of their copies on the corresponding curve in $\mathbf{T}_v$. So, we can connect the copies of the endpoints of any arc of $\widetilde{D}_{\text{in}}$ and $\widetilde{D}_{\text{out}}$ by a monotonic arc without crossing other arcs that have been drawn. After doing that, we have upward embeddings of $\widetilde{D}_{\text{in}}$ and $\widetilde{D}_{\text{out}}$ on the positive portion and the negative portion of $\mathbf{T}_v$, respectively. Now, we replace any path whose internal vertices are dummy vertices by an arc. Finally, we have upward embeddings of $D_{\text{in}}$ and $D_{\text{out}}$ on the positive portion and the negative portion of $\mathbf{T}_v$, respectively.

Now we identify the two copies of the vertices on $c_{\min}$ and $c_{\max}$, if they exist, to gain an upward embedding of $D$ on $\mathbf{T}_v$. Suppose that $t$ is an arbitrary sink of $D$ on $c_{\max}$ and $t'$ and $t''$ are two copies of it on the positive and the negative portion of $\mathbf{T}_v$. Because the copies of the vertices on $c_{\max}$ are uniformly distributed, the points $t'$ and $t''$ are on the same parallel, see Figure 4. Suppose that $t'''$ is the intersection point of this parallel with the curve $c''$, which was defined in Section 2. Assume that $r$ is the number of the incoming arcs to $t'$. If $r = 0$ we just replace $t'$ by $t'''$. Otherwise, let $w_t$ be the resulting curve of the intersection of the positive portion of $\mathbf{T}_v$ with the plane $z = 4 - \varepsilon$.

We choose $\varepsilon > 0$ small enough such that there are two parallels at distance $\varepsilon$ from each other which together with $w_t$ and $c''$ bound a region of $T_v$ that contains $t'$ and intersects only the incoming arcs of $t'$ and intersects no arcs except all incoming arcs of $t'$. We denote this portion by $\mathcal{R}$ (the gray region of Figure 4). Suppose that $x_1, \ldots, x_r$ are the intersection points of the incoming arcs of $t'$ with $w_t$ regarding left to right direction. We denote the incoming arc of $t'$ containing $x_i$ by $a_i$. Next, for every $k = 2, \ldots, r$, we remove $(x_k, t')$ from $a_k$ and continue the arc $a_1$ from $t'$ to $t'''$ on the parallel containing $t'$. Suppose inductively that the arcs $a_1, \ldots, a_l$ have been extended to $t'''$ as incoming arcs of it. The remaining portion of the arc $a_{l+1}$ is continued to $t'''$ by a monotonic curve starting at point $x_{l+1}$ going closely to the extended arc $a_l$, such that the points of the added curve (except $x_{l+1}$ and $t'''$) remain in Int $(\mathcal{R})$. That means we replace $t'$ by $t'''$. If we repeat the above procedure we can replace $t''$, the other copy of $t$, on the negative portion of $\mathbf{T}_v$ by $t'''$. Similarly, we identify the copies of the sources on $c_{\min}$ on $c'$, that was defined in Section 2, properly and obtain an upward embedding of $D$ on $\mathbf{T}_v$.

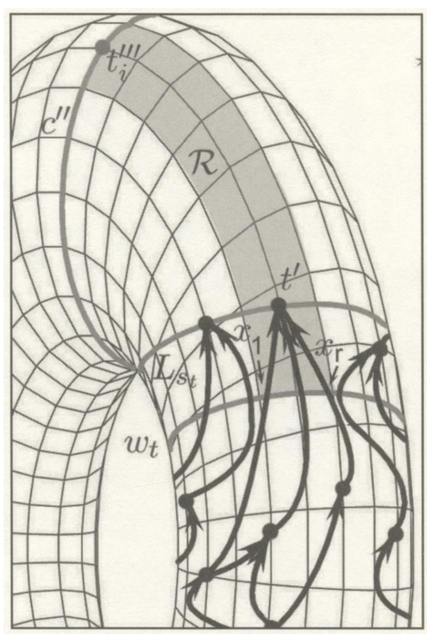


FIGURE 4. The region $\mathcal{R}$.

**6. Conclusion and some open problems.** In this paper, we have introduced horizontal torus $\mathbf{T}_h$ and vertical torus $\mathbf{T}_v$ and have shown that the class of digraphs that have upward embedding on $\mathbf{T}_h$ is a proper subset of the class of digraphs that have upward embedding on $\mathbf{T}_v$. Moreover, we have presented a polynomial time algorithm for upward embedding testing of the single source and single sink digraphs.

The following are some open problems:

1. Is it possible to find a polynomial time algorithm for upward embedding testing of a given digraph on $\mathbf{T}_h$ or on $\mathbf{T}_v$?

2. Characterize all digraphs which admit upward embedding on $\mathbf{T}_h$ or on $\mathbf{T}_v$.

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Department of Mathematics, Shahed University, P.O. Box 18151-159, Tehran, Iran
**Email address: dolati@shahed.ac.ir**

Department of Computer Science, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, P.O. Box 15875-4413, Tehran, Iran
**Email address: hashemi@aut.ac.ir**

Department of Computer Science, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, P.O. Box 15875-4413, Tehran, Iran
**Email address: khosravani.m@gmail.com**