ON THE SQUARE ROOT OF MINKOWSKI SPACE

G.H. DERRICK
School of Physics, University of Sydney, Sydney, NSW, Australia 2006

Received 21 September 1982

A vector space of eight-dimensional complex spinors ψ is constructed which is in a sense a square root of Minkowski space. The usual position, momentum and angular momentum variables of a particle of non-zero rest mass and arbitrary spin are given as expectation values with respect to ψ of appropriate 8×8 matrices.

1. Introduction. Minkowski space M_4 has proved a very useful mathematical tool for modelling the physical world in the absence of gravity. Nevertheless doubts arise whether the space-time continuum concept is adequate on a microscopic scale, where perhaps it needs to be quantised in some sense. One promising approach is the twistor [O(2, 4) spinor] theory of Penrose and co-workers [1] where the basic geometric structure is a complex projective three-space C. A class of points of C corresponds to null straight lines in M_4, while the points in M_4 correspond to particular lines in C. A different approach is taken by Nash [2,3] who replaces M_4 as the basic manifold by what is in a rough sense a square root of M_4, a real space constructed from pairs of real O(3, 3) spinors subject to certain constraints. Nash finds quadratic forms in these sixteen spinor components which model the usual position, momentum and angular momentum coordinates of a free particle of non-zero rest mass and arbitrary spin. Our approach is similar in spirit to that of Nash but differs in that it involves the groups SO(2, 6) and U(4, 4) rather than O(3, 3).

2. Basic postulates and the commutation relations. We assume that the underlying geometric entity is a space of "spinors" ψ, which are column vectors with N complex components ψ^A [‡1]. The choice $N = 8$ will be shown appropriate in section 3. We require the existence of a hermitian "metric" matrix $\beta = \beta^{-1}$ with matrix elements $\beta_{\dot{A}B}$ or $\beta^{A\dot{B}}$, which is used to define the adjoint spinor $\bar{\psi} = \psi^\dagger \beta$, a row vector with components $\bar{\psi}_B = (\psi^A)^* \beta_{\dot{A}B}$. We similarly define the adjoint $\bar{F} = \beta F^\dagger \beta$ for any $N \times N$ matrix F. A particle variable f in Minkowski space is assumed to be the "expectation value" with respect to ψ of some self-adjoint matrix F, defined as follows: $f = \langle F \rangle = \bar{\psi} F \psi / \bar{\psi} E \psi$. The normalisation factor in the denominator involves a self-adjoint matrix E which, as we shall see below, cannot be taken as the unit matrix. The self-adjoint requirement for F and E ensures the reality of $\langle F \rangle$.

Thus to describe a free particle of rest mass m, position x^λ, momentum p^λ and angular momentum $j^{\lambda\mu}$ we seek self-adjoint matrices X^λ, P^λ and $J^{\lambda\mu}$ with $x^\lambda = \langle X^\lambda \rangle$, $p^\lambda = \langle P^\lambda \rangle$ and $j^{\lambda\mu} = \langle J^{\lambda\mu} \rangle$. We now require that P^λ and $J^{\lambda\mu}$ be the infinitesimal generators of Poincaré transformations in the following sense: the spinor transformation

$$\psi' = [I - (i/\hbar)(\tfrac{1}{2}\omega_{\lambda\mu} J^{\lambda\mu} + a_\lambda P^\lambda)]\psi \qquad (1)$$

with infinitesimal Lorentz transformation $\omega_{\lambda\mu}$ and translation a_λ induces a Poincaré transformation of the expectation values:

‡1 Notation. Upper case Latin indices run from 1 to N; Greek indices from 0 to 3; j and k over 0, 1, 2, 3, 5, 6, 7; a, b, c, d, e, f, g, h over 0, 1, 2, 3, 5, 6, 7, 8. A superscript 4 means the combination 5–6, thus $m^{47} = m^{57} - m^{67}$. The metric tensor in M_4 has components $g_{\lambda\mu} = \mathrm{diag}(1, -1, -1, -1)$. The Levi-Civita symbols in 8 and 4 dimensions respectively are $\epsilon^{abcdefgh}$ and $\epsilon^{\kappa\lambda\mu\nu}$ with $\epsilon^{01235678} = \epsilon^{0123} = +1$. The superscripts *, † and T denote the complex conjugate, hermitian conjugate and transpose respectively.

$$(x^\lambda)' = x^\lambda + \omega^\lambda{}_\nu x^\nu - a^\lambda\,, \quad (p^\lambda)' = p^\lambda + \omega^\lambda{}_\nu p^\nu\,,$$
$$(j^{\lambda\mu})' = j^{\lambda\mu} + \omega^\lambda{}_\nu j^{\nu\mu} + \omega^\mu{}_\nu j^{\lambda\nu} - a^\lambda p^\mu + a^\mu p^\lambda\,. \quad (2)$$

Planck's constant $\hbar$ is inserted in (1) for dimensional reasons. For (2) to follow from (1) the following commutation relations must hold:

$$[X^\kappa, P^\lambda] = -\mathrm{i}\hbar g^{\kappa\lambda} E, \quad [P^\kappa, P^\lambda] = 0\,,$$
$$[X^\kappa, J^{\lambda\mu}] = \mathrm{i}\hbar(g^{\kappa\lambda}X^\mu - g^{\kappa\mu}X^\lambda), \quad [E, P^\lambda] = 0\,,$$
$$[E, J^{\lambda\mu}] = 0, \quad [P^\kappa, J^{\lambda\mu}] = \mathrm{i}\hbar(g^{\kappa\lambda}P^\mu - g^{\kappa\mu}P^\lambda)\,,$$
$$[J^{\iota\kappa}, J^{\lambda\mu}] = \mathrm{i}\hbar(g^{\iota\mu}J^{\kappa\lambda} + g^{\kappa\lambda}J^{\iota\mu} - g^{\iota\lambda}J^{\kappa\mu} - g^{\kappa\mu}J^{\iota\lambda})\,. \quad (3)$$

Note that if we had taken E to be the unit matrix, then the first equation of (3) would give a contradiction on taking the trace.

In addition to (3) let us postulate a self-adjoint evolution operator Ξ such that the spinor equation of motion $\mathrm{i}\hbar\,\mathrm{d}\psi/\mathrm{d}s = \Xi\psi$ causes the particle expectation values to satisfy $mc\mathrm{d}x^\lambda/\mathrm{d}s = p^\lambda$, $\mathrm{d}p^\lambda/\mathrm{d}s = 0$, $\mathrm{d}j^{\lambda\mu}/\mathrm{d}s = 0$. This yields the further commutators

$$[X^\kappa, \Xi] = \mathrm{i}\hbar P^\kappa/(mc), \quad [E, \Xi] = 0, \quad [P^\kappa, \Xi] = 0\,,$$
$$[J^{\iota\kappa}, \Xi] = 0\,. \quad (4)$$

3. Realization of the commutation relations. The commutators (3) and (4) do not constitute a Lie algebra because the commutators $[X^\kappa, X^\lambda]$ and $[X^\kappa, E]$ are not prescribed. Nevertheless if we take a representation of the Lie algebra belonging to either SO(3, 4) or SO(2, 5) with infinitesimal generators M^{jk} ($j, k = 0, 1, 2, 3, 5, 6, 7$) and metric $g^{jk} = \mathrm{diag}(1, -1, -1, -1, -1, 1, \pm 1)$ we can realise (3) and (4) with the identification

$$J^{\lambda\mu} = \hbar M^{\lambda\mu}, \quad \lambda, \mu = 0, 1, 2, 3\,,$$
$$P^\lambda = \mu_0(M^{5\lambda} - M^{6\lambda}) \equiv \mu_0 M^{4\lambda}, \quad X^\lambda = (\hbar/\mu_0)M^{\lambda 7}\,,$$
$$E = M^{57} - M^{67} \equiv M^{47}\,, \quad \Xi = \mu_0^2(mc)^{-1}g^{77}E\,, \quad (5)$$

where μ_0 is an arbitrary constant of dimensions momentum. We now construct a spinor representation of SO(3, 4) or SO(2, 5) via the generalised Clifford algebra $\gamma^j\gamma^k + \gamma^k\gamma^j = 2g^{jk}$, with $M^{jk} = (\mathrm{i}/4)(\gamma^j\gamma^k - \gamma^k\gamma^j)$. We shall make the choice SO(2, 5) because this leads to a time like momentum p^λ, whereas SO(3, 4) gives a space like p^λ (section 4). From the general theory of Clifford algebras [4,5] an 8 × 8 representation exists together with real symmetric orthogonal matrices β and C satisfying

$$(\gamma^j)^\dagger = -\beta\gamma^j\beta, \quad (\gamma^j)^\mathrm{T} = -C\gamma^j C, \quad \beta C + C\beta = 0\,.$$

Henceforth we set $N = 8$ and $g^{77} = -1$.

An explicit representation for γ^j may be obtained from three copies of the Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$, (ρ_1, ρ_2, ρ_3), (τ_1, τ_2, τ_3):

$$\gamma^0 = \rho_3\,, \quad (\gamma^1, \gamma^2, \gamma^3) = \mathrm{i}\rho_2\underset{\sim}{\sigma}\,, \quad \gamma^5 = -\mathrm{i}\tau_1\rho_1\,,$$
$$\gamma^6 = \tau_2\rho_1\,, \quad \gamma^7 = \mathrm{i}\tau_3\rho_1\,, \quad \beta = \rho_2\tau_2\,, \quad C = \rho_1\sigma_2\tau_2\,.$$

We can now enlarge the algebra to that of SO(2, 6) by defining further infinitesimal generators $M^{8j} = -M^{j8} = \frac{1}{2}\mathrm{i}\gamma^j$, $g^{88} = -1$. The extended set of generators satisfies

$$[M^{ab}, M^{cd}] = \mathrm{i}[g^{ad}M^{bc} + g^{bc}M^{ad} - g^{ac}M^{bd} - g^{bd}M^{ac}]\,,$$
$$\bar{M}^{ab} \equiv \beta(M^{ab})^\dagger\beta = M^{ab}\,, \quad (6)$$

where $a, b, c, d = 0, 1, 2, 3, 5, 6, 7, 8$.

4. Minkowski space vectors and tensors. The 28 (linearly) independent M^{ab} and 35 independent products

$$M^{abcd} = M^{ab}M^{cd} = -(1/24)\,\epsilon^{abcdefgh}M_{efgh}$$

(a, b, c, d all different), together with the unit matrix, give a total of 64 independent 8 × 8 matrices self-adjoint with respect to β. Since β has eigenvalues 1, 1, 1, 1, −1, −1, −1, −1 these 64 matrices are infinitesimal generators of U(4, 4). Given an arbitrary spinor ψ let us define $m^{ab} = \bar{\psi}M^{ab}\psi$, $m^{abcd} = \bar{\psi}M^{abcd}\psi$, $q = \bar{\psi}\psi$. These 64 real quantities are subject to the identities [6]

$$m^{ab}m_a{}^c = \tfrac{1}{4}[q^2 + |\psi^\mathrm{T}C\psi|^2]g^{bc}\,,$$
$$m^{ab}m^{cd} + m^{bc}m^{ad} + m^{ca}m^{bd} = qm^{abcd}$$
$$+ \mathrm{Re}\{(\psi^\mathrm{T}C\psi)^*\psi^\mathrm{T}CM^{abcd}\psi\} = -\tfrac{1}{8}\,\epsilon^{abcdefgh}m_{ef}m_{gh}\,, \quad (7)$$

cf. the identities (57) of ref. [2]. If we restrict ourselves to the Poincaré subgroup generated by $M^{\lambda\mu}$ and P^λ we can form many tensors and pseudotensors from m^{ab}, m^{abcd} and q [6]. Those which seem to have an obvious physical interpretation are the particle variables already defined, viz. $x^\lambda = (\hbar/\mu_0)m^{\lambda 7}/m^{47}$, $p^\lambda = \mu_0 m^{4\lambda}/m^{47}$, $j^{\lambda\mu} = \hbar m^{\lambda\mu}/m^{47}$, together with $w^\lambda = (\hbar mc/\mu_0)m^{8\lambda}/m^{47}$, $s^{\lambda\mu} = \hbar q m^{\lambda\mu 47}/(m^{47})^2$.

Great simplification occurs if we restrict ψ to lie in the subspace

$$m^{47} = 1\,, \quad m^{48} = 0\,, \quad m^{78} = 0\,, \quad \psi^{\mathrm{T}} C\psi = 0\,. \tag{8}$$

This subspace is invariant under evolution according to $\mathrm{i}\hbar\,\mathrm{d}\psi/\mathrm{d}s = \Xi\psi$. Adopting the constraints (8), (7) yields

$$p^\lambda p_\lambda = \mu_0^2\,, \quad p^\lambda w_\lambda = 0\,,$$
$$\tfrac{1}{2} s^{\lambda\mu} s_{\lambda\mu} = -w^\lambda w_\lambda = \tfrac{1}{4}(\hbar mc/\mu_0)^2 q^2\,,$$
$$s^{\kappa\lambda} = (mc)^{-1}\epsilon^{\kappa\lambda\mu\nu} w_\mu p_\nu\,, \quad j^{\kappa\lambda} = x^\kappa p^\lambda - x^\lambda p^\kappa + s^{\kappa\lambda}\,. \tag{9}$$

Clearly we should take $\mu_0 = mc$ and interpret w^λ as the Pauli–Lubanski spin vector with $s^{\lambda\mu}$ as the spin angular momentum. Note that starting from SO(3, 4) rather than SO(2, 5) in section 3 changes the sign of μ_0^2 in (9), making p^λ space like, which is unphysical.

5. Charge conjugation, time reversal and parity. Analogy with Dirac electron theory suggests that charge conjugation be defined by $\mathcal{C}\psi = C\beta\psi^*$. This operation leaves m^{ab} unchanged, and hence $x^\lambda, p^\lambda, w^\lambda, j^{\lambda\mu}, s^{\lambda\mu}$, but reverses the sign of q and m^{abcd} suggesting that the latter variables represent electromagnetic quantities. Thus q and $m^{\lambda\mu 47}$ might be proportional to the particle's charge and internal magnetic moment tensor respectively. We can also find time reversal and parity operators, which for the particular representation of section 3 take the form $\mathcal{T} = \tau_3 \rho_2 \mathcal{G}$ and $\mathcal{P}\psi = \rho_2\psi$. These operators when applied to a spinor induce the appropriate transformations of the particle variables.

6. Conclusion. We have thus been able to model many of the properties of noninteracting classical particles. The algebra of U(4, 4) is a very rich one, having many subalgebras including those of SU(4) and of the conformal group of Minkowski space time. The identification of particle variables made in section 4 is not the only one possible. A more general one [6], which does not impose the constraint $m^{48} = 0$, may be based on $(mc)x^\lambda - \mathrm{i}w^\lambda = \hbar(m^{\lambda 7} + \mathrm{i}m^{\lambda 8})/(m^{47} + \mathrm{i}m^{48})$, $p^\lambda = (mc)m^{4\lambda}/|m^{47} + \mathrm{i}m^{48}|$.

Unsolved problems within the formalism of this paper are (i) incorporation of interactions and (ii) quantisation.

References

[1] R. Penrose, J. Math. Phys. 8 (1967) 345; Int. J. Theor. Phys. 1 (1968) 61; J. Math. Phys. 10 (1969) 38; in: Quantum gravity, eds. C.J. Isham, R. Penrose and D.W. Sciama (Clarendon, Oxford, 1975) 268;
R. Penrose and M.A.H. MacCallum, Phys. Rep. C6 (1973) 243;
A. Qadir, Phys. Rep. C39 (1978) 133; J. Math. Phys. 21 (1980) 514.
[2] P.L. Nash, J. Math. Phys. 21 (1980) 1024.
[3] P.L. Nash, J. Math. Phys. 21 (1980) 2534; 22 (1981) 983.
[4] E. Cartan, The theory of spinors (Hermann, Paris, 1966).
[5] N. Salingaros, J. Math. Phys. 23 (1982) 1; 22 (1981) 226.
[6] G.H. Derrick, Int. J. Theor. Phys., to be submitted.