

A CLASSIFICATION OF SMOOTH EMBEDDINGS OF 4-MANIFOLDS IN 7-SPACE, II

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ABSTRACT. Let N be a closed, connected, smooth 4-manifold with $H_1(N; \mathbb{Z}) = 0$. Our main result is the following classification of the set $E^7(N)$ of smooth embeddings $N \rightarrow \mathbb{R}^7$ up to smooth isotopy. Haefliger proved that the set $E^7(S^4)$ with the connected sum operation is a group isomorphic to \mathbb{Z}_{12} . This group acts on $E^7(N)$ by embedded connected sum. Boéchat and Haefliger constructed an invariant $BH : E^7(N) \rightarrow H_2(N; \mathbb{Z})$ which is injective on the orbit space of this action; they also described $\text{im}(BH)$. We *determine the orbits* of the action: for $u \in \text{im}(BH)$ the number of elements in $BH^{-1}(u)$ is $\text{GCD}(u/2, 12)$ if u is divisible by 2, or is $\text{GCD}(u, 3)$ if u is not divisible by 2. The proof is based on a new approach using modified surgery as developed by Kreck.

1. INTRODUCTION AND MAIN RESULTS

We work in the smooth category. The main result of this paper is *a complete readily calculable classification of embeddings into \mathbb{R}^7 of closed, smooth 4-manifolds N such that $H_1(N) = 0$* . For such a manifold let $E^7(N)$ denote the set of smooth embeddings $N \rightarrow \mathbb{R}^7$ up to smooth isotopy. We omit \mathbb{Z} -coefficients from the notation of (co)homology groups and denote Poincaré duality by PD .

Classification Theorem 1.1. *Let N be a closed connected 4-manifold such that $H_1(N) = 0$. There is the Boéchat-Haefliger invariant*

$$BH : E^7(N) \rightarrow H_2(N)$$

whose image is

$$\text{im}(BH) = \{u \in H_2(N) \mid u \equiv PDw_2(N) \pmod{2}, u \cap u = \sigma(N)\}.$$

For each $u \in \text{im}(BH)$ there is an injective invariant called the Kreck invariant,

$$\eta_u : BH^{-1}(u) \rightarrow \mathbb{Z}_{\text{GCD}(u, 24)}$$

whose image is the subset of even elements.¹

Corollary 1.2.² (a) *There are exactly twelve isotopy classes of embeddings $N \rightarrow \mathbb{R}^7$ if $N = S^4$ [Hab6] or an integral homology 4-sphere.*

1991 *Mathematics Subject Classification*. Primary: 57R40, 57R52; Secondary: 57R65.

Key words and phrases. Embedding, isotopy, Kreck invariant, Boéchat-Haefliger invariant, surgery.

¹Here $\text{GCD}(u, 24)$ is the maximal integer k such that both $u \in H_2(N)$ and 24 are divisible by k . Thus η_u is surjective if u is not divisible by 2. Note that $u \in \text{im}(BH)$ is divisible by 2 (for some u or, equivalently, for each u) if and only if N is spin.

²For an explicit construction of the embeddings see §3 and Corollary 1.4(c) below.

(b) For each integer u there are exactly $GCD(u, 12)$ isotopy classes of embeddings $f : S^2 \times S^2 \rightarrow \mathbb{R}^7$ with $BH(f) = (2u, 0)$, and the same holds for those with $BH(f) = (0, 2u)$. Other values of \mathbb{Z}^2 are not in the image of BH . (We take the standard basis in $H_2(S^2 \times S^2)$.)

We define the Boéchat-Haefliger invariant and the Kreck invariant in §1 and §2.

The description of $\text{im}(BH)$ in the Classification Theorem 1.1 was already known. So our achievement is to describe the preimages of BH (thus only this part of the proof is presented in this paper). More precisely, in this description our achievement is the transition from the case $N = S^4$ (which was known) to closed connected 4-manifolds N with $H_1(N) = 0$.³ Let us explain what is involved in this transition.

From now on unless otherwise stated, we assume that

N is a closed connected orientable 4-manifold and $f : N \rightarrow \mathbb{R}^7$ is an embedding.

It was known that $E^7(S^4)$ with the embedded connected sum operation is a group isomorphic to \mathbb{Z}_{12} [Ha66]. The group $E^7(S^4)$ acts on the set $E^7(N)$ by connected summation of embeddings $g : S^4 \rightarrow \mathbb{R}^7$ and $f : N \rightarrow \mathbb{R}^7$ whose images are contained in disjoint cubes. It was known that for $H_1(N) = 0$ the orbit space of this action $E^7(S^4) \rightarrow E^7(N)$ maps bijectively under BH (defined in a different way) to $\text{im}(BH)$. This follows by the Section Lemma 3.1 and [BH70, Theorems 1.6 and 2.1] and smoothing theory [BH70, p. 156], cf. [Ha67, Ha, Bo71, Fu94].

Addendum 1.3. Let N be a closed connected 4-manifold such that $H_1(N) = 0$. For each pair of embeddings $f : N \rightarrow \mathbb{R}^7$ and $g : S^4 \rightarrow \mathbb{R}^7$

$$BH(f\#g) = BH(f) \quad \text{and} \quad \eta_{BH(f)}(f\#g) \equiv \eta_{BH(f)}(f) + \eta_0(g) \pmod{GCD(BH(f), 24)}.$$

Here the first equality follows by the definition of the Boéchat-Haefliger invariant, and the second equality is proved in §3.

Definition of the Boéchat-Haefliger invariant. Denote by C_f the closure of the complement in $S^7 \supset \mathbb{R}^7$ to a tubular neighborhood of $f(N)$.

Fix an orientation on N and an orientation on \mathbb{R}^7 . A *homology Seifert surface* A_f for f is the generator of $H_5(C_f, \partial) \cong \mathbb{Z}$ chosen by the fixed orientations of N and \mathbb{R}^7 .⁴

Define $BH(f)$ to be the image of $A_f^2 = A_f \cap A_f$ under the composition $H_3(C_f, \partial) \rightarrow H^4(C_f) \rightarrow H_2(N)$ of the Poincaré-Lefschetz and Alexander duality isomorphisms.

This new definition is equivalent to the original one [BH70] by the Section Lemma 3.1.

The Classification Theorem 1.1 and Addendum 1.3 imply the following *examples of the triviality and the effectiveness of the above action*.

Corollary 1.4. (a) For each embedding $f : \mathbb{C}P^2 \rightarrow \mathbb{R}^7$ and $g : S^4 \rightarrow \mathbb{R}^7$ the embedding $f\#g$ is isotopic to f [Sk05, Triviality Theorem (a)].

(b) Let N be a closed connected 4-manifold such that $H_1(N) = 0$ and the signature $\sigma(N)$ of N is not divisible by the square of an integer $s \geq 2$. Then for each embeddings $f : N \rightarrow \mathbb{R}^7$ and $g : S^4 \rightarrow \mathbb{R}^7$ the embedding $f\#g$ is isotopic to f [Sk05].⁵

(c) If N is a closed connected 4-manifold such that $H_1(N) = 0$ and $f(N) \subset \mathbb{R}^6$ for an embedding $f : N \rightarrow \mathbb{R}^7$, then for each embedding $g : S^4 \rightarrow \mathbb{R}^7$ the embedding $f\#g$ is not isotopic to f . Cf. [Sk05, the Effectiveness Theorem].

³A simpler proof of a particular case of the Classification Theorem 1.1 is given in [Sk05].

⁴More precisely, A_f the image of the fundamental class $[N]$ under the composition $H_4(N) \rightarrow H^2(C_f) \rightarrow H_5(C_f, \partial)$ of the Alexander and Poincaré-Lefschetz duality isomorphisms; this composition is an inverse to the composition $H_5(C_f, \partial) \rightarrow H_4(\partial C_f) \rightarrow H_4(N)$ of the boundary map and the normal bundle map, cf. [Sk08', the Alexander Duality Lemma]; the latter assertion justifies the name 'homology Seifert surface'.

⁵In other words, under the assumption of Corollary 1.4(b) the map BH is injective.

(d) Take an integer u and an embedding $f_u : S^2 \times S^2 \rightarrow \mathbb{R}^7$ constructed below. If $u = 6k \pm 1$, then for each embedding $g : S^4 \rightarrow \mathbb{R}^7$ the embedding $f_u \# g$ is isotopic to f_u .⁶

Sketch of a proof. Part (a) follows from (b).

Part (b) follows by Addendum 1.3 and the Classification Theorem 1.1.

Part (c) follows by the Classification Theorem 1.1 because $BH(f) = 0$ when $f(N) \subset \mathbb{R}^6$, cf. [Sk08', Compression Theorem].

Part (d) follows by the Classification Theorem 1.1 because $BH(f_u) = 2W(f_u) = 2u$ analogously to [Sk08', Boéchat-Haefliger Invariant Theorem], where $W(f_u)$ is defined analogously to [Sk08', definition of the Whitney invariant]. \square

The first construction of f_u . Let $\bar{f}_u : S^2 \rightarrow V_{5,3}$ be a map representing u times the generator of $\pi_2(V_{5,3}) \cong \mathbb{Z}$. This map f_u can be seen as a map from S^2 to the space of linear orthogonal embeddings $\mathbb{R}^3 \rightarrow \mathbb{R}^5$. By the exponential law this gives a map $\hat{f}_u = \text{pr}_1 \times \bar{f}_u : S^2 \times \mathbb{R}^3 \rightarrow S^2 \times \mathbb{R}^5$, where pr_1 is the projection onto the first factor. Let f_u be the composition $S^2 \times \partial D^3 \rightarrow S^2 \times \partial D^5 \rightarrow \mathbb{R}^7$ of the restriction of \hat{f}_u and the standard inclusion.

The second construction of f_u . Take the standard embeddings $2D^5 \times S^2 \subset \mathbb{R}^7$ (where 2 is multiplication by 2) and $\partial D^3 \subset \partial D^5$. Take u copies $(1 + \frac{1}{n})\partial D^5 \times x$ ($n = 1, \dots, u$) of 4-sphere outside $D^5 \times S^2$ ‘parallel’ to $\partial D^5 \times x$. Join these spheres by tubes so that the homotopy class of the resulting embedding $S^4 \rightarrow S^7 - D^5 \times S^2 \simeq S^7 - S^2 \simeq S^4$ will be $u \in \pi_4(S^4) \cong \mathbb{Z}$. Let f be the connected sum of this embedding with the standard embedding $\partial D^3 \times S^2 \subset \mathbb{R}^7$.

It follows from the Classification Theorem 1.1 that if $f_k : N_k \rightarrow \mathbb{R}^7$ are embeddings of closed connected 4-manifolds such that $H_1(N_k) = 0$ and $a_k := BH_{N_k}(f_k)$, then

$$\#BH_{N_1 \# N_2}^{-1}(a_1 \oplus a_2) = \begin{cases} GCD(a_1, a_2, 3) & \text{if either } a_1 \text{ or } a_2 \text{ is not divisible by 2,} \\ GCD(a_1/2, a_2/2, 12) & \text{if both } a_1 \text{ and } a_2 \text{ are divisible by 2.} \end{cases}$$

The General Knotting Problem.

This subsection is not used in the proof of the Classification Theorem 1.1. This paper concerns the classical Knotting Problem: *given an n -manifold N and a number m , describe $E^m(N)$, the set of isotopy classes of embeddings $N \rightarrow \mathbb{R}^m$.*⁷ For recent surveys see [RS99, Sk08]; whenever possible we refer to these surveys not to original papers.

The Knotting Problem is more accessible for $2m \geq 3n + 4$ [RS99, Sk08]. It is much harder for

$$2m < 3n + 4 :$$

if N is a closed manifold that is not a disjoint union of spheres, then until recently no complete readily calculable descriptions of isotopy classes was known, in spite of the existence of interesting approaches of Browder-Wall and Goodwillie-Weiss [Wa70, GW99, CRS04].⁸ For recent results see [Sk06, Sk08']; for *rational* and *piecewise linear* classification see [CRS07, CRS] and [Sk06, Sk07, Sk08, §2, §3 and §5, Sk], respectively.

⁶For a general integer u the number of isotopy classes of embeddings $f_u \# g$ is $GCD(u, 12)$.

⁷The classification of embeddings into S^m is the same because *if the compositions with the inclusion $i : \mathbb{R}^m \rightarrow S^m$ of two embeddings $f_0, f_1 : N \rightarrow \mathbb{R}^m$ of a compact n -manifold N are isotopic, then f_0 and f_1 are isotopic* (in spite of the existence of orientation-preserving diffeomorphisms $S^m \rightarrow S^m$ not isotopic to the identity). Indeed, since f_0 and f_1 are isotopic, by general position $i \circ f_0$ and $i \circ f_1$ are non-ambiently isotopic. Since every non-ambient isotopy extends to an ambient one [Hi76, Theorem 1.3], $i \circ f_0$ and $i \circ f_1$ are isotopic.

⁸We are grateful to M. Weiss for indicating that the approach of [GW99] does give explicit results on higher homotopy groups of the space of embeddings $S^1 \rightarrow \mathbb{R}^n$.

In particular, a complete, readily calculable classification of embeddings of a closed connected 4-manifold N into \mathbb{R}^m was only known only for $m \geq 8$ (Wu, Haefliger, Hirsch and Bausum) or for $N = S^4$ and $m = 7$ (Haefliger):

$$\# E^m(N) = 1 \quad \text{for } m \geq 9.$$

$$E^8(N) = \begin{cases} H_1(N; \mathbb{Z}_2) & N \text{ orientable,} \\ \mathbb{Z} \oplus \mathbb{Z}_2^{s-1} & N \text{ non-orientable and } H_1(N; \mathbb{Z}_2) \cong \mathbb{Z}_2^s. \end{cases}$$

$$E^7(S^4) \cong \mathbb{Z}_{12}.$$

Here $E^m(N)$ is the set of smooth embeddings $N \rightarrow \mathbb{R}^m$ up to smooth isotopy; the equality sign between sets denotes the existence of a bijection; the isomorphism is a group isomorphism for certain geometrically defined group structures. See references in [Sk08, §2, §3]; cf. [Sk06, Sk].

The ‘connected sum’ group structure on $E^m(S^n)$ was defined in [Ha66]. By [Ha61, Ha66, Corollary 6.6, Sk08, §3], $E^m(S^n) = 0$ for $2m \geq 3n + 4$. However, $E^m(S^n) \neq 0$ for many m, n such that $2m < 3n + 4$,⁹ e.g. $E^7(S^4) \cong \mathbb{Z}_{12}$.

In this paragraph assume that N is a closed n -manifold and $m \geq n + 3$. The group $E^m(S^n)$ acts on the set $E^m(N)$ by connected summation of embeddings $g : S^n \rightarrow \mathbb{R}^m$ and $f : N \rightarrow \mathbb{R}^m$ whose images are contained in disjoint cubes.¹⁰ Various authors have studied analogous connected sum action of the group of homotopy n -spheres on the set of n -manifolds topologically homeomorphic to given manifold [Le70]. The quotient of $E^m(N)$ modulo the above action of $E^m(S^n)$ is known in some cases.¹¹ Thus in these cases the knotting problem is reduced to *the determination of the orbits of this action*. This is as non-trivial a problem: until recently *no results were known* on this action for $m \geq n + 3$, $E^m(S^n) \neq 0$ and N not a disjoint union of spheres. For recent results see [Sk08’, Sk06]; for a *rational* description see [CRS07, CRS]; for $m = n + 2$ see [Vi73].

Acknowledgements. These results are based on ideas of and discussion with Matthias Kreck and were presented at the International Pontryagin Conference (Moscow, 2008).

2. AN OVERVIEW OF THE PROOF

This section consists of four subsections. The first discusses the general strategy we use. The second states the preliminary results needed to apply this strategy to calculate $E^7(N)$. The third defines the key invariant, the Kreck invariant. The final section gives the proof of the Classification Theorem 1.1.

A general strategy for the embedding problem.

The proof of the Classification Theorem 1.1 is based on the ideas we explain below which are useful in a wider range of dimensions [Sk08’] and for solving problems other than the action of $E^m(S^n)$ on $E^m(N)$ [FKV87, FKV88].

⁹This differs from the Zeeman-Stallings Unknotting Theorem: *for $m \geq n+3$ any PL or TOP embedding $S^n \rightarrow S^m$ is PL or TOP isotopic to the standard embedding.*

¹⁰Since $m \geq n + 3$, the connected sum is well-defined, i.e. does not depend on the choice of an arc between gS^n and fN . If N is not connected, we assume that a component of N is chosen and we consider embedded connected summation with this chosen component.

¹¹In those cases when this quotient coincides with $E_{PL}^m(N)$ and when the latter set was known [Hu69, §12, Vr77, Sk97, Sk02, Sk07, Sk06].

In this subsection N is a closed connected n -manifold and $f : N \rightarrow \mathbb{R}^m$ is an embedding. Let ν_f be the normal vector bundle of $f(N)$ and let C_f be the closure of the complement in $S^m \supset \mathbb{R}^m$ of a tubular neighbourhood of $f(N)$. We identify the boundary of C_f , ∂C_f , with the total space of the sphere bundle of ν_f . In this paper a bundle isomorphism is always the restriction of a linear bundle isomorphism to the sphere bundle.

The following classical lemma reduces the classification of embeddings to the relative classification of manifolds.

Lemma 2.1. *For a closed connected manifold N embeddings $f_0, f_1 : N \rightarrow \mathbb{R}^m$ are isotopic if and only if there is a bundle isomorphism $\varphi : \partial C_{f_0} \rightarrow \partial C_{f_1}$ which extends to an orientation-preserving diffeomorphism $C_{f_0} \rightarrow C_{f_1} \# \Sigma$ for some homotopy n -sphere Σ .*

Proof. The ‘only if’ part is obvious, so let us prove the ‘if’ part. The bundle isomorphism φ also extends to an orientation-preserving diffeomorphism $S^m - \text{Int } C_{f_0} \rightarrow S^m - \text{Int } C_{f_1}$. Therefore $\Sigma \cong S^m \# \Sigma \cong S^m$. So φ extends to an orientation-preserving diffeomorphism $C_{f_0} \cong C_{f_1}$. Since any orientation-preserving diffeomorphism of \mathbb{R}^m is isotopic to the identity, it follows that f_0 and f_1 are isotopic. \square

Remark. *Lemma 2.1 has been used to obtain embedding theorems in terms of Poincaré embeddings [Wa70]. But ‘these theorems reduce geometric problems to algebraic problems which are even harder to solve’ [Wa70]. One of the main problems is that in general (i.e. not in simpler cases like that of [Sk05, the Effectiveness Theorem]) it is hard to work with the homotopy type of the pair $(C_f, \partial C_f)$ (which is sometimes unknown even when the classification of embeddings is known [Sk06]).*

The main idea of our proof is to apply the modification of surgery [Kr99] which allows to classify m -manifolds using their homotopy type just below dimension $m/2$.¹² Applying modified surgery we prove a diffeomorphism criterion for certain 7-manifolds with boundary: the Almost Diffeomorphism Theorem 2.6 (cf. the Diffeomorphism Theorem 4.7) which is a new, non-trivial version of [KS91, Theorem 3.1] and of [Kr99, Theorem 6] for 7-manifolds M with non-empty boundary and without the assumption that $H_4(M)$ is finite.

Preparatory results.

In order to let the reader understand the main ideas before going into details, we sometimes apply a result before presenting its proof. In such cases the proof is given in §3 (except for the proof of ‘if part’ of the Almost Diffeomorphism Theorem 2.6 which is given in §4).

Remark. For some readers it would be more convenient to replace homology by cohomology using Poincaré-Lefschetz duality (these readers would have to pass back to homology at the decisive step of the proof because in geometric situations like in this paper cup-products are anyway calculated by passing to cap-products). For some readers it would be more convenient to replace for a manifold Q a homology class $z \in H_{n-2}(Q, \partial Q)$ by a homotopy class of a map $Q \rightarrow \mathbb{C}P^\infty$ (then sewing two maps would be a bit more technical) and a spin structure on Q by a map $Q \rightarrow BSpin$.

Recall that unless otherwise stated

N is a closed connected orientable 4-manifold and $f : N \rightarrow \mathbb{R}^7$ is an embedding.

Lemma 2.2. *The normal bundle of f , ν_f , does not depend on f .*

¹²The realization of this idea is close to, but different from the realization of [Sk05]. Here we use $BSpin \times \mathbb{C}P^\infty$ -surgery while in [Sk05] $BO(5) \times \mathbb{C}P^\infty$ -surgery is used.

Proof. The lemma follows because $\nu = \nu_f$ is completely defined by its characteristic classes [DW59]. We have $e(\nu) = 0$, $w_2(\nu) = w_2(N)$ by the Wu formula and $p_1(\nu) = p_1(N)$ by the analogue of the Wu formula for real Pontryagin classes. \square

Take two embeddings $f_0, f_1 : N \rightarrow S^7$. By Lemma 2.2 there is a bundle isomorphism $\varphi : \partial C_{f_0} \rightarrow \partial C_{f_1}$. By Lemma 2.1 embeddings f_0 and f_1 are isotopic if and only if there is an extension $\bar{\varphi} : C_{f_0} \rightarrow C_{f_1} \# \Sigma$. In this situation we may assume:

- that $\bar{\varphi}$ preserves the spin structures s, s' coming from S^7 and
- that $\bar{\varphi}$ sends the generator $A_{f_0} \in H_5(C_{f_0}, \partial)$ to the generator $A_{f_1} \in H_5(C_{f_1}, \partial)$.

The first property is fulfilled because $H_1(N) = 0$. A necessary condition for the second property is $\varphi_* \partial A_{f_0} = \partial A_{f_1}$.

Agreement Lemma 2.3. *Suppose that $H_1(N)$ has no 2-torsion,¹³ $f_0, f_1 : N \rightarrow S^7$ are embeddings and $\varphi : \partial C_{f_0} \rightarrow \partial C_{f_1}$ is a bundle isomorphism. We have $\varphi_* \partial A_{f_0} = \partial A_{f_1}$ if and only if $BH(f_0) = BH(f_1)$.*

Now suppose that $BH(f_0) = BH(f_1)$. There is a *spin bordism* between (C_{f_0}, A_{f_0}) and (C_{f_1}, A_{f_1}) relative to the boundaries identified by φ (because by the Remark the obstruction to the existence of such a cobordism assumes values in $\Omega_7^{Spin}(\mathbb{C}P^\infty) = 0$ [KS91, Lemma 6.1]). It remains to replace the bordism by an h -cobordism. This problem is solved by modified surgery [Kr99]. The heart of our argument is to analyse the dependence of the surgery obstructions which arise from various choices of the bordism and the bundle isomorphism φ . We call the resulting obstruction the Kreck invariant.

The definition of the Kreck invariant.

For any manifold Q we abbreviate $H_i(Q, \partial Q)$ to $H_i(Q, \partial)$ and denote Poincaré-Lefschetz duality by

$$PD : H^i(Q) \rightarrow H_{q-i}(Q, \partial) \quad \text{and} \quad PD : H_i(Q) \rightarrow H^{q-i}(Q, \partial).$$

Recall that for an abelian group G the divisibility $d(0)$ of zero is zero and the divisibility

$$d(x) \quad \text{of} \quad x \in G - \{0\} \quad \text{is} \quad \max\{k \in \mathbb{Z} \mid \text{there is } x_1 \in G : x = kx_1\}.$$

A sentence involving k holds for each $k = 0, 1$.

A set $X = (C_0, C_1, A_0, A_1, \varphi)$ consisting of compact connected spin 7-manifolds C_0 and C_1 , generators $A_k \in H_5(C_k, \partial) \cong \mathbb{Z}$ and a spin diffeomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ is called *admissible* if

$$\partial A_1 = \varphi_* \partial A_0, \quad H_3(\partial C_0) = 0, \quad p_1(C_0) = p_1(C_1) = 0 \quad \text{and} \quad d(A_0^2) = d(A_1^2).$$

According to our strategy we first define the obstruction η_X to extending φ to a diffeomorphism carrying A_0 to A_1 .¹⁴

Denote $M_\varphi := C_0 \cup_\varphi (-C_1)$. For $y \in H_5(M_\varphi)$ and an orientable n -submanifold $C \subset M_\varphi$ we denote¹⁵

$$y \cap C := PD[(PDy)|_C] \in H_{n-2}(C, \partial).$$

¹³We conjecture that this assumption is superfluous when φ is a *spin* bundle isomorphism.

¹⁴A more general situation makes things simpler, but a reader who do not wish to keep in mind the properties of C_k, A_k, φ may assume that $C_k = C_{f_k}$, $A_k = A_{f_k}$ and φ is any spin bundle isomorphism.

¹⁵If y is represented by a closed oriented 6-submanifold $Y \subset M_\varphi$ transverse to C , then $y \cap C$ is represented by $Y \cap C$. If $C = \partial C_0$, then $y \cap C = y \cap [C]$. If $C = C_0$, then $y \cap C = y \cap [C']$, where C' is the image of $[M_\varphi]$ under the excision isomorphism $H_n(M_\varphi, C_1) \rightarrow H_n(C_0, \partial)$.

Null-bordism Lemma 2.4. *Each admissible set has a null-bordism, i.e. a compact connected spin 8-manifold W and $z \in H_6(W, \partial)$ such that $\partial W \underset{spin}{=} M_\varphi$ and $(\partial z) \cap C_k = A_k$.*

Proof. Look at the segment of (the Poincaré-Lefschetz dual to) the Mayer-Vietoris sequence:

$$H_5(\partial C_0) \rightarrow H_5(M_\varphi) \xrightarrow{\Psi_1 \oplus \Psi_2} H_5(C_0, \partial) \oplus H_5(C_1, \partial) \xrightarrow{\partial_1 - \partial_2} H_4(\partial C_0).$$

Here the unmarked arrow is induced by inclusion and $\Psi_k x := x \cap C_k$.

Since $\partial A_1 = \varphi_* \partial A_0$, there is $A \in H_5(M_\varphi)$ such that $A \cap C_k = A_k$.¹⁶

Since $\Omega_7^{Spin}(\mathbb{C}P^\infty) = 0$ [KS91, Lemma 6.1], there are a compact spin 8-manifold W and a class $z \in H_6(W, \partial)$ such that $\partial W \underset{spin}{=} M_\varphi$ and $\partial z = A$. \square

Let W be a compact spin 8-manifold. Consider the following fragment of the exact sequence of pair:

$$H_4(\partial W) \xrightarrow{i_W} H_4(W) \xrightarrow{j_W} H_4(W, \partial) \xrightarrow{\partial_W} H_3(\partial W)$$

(with any coefficients). Let $p_1(W)$ be the first Pontryagin class of W . Denote

$$p_W := PDp_1(W).$$

It is known that p_W is divisible by 2, see Lemma 2.9.

Denote by ρ_m the reduction modulo m .

Definition: the Kreck obstruction $\eta_{W,z}$. Take a null-bordism (W, z) of an admissible set X . Denote $d := d(\partial_W z^2)$. Then there is $\overline{z^2} \in H_4(W; \mathbb{Z}_d)$ such that $j_W \overline{z^2} = \rho_d z^2$. Define

$$\eta_{W,z} := \overline{z^2} \cap \rho_d(z^2 - \frac{1}{2}p_W) \in \mathbb{Z}_d.$$

The proof of the independence of $\eta_{W,z}$ on the choice of $\overline{z^2}$. We have $\overline{z^2} - \overline{z^2}' = i_W a$ for some $a \in H_4(\partial W; \mathbb{Z}_d)$. By Lemma 2.5 below there is $\overline{\frac{1}{2}p_W} \in H_4(W)$ such that $j_W \overline{\frac{1}{2}p_W} = \frac{1}{2}p_W$. Then

$$\eta_{W,z}(\overline{z^2}) - \eta_{W,z}(\overline{z^2}') = i_W a \cap (z^2 - \frac{1}{2}p_W) = i_W a \cap (\overline{z^2} - \rho_d \overline{\frac{1}{2}p_W}) = 0. \quad \square$$

Lemma 2.5. *If (W, z) is a null-bordism of an admissible set X , then $\partial_W p_W = 0$ and $d(A_0^2) = d(\partial_W z^2)$.*

Proof. Consider the segment of the Mayer-Vietoris sequence

$$H_3(\partial C_0) \rightarrow H_3(\partial W) \rightarrow H_3(C_0, \partial) \oplus H_3(C_1, \partial) \rightarrow H_2(\partial C_0).$$

Since $(\partial_W p_W) \cap C_k = PDp_1(C_k) = 0$ and $H_3(\partial C_0) = 0$, we have $\partial_W p_W = 0$.

We have $(\partial_W z^2) \cap C_k = (\partial(z \cap C_k))^2 = A_k^2$. Hence $d(A_0^2)$ is divisible by $d(\partial_W z^2)$, and in the above segment of the Mayer-Vietoris sequence $\partial_W z^2$ is mapped to $A_0^2 \oplus A_1^2$. If A_0^2 is divisible by an integer d , then A_1^2 is. Since $H_3(\partial C_0) = 0$, we obtain that $\partial_W z^2$ is divisible by $d(A_0^2)$. This proves $d(A_0^2) = d(\partial_W z^2)$. \square

¹⁶If $C_0 = C_{f_0}$ and $C_1 = C_{f_1}$, then $H_5(\partial C_0) = 0$, hence such a class A is unique. This class is called a *joint homology Seifert surface* for f_0 and f_1 .

For an admissible set X by Lemma 2.5 we can define

$$\eta_X := \rho_{GCD(A_0^2, 24)} \eta_{W, z} \in \mathbb{Z}_{GCD(A_0^2, 24)}.$$

The proof of the independence of η_X on the choice of (W, z) . The independence on the choice of (W, z) within a cobordism class relative to the boundary is standard. Change of the cobordism class (relative to $\partial W = M_\varphi$) of W changes $\eta_{W, z}$ by adding $v^2(v^2 - \frac{1}{2}p_1(V))$, where V is a closed spin 8-manifold and $v \in H_6(V)$. This is divisible by 24 by the smooth spin case of [KS91, Proposition 2.5]. \square

Definition: the Kreck invariant η_u . Assume that $H_1(N) = 0$. Take two embeddings $f_0, f_1 : N \rightarrow S^7$ such that $BH(f_0) = BH(f_1) = u$. By Lemma 2.2 there is a bundle isomorphism $\varphi : \partial C_{f_0} \rightarrow \partial C_{f_1}$. The difference between spin structures on ∂C_{f_0} is in $H_5(\partial C_{f_0}; \mathbb{Z}_2) = H^1(\partial C_{f_0}; \mathbb{Z}_2) = 0$, so we may assume that φ is spin. Then by the Alexander duality and the Agreement Lemma 2.3 the set $X = (C_{f_0}, C_{f_1}, A_{f_0}, A_{f_1}, \varphi)$ is admissible. Define

$$\eta_u(f_0, f_1) := \eta_X \in \mathbb{Z}_{GCD(A_0^2, 24)}.$$

This is well-defined by the (non-trivial) Framing Theorem 2.7(η).

For $u \in H_2(N)$ fix an embedding $f_0 : N \rightarrow \mathbb{R}^7$ such that $BH(f_0) = u$ and define $\eta_u(f) := \eta_u(f, f_0)$. (We write $\eta_u(f)$ not $\eta_{f_0}(f)$ for simplicity.)¹⁷

The outline of the proof.

Definition of the framing invariant η'_X . Take an admissible set $X = (C_0, C_1, A_0, A_1, \varphi)$ such that A_0^2 and A_1^2 are divisible by 2. Define $\overline{z^2} \in H_4(W; \mathbb{Z}_2)$ analogously to $\overline{z^2} \in H_4(W; \mathbb{Z}_d)$ in the definition of $\eta_{W, z}$. Define¹⁸

$$\eta'_X := \overline{z^2} \cap \rho_2 z^2 \in \mathbb{Z}_2.$$

Almost Diffeomorphism Theorem 2.6. *Let $X = (C_0, C_1, A_0, A_1, \varphi)$ be an admissible set such that $\pi_1(C_k) = H_3(C_k) = H_4(C_k, \partial) = 0$ and $H_2(\partial C_0)$ is free. For some homotopy 7-sphere Σ there is a diffeomorphism $C_0 \rightarrow C_1 \# \Sigma$ extending φ if and only if*

$$\eta_X = 0 \quad \text{and, for } A_0^2 \text{ divisible by 2, } \eta'_X = 0.$$

The ‘only if’ part is simple (take $W = C_0 \times I \cup_{\overline{\varphi}} (C_1 \# \Sigma)$, where $\overline{\varphi} : C_0 \times 1 = C_0 \rightarrow C_1 \# \Sigma$ is given extension) and is not used in the proof of the Classification Theorem 1.1.

Framing Theorem 2.7. *Let $X = (C_0, C_1, A_0, A_1, \varphi)$ be an admissible set such that ∂C_0 is an S^2 -bundle over a closed 4-manifold N with $H_1(N) = 0$. Then*

(η) η_X is independent of the choice of φ (preserving C_k, A_k and admissibility).¹⁹

(φ) If A_0^2 is divisible by 2, then we can change φ (preserving C_k, A_k and admissibility) so as to obtain $\eta'_X = 0$.

Lemma 2.8. *If $f, f_1, f_2 : N \rightarrow \mathbb{R}^7$ are embeddings with the same value of the Boéchat-Haefliger invariant, u , then $\eta_u(f, f_1) + \eta_u(f_1, f_2) = \eta_u(f, f_2)$.*

Proof of the injectivity of η_u . By Lemma 2.8 it suffices to prove that

¹⁷In general η_u depends on the choice of an orientation on N , but $E^7(N)$ by definition does not.

¹⁸This is independent on the choice of W, z analogously to η_X using the smooth spin case of [KS91, Proposition 2.5] (because $12S_3 - 48S_2 = 6z^4$ is divisible by 12, so z^4 is divisible by 2 for closed manifolds).

¹⁹The change of φ is only possible together with certain changes of W, z .

if $BH(f) = BH(f')$ and $\eta_{BH(f)}(f, f') = 0$, then f is isotopic to f' .

In order to prove this assertion construct an admissible set X as in the definition of the Kreck invariant $\eta_u(f, f')$. Since $\eta_u(f, f') = 0$, we have $\eta_X = 0$.

If A_f^2 is divisible by 2, by the Framing Theorem 2.7(φ) we can change φ so as to obtain $\eta'_X = 0$. By the Framing Theorem 2.7(η) η_X will be preserved.

Therefore by the Almost Diffeomorphism Theorem 2.6 φ extends to a diffeomorphism $C_f \rightarrow C_{f'} \# \Sigma$ for a certain homotopy 7-sphere Σ . Hence f is isotopic to f' by Lemma 2.1. \square

The description of $\text{im } \eta_u$ holds by the second equality of the Addendum 1.3 and the following two partially known results proved in §3.

Lemma 2.9. *Let W be a compact spin 8-manifold. Then p_W is divisible by 2 and $(p_W/2) \cap x - x \cap x$ is divisible by 2 for each $x \in H_4(W)$.*

Realization Theorem 2.10. *There is an embedding $g_1 : S^4 \rightarrow \mathbb{R}^7$ such that $\eta_0(g_1) = 2$.*

This holds by the injectivity of η_0 (proved above) because there exist 12 pairwise non-isotopic embeddings $S^4 \rightarrow S^7$ [Ha66]. We present an alternative direct proof in §3.

Sections §3 and §4 depend on §2 but are independent of each other.

3. THE DETAILS OF THE PROOF

Proof of the Agreement Lemma 2.3.

Let $N_0 := \text{Cl}(N - B^4)$, where B^4 is a closed 4-ball in N . Denote $\nu = \nu_f$.

For a section $\xi : N_0 \rightarrow \partial C_f$ we denote by ξ^\perp the oriented 2-bundle that is the orthogonal complement to ξ in $\nu|_{N_0}$. Denote by $|\cdot, \cdot|$ the distance in N such that B^4 is a ball of radius 2. By ‘a section $\xi : N_0 \rightarrow \partial C_f$ ’ we would mean ‘a section over N_0 of the normal bundle $\partial C_f \rightarrow N$ ’. For a section $\xi : N_0 \rightarrow \partial C_f$ define a map

$$\bar{\xi} : N \rightarrow S^7 - fN_0 \quad \text{by} \quad \bar{\xi}(x) = \begin{cases} \xi(x) & x \in N_0 \\ f(x) & |x, N_0| \geq 1 \\ |x, N_0|f(x) + (1 - |x, N_0|)\xi(x) & |x, N_0| \leq 1. \end{cases}$$

A section $\zeta : N_0 \rightarrow \partial C_f$ is called *unlinked* if $\bar{\zeta}_*[N] = 0 \in H_4(S^7 - fN_0)$ [BH70].

For a map $\xi : P \rightarrow Q$ between a p - and a q -manifold denote the ‘preimage’ homomorphism by

$$\xi^! := PD \circ \xi^* \circ PD : H_i(Q, \partial) \rightarrow H_{p-q+i}(P, \partial).$$

Section Lemma 3.1. *If ζ is an unlinked section, then $BH(f) = PDe(\zeta^\perp) = \zeta^! \partial A_f$.*

Proof. Since ζ is unlinked, there is a 5-chain a in $S^7 - fN_0$ such that ∂a is represented by $\bar{\zeta}N$. We may assume that the support of a is in general position to ∂C_f , so 5-chain $a \cap C_f$ and 4-chain $a \cap \partial C_f$ are defined.

Take 5-chain b in S^7 represented by the union of segments $f(x)\bar{\zeta}(x)$, $x \in N$. By Alexander duality $A_f = [(a + b) \cap C_f] = [a \cap C_f]$. By pushing out of $\nu^{-1}N_0$ we may assume that the support of a intersects $\nu^{-1}N_0$ by ζN_0 . Hence

$$A_f \cap \nu^{-1}N_0 = [a \cap C_f \cap \nu^{-1}N_0] = [\zeta N_0] \in H_4(\nu^{-1}N_0, \nu^{-1}\partial N_0).$$

Identify the groups $H_2(N)$ and $H_2(N_0)$ by the restriction isomorphism. Then

$$\zeta^! \partial A_f = \zeta^!(A_f \cap \nu^{-1}N_0) = \zeta^![\zeta N_0] = PDe(\zeta^\perp) \in H_2(N).$$

Here the last equality holds because the normal bundle of $\zeta : N_0 \rightarrow \partial C_f$ is isomorphic to ζ^\perp .

Also

$$BH(f) \stackrel{(1)}{=} \nu_* \partial(A_f \cap A_f) = \nu_*(\partial A_f \cap \partial A_f) = \nu_{0,*}(A_f \cap \nu^{-1} N_0)^2 = \nu_{0,*}[\zeta N_0]^2 \stackrel{(5)}{=} PDe(\zeta^\perp).$$

Here

- (1) follows by Alexander duality, cf. [Sk08', the Alexander Duality Lemma];
- $\nu_0 := \nu|_{\nu^{-1} N_0}$ and the square means intersection square in $H_4(\nu^{-1} N_0, \nu^{-1} \partial N_0)$;
- (5) holds because the normal bundle of $\zeta : N_0 \rightarrow \partial C_f$ is isomorphic to ζ^\perp . \square

Proof of the Agreement Lemma 2.3. Denote $f = f_0$. Consider the following fragment of the Gysin sequence for the bundle ν having trivial Euler class:

$$0 \rightarrow H_2(N) \xrightarrow{\nu^!} H_4(\partial C_f) \xrightarrow{\nu_*} H_4(N) \rightarrow 0.$$

We see that for each section $\zeta : N_0 \rightarrow \partial C_f$ the map

$$\nu_* \oplus \zeta^! : H_4(\partial C_f) \rightarrow H_4(N) \oplus H_2(N)$$

is an isomorphism. By definition of A_f we have $\nu_* \partial A_f = [N] = \nu_{f_1*} \partial A_{f_1}$.

There exist unlinked sections ζ and ζ_1 for f and f_1 [HH63, 4.3, BH70, Proposition 1.3, Sk08', the Unlinked Section Lemma (a)]. We have $e((\varphi\zeta)^\perp) = e(\zeta^\perp) = BH(f) = BH(f_1)$, where the second equality holds by (the first equality of) the Section Lemma 3.1.

For sections

$$\xi, \eta : N_0 \rightarrow \partial C_{f_1} \quad \text{we have} \quad PDe(\xi^\perp) - PDe(\eta^\perp) = \pm 2d(\xi, \eta),$$

where $d(\xi, \eta) \in H_2(N)$ is the difference element [BH70, Lemme 1.7, Bo71, Lemme 3.2.b].

Since $H_2(N)$ has no 2-torsion, the previous two sentences together with (the first equality of) the Section Lemma 3.1 imply that the section $\varphi\zeta$ is unlinked for f_1 . Hence by (the second equality of) the Section Lemma 3.1

$$(\varphi\zeta)^\perp \partial A_{f_1} = PDe((\varphi\zeta)^\perp) = PDe(\zeta^\perp) = \zeta^\perp \partial A_f, \quad \text{so} \quad \varphi_* \partial A_f = \partial A_{f_1}. \quad \square$$

Proof of the Framing Theorem 2.7.

Lemma 3.2. *Define $i : S^1 = SU_1 \rightarrow SU_3$ by $i(z) = \text{diag}(z, \bar{z}, 1)$. Then the homogeneous space $SU_3/i(S^1)$ is the total space of the non-trivial S^2 -bundle over S^5 (i.e. the bundle corresponding to the non-trivial element of $\pi_4(SO_3) \cong \mathbb{Z}_2$).*

Proof. Since $i(S^1) \subset SU_2$, the standard bundle $SU_2 \rightarrow SU_3 \rightarrow S^5$ gives a bundle

$$S^2 \cong SU_2/i(S^1) \rightarrow SU_3/i(S^1) \rightarrow S^5. \quad (*)$$

Here the diffeomorphism is given by a free action of SU_2 on $\mathbb{C}P^1 = S^2$ whose stabilizer subgroup is $i(S^1)$.

(In order to define such an action, identify SU_2 with the group of unit length quaternions. Define the Hopf map

$$h : SU_2 \rightarrow \mathbb{C}P^1 \quad \text{by} \quad h(z + jw) := (z : w) \quad \text{for} \quad z, w \in \mathbb{C} \quad \text{and} \quad |z|^2 + |w|^2 = 1.$$

The required action is well-defined by $uh(v) := h(uv)$. The action of SU_2 on $\mathbb{C}^2 = \mathbb{H}$ is given by $(z + jw)(p + jq) = zp + \bar{w}q + j(wp + \bar{z}q)$. Hence $z + jw$ corresponds to matrix $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$. Thus the stabilizer subgroup is $\{z + j0 \mid z \in \mathbb{C}\} = i(S^1)$.

Since $\pi_4(SU_3) = 0$ (by $\pi_4(SU_3) \cong \pi_4(SU)$ and the Bott periodicity), we have $\pi_4(SU_3/i(S^1)) = 0 \neq \mathbb{Z}_2 \cong \pi_4(S^2 \times S^5)$. Hence $SU_3/i(S^1) \not\cong S^2 \times S^5$. Therefore the bundle (*) is non-trivial. \square^{20}

Proof of the Framing Theorem 2.7. Take a closed 4-ball $B^4 \subset N$. Since $H_1(N) = 0$, the bundle isomorphism φ is iniquely defined over $Cl(N - B^4)$ by the condition that φ is spin. If we change φ on B^4 , then analogously to [Sk08', proof of the Independence Lemma] and by Lemma 3.2 ($M_\varphi, A_0 \cup_\varphi A_1$) would change by interior connected sum with $(SU_3/i(S^1), A)$, where $A \in H_5(SU_3/i(S^1)) \cong \mathbb{Z}$. It suffices to consider the case when A is a generator.

We have $SU_3/i(S^1)$ is $N_{1,-1}$ defined in [KS91, §1]; the assumption $k+l \neq 0$ is not used for the definition (but required for the positive curvature property). By [KS91, Proposition 2.2] $(SU_3/i(S^1), A) \stackrel{spin}{=} \partial(W, z)$ for some spin 8-manifold W and $z \in H_6(W, \partial)$. By Lemma 3.2 $H_3(\partial W) = H_4(\partial W) = 0$. Hence we may identify z^2 and p_W with elements of $H_4(W)$ (which elements are denoted by the same letters). In [KS91, proof of Lemma 4.4] the assumption $k+l \neq 0$ was not used.²¹ So by [KS91, (2.4), Lemma 4.4 and bottom of p. 475] with

$$k = m = 1, \quad l = -1, \quad n = 0 \quad \text{we have} \quad z^4 = -1 \quad \text{and} \quad N = P = S = 1,$$

$$\text{so} \quad -z^2 p_W + 2z^4 = 48s_2(N_{1,-1}) = 2(-P + NS)/N = 0.$$

Thus change of φ together with certain corresponding change of W, z preserves η_X and, for A_0^2 divisible by 2, changes η'_X by 1. \square

Proof of Lemma 2.8 and the second equality of the Addendum 1.3.

Proof of Lemma 2.8. Assume that (W_k, z_k) is a null-bordism of the admissible set $(C_f, C_{f_k}, A_f, A_{f_k}, \varphi_k)$.

Take $\varphi := \varphi_2 \varphi_1^{-1}$. Then $X = (C_{f_1}, C_{f_2}, A_{f_1}, A_{f_2}, \varphi)$ is admissible.

Take $W := W_2 \cup_{C_f} (-W_1)$. From the Mayer-Vietoris sequence

$$H_6(C_f) \rightarrow H_6(W, \partial) \xrightarrow{\Psi} H_6(W_1, \partial) \oplus H_6(W_2, \partial) \rightarrow H_5(C_f)$$

we see that Ψ is an isomorphism. Take $z := \Psi^{-1}(z_1 \oplus z_2)$. Then (W, z) is a null-bordism of X .

Consider the maps

$$(\cdot \cap W_1) \oplus (\cdot \cap W_2) : H_4(W, \partial) \rightarrow H_4(W_1, \partial) \oplus H_4(W_2, \partial) \quad \text{and}$$

$$i_1 \oplus i_2 : H_4(W_1; \mathbb{Z}_d) \oplus H_4(W_2; \mathbb{Z}_d) \rightarrow H_4(W; \mathbb{Z}_d).$$

²⁰ An alternative proof of the non-triviality of the bundle (*). If (*) is trivial, then there is a bundle $S^1 \rightarrow SU_3 \rightarrow S^2 \times S^5$ whose first Chern class is a generator of $H^2(S^2 \times S^5) \cong \mathbb{Z}$. Then $SU_3 \cong S^3 \times S^5$ which is a contradiction because $\pi_4(SU_3) = 0 \neq \mathbb{Z}_2 \cong \pi_4(S^3 \times S^5)$.

²¹ There is a typographical error in the expression for s_3 which should read $s_3(N_{k,l}) = (-4P + NS)/6N$ and in the expression for P where $-6m^2n^2$ should read $-6lm^2n^2$; we do not use these corrections.

Clearly, $p_{W_i} = p_W \cap W_i$ and $z_i^2 = z^2 \cap W_i$. Take $\overline{z^2} := i_1 \overline{z_1^2} \oplus i_2 \overline{z_2^2}$. Since

$$(i_1 x_1 \oplus i_2 x_2) \cap y = x_1 \cap (y \cap W_1) + x_2 \cap (y \cap W_2) \quad \text{we have} \quad \eta_{W,z} = \eta_{W_1,z_1} - \eta_{W_2,z_2}.$$

Hence $\eta_u(f_1, f_2) = \eta_u(f, f_2) - \eta_u(f, f_1)$. \square

Proof of the second equality of the Addendum 1.3. It suffices to prove that $\eta_u(f \# g, f_0 \# g_0) = \eta_u(f, f_0) + \eta_u(g, g_0)$, where $u = BH(f_0)$ and $g_0 : S^4 \rightarrow \mathbb{R}^7$ is the standard embedding.

Assume that (W_f, z_f) is a null-bordism of an admissible set $(C_f, C_{f_0}, A_f, A_{f_0}, \varphi_f)$ and the same for f, f_0 replaced by g, g_0 .

We may assume that φ_f is the identity outside $B^4 \subset N$ and that $\nu_f = \nu_{f \# g}$ outside $B^4 \subset N$. Then take any spin bundle isomorphism $\varphi : \partial C_{f \# g} \rightarrow \partial C_{f_0 \# g_0}$ that is the identity outside B^4 .

Identify $B^4 \times S^2$ and $\nu_f^{-1} B^4 \subset \partial C_f$ by some bundle isomorphism. The same for f replaced by f_0, g, g_0 . We have

$$C_{f \# g} = C_f \cup_{B^4 \times S^2} C_g \quad \text{and} \quad C_{f_0 \# g_0} = C_{f_0} \cup_{B^4 \times S^2} C_{g_0}.$$

Then $(C_{f \# g}, C_{f_0 \# g_0}, A_f, A_{f_0}, \varphi)$ is an admissible set.

By $B^5 = B_+^5 \cup_{B^4} B_-^5$ we denote the standard decomposition. Take an embedding $B^5 \times S^2 \rightarrow \partial W_f = C_f \cup_{\varphi_f} C_{f_0}$ whose image intersects

$$C_f, \quad C_{f_0} \quad \text{and} \quad \partial C_f \stackrel{\varphi_f}{=} \partial C_{f_0} \quad \text{by} \quad B_+^5 \times S^2, \quad B_-^5 \times S^2 \quad \text{and} \quad B^4 \times S^2,$$

respectively. Take the analogous embedding $B^5 \times S^2 \rightarrow \partial W_g$. Then take

$$W := W_f \cup_{B^5 \times S^2} W_g.$$

Consider the Mayer-Vietoris sequence:

$$H_6(B^5 \times S^2) \rightarrow H_6(W, \partial) \rightarrow H_6(W_f, \partial) \oplus H_6(W_g, \partial) \rightarrow H_5(B^5 \times S^2, \partial).$$

Identify ∂W and $C_{f \# g} \cup_{\varphi} C_{f_0 \# g_0}$ by the easily constructed homeomorphism. We have $\partial A_f \cap B^4 \times S^2 = [B^4 \times x] \in H_4(B^4 \times S^2, \partial)$, and the same for f replaced by f_0, g, g_0 . Hence

$$\partial z_f \cap B^5 \times S^2 = \partial z_g \cap B^5 \times S^2 = [B^5 \times x] \in H_5(B^5 \times S^2, \partial).$$

Therefore there is a unique $z \in H_6(W, \partial)$ such that (W, z) is a null-bordism of $(C_{f \# g}, C_{f_0 \# g_0}, A_f, A_{f_0}, \varphi)$.

Since $H_c(B^5 \times S^2) = H_c(B^5 \times S^2, \partial) = 0$ for $c = 3, 4$, by the exact sequence of pair and the Mayer-Vietoris sequence we have orthogonal isomorphisms Ψ and Ψ_{∂} appearing in the following commutative diagram:

$$\begin{array}{ccc} H_4(W) & \xrightarrow{\Psi \cong} & H_4(W_f) \oplus H_4(W_g) \\ \downarrow j & & \downarrow j_f \oplus j_g \\ H_4(W, \partial) & \xrightarrow{\Psi_{\partial} \cong} & H_4(W_f, \partial) \oplus H_4(W_g, \partial) \end{array}.$$

Clearly, $\Psi_{\partial} z^2 = z_f^2 \oplus z_g^2$ and $\Psi_{\partial} p_W = p_{W_f} \oplus p_{W_g}$. So we can take $\overline{z^2} := \Psi^{-1}(\overline{z_f^2} \oplus \overline{z_g^2})$, where Ψ denotes the isomorphism analogous to Ψ with coefficients \mathbb{Z}_d . Then clearly $\eta_{W,z} = \eta_{W_f, z_f} + \eta_{W_g, z_g}$. This implies the required statement. \square^{22}

²²We conjecture that $\eta_{u_1 \oplus u_2}(f_1 \# f_2, f'_1 \# f'_2) = \eta_{u_1}(f_1, f'_1) + \eta_{u_2}(f_2, f'_2)$, where $f_k, f'_k : N_k \rightarrow \mathbb{R}^7$ are embeddings such that $BH(f_k) = BH(f'_k) = u_k$.

Proof of Lemma 2.9.

Consider the fibration $\mathbb{R}P^\infty \rightarrow BSpin \rightarrow BSO$. The 4-line of the cohomology Leray-Serre spectral sequence of this fibration is the same at the E_2 term and at the E_∞ term. The 4-line has $\mathbb{Z} = H^4(BSO)$ in the $(4, 0)$ position and also a $\mathbb{Z}_2 = H^2(BSO; \mathbb{Z}_2)$ in the $(2, 2)$ position. Therefore $H^4(BSO)$ maps into $H^4(BSpin)$ as a subgroup of index 2. Hence the pullback, $p_1 \in H^4(BSpin)$, of the universal first Pontryagin class in $H^4(BSO)$ equals to twice the generator of $H^4(BSpin) \cong \mathbb{Z}$. (This fact is also proved in [KS91, proof of Lemma 6.5].)

Take the map $\bar{\nu} : W \rightarrow BSpin$ corresponding to the given spin structure on W . We have $p_W = PD\bar{\nu}^*p_1$. Hence p_W is divisible by 2.

Let $w_4 \in H^4(BSpin; \mathbb{Z}_2)$ be the pullback of the universal 4-th Stiefel-Whitney class in $H^4(BSO; \mathbb{Z}_2)$. Since w_4 generates $H^4(BSpin; \mathbb{Z}_2)$ and the mod 2 reduction $\rho_2 : H^4(BSpin) \rightarrow H^4(BSpin; \mathbb{Z}_2)$ is onto, we have $\rho_2(p_1/2) = w_4$. Also $w_4(W) = \bar{\nu}^*w_4$. Hence $\rho_2(p_W/2) = PDw_4(W)$. Let us prove that this implies the remaining divisibility by 2.

If W is closed, then the required divisibility follows because $w_4(W) = v_4(W) + \text{Sq}^1 v_3(W) = v_4(W)$. Here the first equality holds by the Wu formula and the second because $\text{Sq}^1 v_3(W) = \text{Sq}^1 w_3(W) = 0$ since W is spin (or else because $v_3(W) = w_3(W) = 0$ since W is spin and $BSpin$ is 3-connected).

If W has a non-empty boundary, then let $Y := W \cup_{\partial W} (-W)$. Since

$$p_W = p_Y \cap W, \quad \text{we have} \quad (p_W/2) \cap_W x = (p_Y/2) \cap_Y i_Y x \equiv_{\text{mod } 2} i_Y x \cap_Y i_Y x = x \cap_W x,$$

where i_Y is the inclusion-induced map $H_4(W) \rightarrow H_4(Y)$. \square

Proof of the Realization Theorem 2.10.

Construction of $g_1 : S^4 \rightarrow \mathbb{R}^7$. By general position, there is an embedding $\eta'' : S^3 \rightarrow S^2 \times D^5$ whose composition with the projection onto S^2 is the Hopf map.²³ Take an embedding $\psi : D^4 \rightarrow S^2 \times D^5$ whose image intersects $\eta''(S^3)$ transversally at exactly one point of sign +1. Let $\psi' := \psi|_{\partial D^4}$.

Since each embedding $S^3 \rightarrow S^7$ is unknotted, it extends to an embedding $D^4 \rightarrow D^8 \supset S^7$. Since D^4 is contractible, it has a unique framing. Therefore there is a unique framing of $S^3 \subset S^7$ which extends to a framing of some extension $D^4 \rightarrow D^8$. Define this framing to be the zero framing. This and the isomorphism $\pi_3(SO_4) \cong \mathbb{Z} \oplus \mathbb{Z}$ [Mi56] gives a 1-1 correspondence between normal framings on an embedding $S^3 \rightarrow \mathbb{R}^7$ (up to homotopy) and $\mathbb{Z} \oplus \mathbb{Z}$.

Assume that $S^2 \times D^5 \subset S^7$ is standardly embedded as a complement to the tubular neighborhood of the standard $S^4 \subset S^7$. Take the framing on η'' corresponding to $(0, 0)$ and the framing on ψ' corresponding to $(1, -1)$. Let M be the closed 7-manifold obtained from S^7 by surgery along framed embeddings ψ' and η'' . Then M is a homotopy sphere containing the above S^4 . In the ‘proof of the Realization Theorem 2.10’ below we prove that $M \cong S^7$. Let g_1 be the composition of the inclusion $S^4 \rightarrow M$ and any diffeomorphism $M \rightarrow S^7$.

In this subsection let $i : S^2 \times D^5_- \rightarrow S^7 = \partial D^8$ be the standard embedding. For a D^4 -bundle $\tilde{\alpha}$ over S^4 denote by $e(\tilde{\alpha}) \in \mathbb{Z}$ the Euler number of this bundle.

²³An explicit construction of η'' [Sk]: Define an embedding $\eta' : S^3 \rightarrow S^2 \times D^2$ by $\eta'(z_1, z_2) := ((z_1 : z_2), z_1)$. The composition of η' with the projection onto S^2 is the Hopf map. Let η'' be the composition of η' and the standard inclusion $S^2 \times D^2 \rightarrow S^2 \times D^5$.

Lemma 3.3. *Let W be the 8-manifold obtained by adding 4-handles to $S^2 \times D^6$ via embeddings*

$$\alpha_1, \dots, \alpha_n : S^3 \times D^4 \rightarrow S^2 \times D^5_- \subset \partial(S^2 \times D^6)$$

with disjoint images. Denote by $[\alpha_1], \dots, [\alpha_n] \in H_4(W)$ the basis corresponding to the 4-handles. Denote by $\tilde{\alpha}_m$ the D^4 -bundle over S^4 corresponding to α_m (i.e. the projection to S^4 from the 8-manifold obtained from D^8 by adding a 4-handle along $i\alpha_m$). Then

$$[\alpha_m] \cap [\alpha_l] = \begin{cases} \text{lk}_{S^7}(i\alpha_m, i\alpha_l) & m \neq l \\ e(\tilde{\alpha}_m) & m = l \end{cases} \quad \text{and} \quad p_W \cap [\alpha_m] = p_1(\tilde{\alpha}_m)([S^4]).$$

Proof. Cf. [Sc02].

The equality $[\alpha_i] \cap [\alpha_j] = \text{lk}_{S^7}(\alpha_i, \alpha_j)$ for $i \neq j$ follows analogously to [Ma80, 3.2].

For the other equalities we may assume that $m = l = 1$ and replace W by the 8-manifold W' obtained from D^8 by adding a 4-handle along embedding $\alpha = i\alpha_1$.

Since every embedding $S^3 \rightarrow S^7$ is isotopic to the standard embedding, there is a 4-sphere $X \subset W'$ representing $[\alpha] \in H_4(W')$. Then X is homologous in W' to the zero section $X' \subset \partial W'$ of $\tilde{\alpha}$. Hence $\nu_{W'}(X) = \tilde{\alpha}$. Thus the characteristic classes of $\nu_{W'}(X)$ and of $\tilde{\alpha}$ coincide.

We have $[\alpha] \cap [\alpha] = e(\tilde{\alpha})$ because the self-intersection of a homology class represented by a submanifold equals to the Euler class of the normal bundle of the submanifold in the manifold (this is easily proved directly or else deduced from [MS74, Exercise 11-C in p. 134]).

We have $p_{W'} \cap [\alpha] = PDp_1(\tau_{W'}|_X) = PDp_1(\tilde{\alpha})$, where the second equality holds because $\tau_{W'}|_X \cong \tau_X \oplus \nu_{W'}(X)$ is stably equivalent to $\nu_{W'}(X) = \tilde{\alpha}$ since $X \cong S^4$ is stably parallelizable. \square

Proof of the Realization Theorem 2.10. Let $S^2 \times \partial D^6 = S^2 \times D^5_+ \cup_{S^2 \times S^4} S^2 \times D^5_-$ be the standard decomposition corresponding to the standard decomposition $\partial D^6 = D^5_+ \cup_{S^4} D^5_-$. Let W be the 8-manifold obtained from $S^2 \times D^6$ by adding 4-handles along the framed embeddings ψ' and η'' into $S^2 \times D^5_-$. Let $C_0 := S^2 \times D^5_+ \subset \partial W$. Let $C_1 \subset \partial W$ be the 7-manifold obtained from $S^2 \times D^5_-$ by surgery along framed embeddings ψ' and η'' into $S^2 \times D^5_-$. Take the identity diffeomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$.

For the W constructed both maps of the composition $H_6(W, \partial) \rightarrow H_5(\partial W) \rightarrow H_5(C_k, \partial)$ (the boundary map and the map $x \mapsto x \cap \partial C_k$) are isomorphisms. Hence for the generator $z_W \in H_6(W, \partial)$ we have that ∂z_W is a generator of $H_5(\partial W)$ and $A_k := \partial z_W \cap C_k$ is a generator of $H_5(C_k, \partial)$. Then $X = (C_0, C_1, A_0, A_1, \varphi)$ is an admissible set and W, z_W is a null-bordism of X .

Identify $H_4(W)$ with $H_4(W, \partial)$ (and the same for W replaced by W' defined below) by the isomorphism from the exact sequence of pair.

Take a basis x, y of $H_4(W) \cong \mathbb{Z}^2$ with x and y corresponding to the handle attached by ψ' and by η'' , respectively. By Lemma 3.3 and [Mi56]

$$x \cap y = 1, \quad x \cap x = p_W \cap x = 0, \quad y \cap y = 1 + (-1) = 0 \quad \text{and} \quad p_W \cap y = 2(1 - (-1)) = 4.$$

Hence $p_W = 4x$.

Denote by W' the 8-manifold obtained from D^8 by adding 4-handles along framed embeddings $i\psi'$ and $i\eta''$ into ∂D^8 . Recall that $M = \partial W'$ for the 7-manifold M defined in the ‘construction of g_1 ’. Analogously to above there is a basis x, y of $H_4(W') \cong \mathbb{Z}^2$

in which the intersection form of W' has matrix H_+ , and $p_{W'} = 4x$. Then $4\sigma(W') = 0 \equiv_{\text{mod } 28 \cdot 32} 0 = p_{W'} \cap p_{W'}$. Hence $\partial W' \cong S^7$ [EK62, §6].

We have $z_W^2 = y$. (Indeed, $W \simeq S^2 \cup (e_x^4 \cup e_y^4)$, where \simeq means ‘homotopy equivalent up to dimension 4’. Homotopy classes of the attaching maps for e_x^4 and for e_y^4 equal to the homotopy classes of η'' and ψ' . So the attaching maps are homotopic to the Hopf map and trivial map $S^3 \rightarrow S^2$, respectively. It follows that $W \simeq \mathbb{C}P^2 \vee S^4$. Thus we obtain the cohomology ring of W up to dimension 4. By duality we obtain the homology groups of W and relevant intersection products above dimension 3. Hence $z_W^2 \cap x = 1$ and $z_W^2 \cap y = 0$ for a generator $z_W \in H_6(W)$. By Poincaré duality $z_W^2 = y$.)

Then $\eta(g_1, g_0) = \eta_{W, z_W} = 2$. \square

4. PROOF OF THE ‘IF’ PART OF THE ALMOST DIFFEOMORPHISM THEOREM 2.6

The Kreck Theorem 4.1. *Let*

- W be a compact $4l$ -manifold such that $\partial W = C_0 \cup C_1$ for compact $(4l - 1)$ -manifolds $C_0, C_1 \subset \mathbb{R}^{8l}$ with common boundary;

- $p : B \rightarrow BO$ be a fibration such that $\pi_i(p) = 0$ for $i \geq 2l$ and $\pi_1(B) = 0$;

- $\bar{v} : W \rightarrow B$ is a $2l$ -connected map such that $p\bar{v}|_{C_k}$ is the classifying map of the normal bundle of C_k and $\bar{v}|_{C_k}$ is $(2l - 1)$ -connected.

Then \bar{v} is bordant (relative to the boundary) to a product of $\bar{v}|_{C_0}$ with the interval if²⁴ there is a subgroup $U \subset H_{2l}(W)$ such that

- $U \cap U = 0$ and $\bar{v}_*U = 0 \subset H_{2l}(B)$,

- $j_k|_U$ is an isomorphism onto a direct summand in $V_k := H_{2l}(W, C_k)$, and

- the quotient $j_0U \times V_1/j_1U \rightarrow \mathbb{Z}$ of the intersection pairing $\cap : V_0 \times V_1 \rightarrow \mathbb{Z}$ is unimodular.

Proof. Denote $K := \ker(\bar{v}_* : H_{2l}(W) \rightarrow H_{2l}(B))$. The form $\cap : K \times K \rightarrow \mathbb{Z}$ is even because²⁵

$$x \cap x = \langle w_{4l}(W), x \rangle = \langle p^*\bar{v}^*w_{4l}, x \rangle = \langle w_{4l}, p_*\bar{v}_*x \rangle = 0 \pmod{2},$$

where $x \in K$ and $w_4 \in H^4(BO)$ is the Stiefel Whitney class. So in [Kr99, p. 725] we can take $\mu(x) := x \cap x/2$ for $x \in K$ (because $2l$ is even). We have $Wh(\pi_1(B)) = 0$ and so an isomorphism is a simple isomorphism. Hence the hypothesis on U implies that $\theta(W, \bar{v})$ is ‘elementary omitting the bases’ [Kr99, Definition in p. 730 and the second remark on p. 732].²⁶ Thus the result follows by the h -cobordism theorem and [Kr99, Theorem 3 and second remark in p. 732]. \square

The Bordism Theorem.

Lemma 4.2. *Let C_k be compact connected 7 -manifolds such that $H_3(C_0) = H_3(C_1) = 0$, $\varphi : \partial C_0 \rightarrow \partial C_1$ a diffeomorphism and W a compact 8 -manifold such that $\partial W = M_\varphi$. Denote*

$$V_0 := H_4(W, C_0) \quad \text{and let} \quad j_0 : H_4(W) \rightarrow V_0$$

²⁴and only if

²⁵In the situation of the Almost Diffeomorphism Theorem 2.6 this form is even by Lemma 2.9.

²⁶In [Kr99, Definition on p. 729] $\theta(W, \bar{v})$ was only defined for a q -connected map $\bar{v} : W \rightarrow B$. (Indeed, on p. 725 in [Kr99] there is a paragraph beginning “The objects in $l_{2q}(\pi, \omega)$ are represented ...”. In condition (i) V_0 and V_1 are based. This means in particular that they are stably free. Now for a bordism $(W, \bar{v}; M_0, M_1)$ we have by definition $V_0 = H_q(W, M_0)$ and this is only a stably free module if $\bar{v} : W \rightarrow B$ is q -connected.) If \bar{v} is not q -connected, then it is bordant to a q -connected map $\bar{v}_1 : W_1 \rightarrow B$ and we can define $\theta(W, \bar{v}) := \theta(W_1, \bar{v}_1)$. This is well-defined by [Kr99, the first sentence in p. 730].

be the map from the exact sequence of pair. There is a well-defined bilinear map

$$\cdot : V_0 \times V_0 \rightarrow \mathbb{Z} \quad \text{given by} \quad x \cdot x' := j_0^{-1}x \cap x'$$

which is symmetric and unimodular and where $j_0^{-1}x$ denotes any element in $j_0^{-1}x$.²⁷

Proof. Since $H_3(C_0) = 0$, the map j_0 is epimorphic.

If $y, y' \in j_0^{-1}x$, then we may assume that the support of $y - y'$ is in C_0 . Then $(y - y') \cap x' = (y - y') \cap_{C_0} \partial x' = 0$ because $H_3(C_0) = 0$. So \cdot is well-defined.

This form is symmetric because of the symmetry of linking coefficients of 3-cycles in C_0 . In order to prove the unimodularity of \cdot take primitive $x_0 \in V_0$. By Poincaré-Lefschetz duality there is $x_1 \in V_1$ such that $x_1 \cap x_0 = 1$. Since $H_3(C_1) = 0$, there is $y \in H_4(W)$ such that $j_1 y = x_1$. We have $x_0 \cdot j_0 y = x_0 \cap y = x_0 \cap x_1 = 1$. \square

Bordism Theorem 4.3. *Let (W, z) be a null-bordism of an admissible set*

$$X = (C_0, C_1, A_0, A_1, \varphi) \quad \text{such that} \quad \pi_1(C_k) = H_3(C_k) = H_4(C_k, \partial) = 0.$$

The pair (W, z) is spin bordant (relative to the boundary) to a product with the interval if²⁸ there is a left inverse s of the map

$$j : V_0 \rightarrow H_4(W, \partial)$$

from the exact sequence of triple $(sj = \text{id})$ such that

$$\sigma(W) = sp_W \cdot sp_W = sz^2 \cdot sp_W = sz^2 \cdot sz^2 = 0.$$

Beginning of the proof of the Bordism Theorem 4.3. Recall that $BSpin = BO \langle 4 \rangle$ is the (unique up to homotopy) 3-connected space for which there exists a fibration $BSpin \rightarrow BO$ inducing an isomorphism on π_i for $i \geq 4$. Denote $B := BSpin \times \mathbb{C}P^\infty$. Define $p : B \rightarrow BO$ to be the composition of the projection to $BSpin$ and the map $BSpin \rightarrow BO$ inducing an isomorphism on π_i for $i \geq 4$. Take the map $\bar{v} : W \rightarrow B$ corresponding to the given spin structure on W and to $z \in H_6(W, \partial) \cong [W, \mathbb{C}P^\infty]$.

Since X is admissible and $H_4(C_k, \partial) = 0$, by Poincaré-Lefschetz duality the map $(\bar{v}|_{C_k})_* : H_2(C_k) \rightarrow H_2(\mathbb{C}P^\infty)$ is an isomorphism. This and $\pi_1(C_k) = H_3(C_k) = 0$ imply that the map $\bar{v}|_{C_k}$ is 3-connected. Making B -surgery below the middle dimension we can change \bar{v} relative to the boundary and assume that \bar{v} is 4-connected [Kr99, Proposition 4]. This surgery together with the obvious corresponding change of s preserves $\sigma(W)$, $sp_W \cdot sp_W$, $sz^2 \cdot sp_W$ and $sz^2 \cdot sz^2$. Hence it suffices to construct U as in the Kreck Theorem 4.1.

Since $BSpin$ is 3-connected, we have

$$H_4(B) \cong H_4(BSpin) \oplus H_4(\mathbb{C}P^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

This isomorphism carries $\bar{v}_* u$ to $(u \cap p_W / 2, u \cap z^2)$ (where $a \in H^2(\mathbb{C}P^\infty)$ is a generator and p_W is even by Lemma 2.9). So ' $\bar{v}_* U = 0 \in H_4(B)$ ' is equivalent to ' $U \cap z^2 = U \cap p_W = 0$ '.

Let

$$\widehat{U} = \{u \in V_0 \mid du = msz^2 + nsp_W \text{ for some integers } d, m, n\}.$$

²⁷Of course 'geometrically $j_0^{-1}x \cap x' = x \cap x'$ ', but the first intersection assumes values in $H_0(W) = \mathbb{Z}$ while the second one in $H_0(W, C_0) = 0$.

²⁸and only if

(Note that $\text{rk } \widehat{U}$ is 1 or 2.) Since

$$sp_W \cdot sp_W = sz^2 \cdot sp_W = sz^2 \cdot sz^2 = 0, \quad \text{we have } \widehat{U} \cdot \widehat{U} = 0.$$

Since the form \cdot is unimodular, there is

$$X \subset V_0 \quad \text{such that} \quad \widehat{U} \subset X, \quad \text{rk } X = 2 \text{rk } \widehat{U} \quad \text{and} \quad \cdot|_X \text{ is unimodular.}$$

Then²⁹ $V_0 \cong X \oplus X^\perp$ and $\sigma(X) = 0$.

The map $j_0 : H_4(W) \rightarrow V_0$ is onto and carries \cap to \cdot . Therefore $\sigma(X^\perp) = \sigma(\cdot) = \sigma(W) = 0$. Hence there is a direct summand $\widetilde{U} \subset X^\perp$ such that $\widetilde{U} \cdot \widetilde{U} = 0$. Let $U := s^*(\widehat{U} \oplus \widetilde{U})$, where s^* is given by the following Lemma 4.4.

Lemma 4.4. *Under the assumptions of Lemma 4.2 for each left inverse s of j a right inverse $s^* : V_0 \rightarrow H_4(W)$ of j_0 is well-defined by*

$$s^*x \cap y = x \cdot sy \quad \text{for each } y \in H_4(W, \partial).$$

The map $j_1 s^* : V_0 \rightarrow V_1$ is an isomorphism carrying the product $\cap : V_0 \times V_1 \rightarrow \mathbb{Z}$ to \cdot , i.e. $x \cdot x' = j_1 s^* x \cap x'$ for each $x, x' \in V_0$.³⁰

Proof. Define a homomorphism $\bar{x} : H_4(W, \partial) \rightarrow \mathbb{Z}$ by $\bar{x}(y) := x \cdot sy$. Now the existence and uniqueness of such an element s^*x follows by Poincaré-Lefschetz duality.

Clearly, s^* is a homomorphism.

We have

$$j_0 s^* x \cdot x' = s^* x \cap x' = s^* x \cap j x' = x \cdot s j x' = x \cdot x' \quad \text{for each } x, x' \in V_0.$$

Since the form \cdot is unimodular, $j_0 s^* x = x$.

We have $x \cdot x' = s^* x \cap x' = j_1 s^* x \cap x'$. (Cf. the end of the proof of Lemma 4.2.)

The map s^* is injective. For $x, x' \in V_0$ if

$$j_1 s^* x = j_1 s^* x', \quad \text{then} \quad x \cap a = j_1 s^* x \cap a = j_1 s^* x' \cap a = x' \cap a \quad \text{for each } a \in V_1.$$

Hence by Poincaré-Lefschetz duality $x = x'$. Thus $j_1 s^*$ is injective. So it is an isomorphism. \square

Completion of the proof of the Bordism Theorem 4.3: checking of the required properties of U . Clearly, \widehat{U} is a direct summand in X .

Let $U' := \widehat{U} \oplus \widetilde{U}$. Then

$$j_0 U = U', \quad U' \cdot U' = U' \cdot sz^2 = U' \cdot sp_W = 0$$

and U' is a direct summand in V_0 .

By Lemma 4.4

$$U \cap U = U \cap j j_0 U = s^* U' \cap j U' = U' \cdot s j U' = U' \cdot U' = 0, \quad U \cap x = U' \cdot s x = 0 \quad \text{for } x \in \{z^2, p_W\}$$

and $j_0|_U$ is an isomorphism onto the direct summand $U' \subset V_0$.

²⁹Since both V_0 and $X \subset V_0$ are unimodular, we have $X \cap X^\perp = 0$ and $\text{rk } X^\perp = \text{rk } V_0 - \text{rk } X$. Then $V_0 = X \oplus X^\perp$.

³⁰The second statement holds for each right inverse of j_0 , not necessarily the one obtained from s .

Since $U \subset \text{im } s^*$, by Lemma 4.4 $j_1|_U$ is monomorphic.

Since $U' \subset V_0$ is a direct summand, we have $V_0 \cong U' \oplus U''$ (additive) for some $U'' \subset V_0$. Suppose that $j_1 s^* u' = j_1 s^* u''$ for some $u' \in U'$ and $u'' \in U''$. By excision $H_4(\partial W, C_1) \cong H_4(C_0, \partial) = 0$, so by the exact sequence of pair the inclusion-induced map $H_4(C_1) \rightarrow H_4(\partial W)$ is surjective. Hence for the inclusion-induced maps

$$i : H_4(\partial W) \rightarrow H_4(W) \quad \text{and} \quad i_k : H_4(C_k) \rightarrow H_4(W) \quad \text{we have} \quad \text{im } i = \text{im } i_1.$$

Analogously $\text{im } i = \text{im } i_0$. Hence

$$s^* u' - s^* u'' \in \text{im } i_1 = \text{im } i_0, \quad \text{so} \quad u' - u'' = j_0(s^* u' - s^* u'') = 0, \quad \text{hence} \quad u' = u'' = 0.$$

Thus $j_1 U \cap j_1 s^* U'' = 0$. Therefore by dimension considerations $V_1 \cong j_1 U \oplus j_1 s^* U''$ (additively). So $j_1 U$ is a direct summand.

The pairing $\cap : j_0 U \times V_1/j_1 U \rightarrow \mathbb{Z}$ is isomorphic to the pairing $\cap : U' \times j_1 s^* U'' \rightarrow \mathbb{Z}$ and (by Lemma 4.4) to the pairing $\cdot : U' \times U'' \rightarrow \mathbb{Z}$. Since the form $\cdot : V_0 \times V_0 \rightarrow \mathbb{Z}$ is unimodular and $U' \cdot U' = 0$, the latter pairing is unimodular. \square

Proof of the ‘if’ part of the Almost Diffeomorphism Theorem 2.6.

Beginning of the proof. Take a null-bordism (W, z) of X given by the Null-bordism Lemma 2.4. The idea is to modify (W, z) and apply the Bordism Theorem 4.3. Define B, p and a 4-connected map $\bar{v} : W \rightarrow B$ as in the beginning of the proof of the Bordism Theorem 4.3.

Since $H_3(C_0) = 0$, we can take the product \cdot given by Lemma 4.2.

By excision $H_4(\partial W, C_0) \cong H_4(C_1, \partial) = 0$. Then, by the exact sequence of a triple, j is injective.

Take $x \in V_0$. We have $x' \cdot x = y \cap x = y \cap jx$ for each $x' \in V_0$ and $y \in j_0^{-1} x'$. If jx is divisible by an integer d , then $x' \cdot x$ is divisible by d for each $x' \in V_0$. Hence the unimodularity of \cdot implies that jx is primitive for each primitive $x \in V_0$. So there exists a left inverse s of j (because \bar{v} is 4-connected and so $\text{Tors } H_4(W, \partial) = \text{Tors } H_3(W) = 0$).

Denote $d := d(\partial_W z^2)$. Recall the definition of $\overline{p_W} \in H_4(W)$ and $\overline{z^2} \in H_4(W; \mathbb{Z}_d)$ from the definition of η_X in §2. Since $j_0 \overline{p_W} = sp_W$, we have $\overline{p_W} \cap p_W = sp_W \cdot sp_W$. Since

$$j_0 \overline{z^2} = \rho_d s z^2, \quad \text{we have} \quad \overline{z^2} \cap p_W = \rho_d s z^2 \cdot sp_W \in \mathbb{Z}_d \quad \text{and} \quad \overline{z^2} \cap z^2 = \rho_d s z^2 \cdot s z^2 \in \mathbb{Z}_d.$$

Denote $\widehat{\eta}_{W,z,s} = s z^2 \cdot (s z^2 - s \frac{p_W}{2}) \in \mathbb{Z}$. Thus $\eta_X = \rho_d \widehat{\eta}_{W,z,s}$.³¹

Analogously for A_0^2 divisible by 2, $\eta'_X = \rho_2(s z^2 \cdot s z^2)$.

For completion of the proof we need two lemmas. Let W be a compact spin 8-manifold such that $\partial_W p_W = 0$. Define $\frac{\overline{p_W}}{2} \in H_4(W)$ analogously to $\overline{p_W}$. (It is clear that the intersections below do not depend on the choice of $\frac{\overline{p_W}}{2}$, which choice is in $H_4(\partial W)$.) By Lemma 2.9

$$\sigma(W) \equiv_{\text{mod } 8} \frac{\overline{p_W}}{2} \cap \frac{p_W}{2} \quad \text{so} \quad \alpha_W := \frac{4\sigma(W) - \overline{p_W} \cap p_W}{32} \quad \text{is an integer.}$$

Lemma 4.5. *For each of the four quadruples*

$$(1, 0, 0, 0), \quad (0, 28, 0, 0), \quad (0, 0, 2, 0), \quad (0, 0, 0, 12)$$

³¹Note that $\rho_d(\overline{p_W} \cap z^2) = \overline{z^2} \cap \rho_d p_W = \rho_d(sp_W \cdot s z^2)$ but $\overline{p_W} \cap z^2 \neq sp_W \cdot s z^2 = s^* j_0 \overline{p_W} \cap z^2$.

there is a closed compact spin 8-manifold W and $z \in H_6(W)$ such that the quadruple $Q_{W,z} := (\sigma(W), \alpha_W, z^4, z^4 - z^2 \frac{1}{2} p_W)$ coincides with the given quadruple.³²

Lemma 4.6. *Let (W, z) be a null-bordism of an admissible set X such that $H_3(C_k) = H_3(W) = H_5(W, \partial) = 0$ and $H_2(\partial C_0)$ is free. Let s be a left inverse of j . By connected sum of W with a null-bordant closed 3-connected 8-manifold and certain corresponding change of z, s one can change*

- $sz^2 \cdot sz^2$ by adding an odd number, provided A_0^2 is not divisible by 2.
- $\widehat{\eta}_{W,z,s}$ by adding $2d/\text{GCD}(d, 2)$, where $d := d(\partial_W z^2)$, and preserving $\rho_2(sz^2 \cdot sz^2)$.

The lemmas are proved in the next subsection (Lemma 4.5 is known).

Completion of the proof of the ‘if’ part of the Almost Diffeomorphism Theorem 2.6. Take a 3-connected parallelizable 8-manifold \overline{E}_8 whose boundary is a homotopy sphere and whose signature is 8. Then $p_{\overline{E}_8} = 0$. The boundary connected sum of \overline{v} with a constant map $\overline{E}_8 \rightarrow \mathbb{C}P^\infty$ changes α_W by 1 and preserves the 4-connectedness of \overline{v} .³³ Thus we may assume that $\alpha_W = 0$.

For a null-bordism W, z of an admissible set X such that $H_3(C_k) = 0$ and a left inverse s of j denote $Q_{W,z,s} := (\sigma(W), \alpha_W, sz^2 \cdot sz^2, \widehat{\eta}_{W,z,s})$. For a closed spin 8-manifold W_0 and $z_0 \in H_6(W_0)$ we have $Q_{W \# W_0, z \oplus z_0, s \oplus \text{id}} = Q_{W,z,s} + Q_{W_0,z_0}$. Since z is primitive, $z \oplus z_0$ is primitive. So we may spin surger $W \# W_0$ and assume that the map $\overline{v}' : W \# W_0 \rightarrow B$ corresponding to $z \oplus z_0$ and the ‘connected sum’ spin structure on $W \# W_0$ is 4-connected. So by Lemma 4.5 we may change the quadruple $Q_{W,z,s}$ by any of the four quadruples of Lemma 4.5, and \overline{v} would remain 4-connected.

Thus we may assume that $\sigma(W) = \alpha_W = 0$.

Connected sum of \overline{v} with the constant map from a null-bordant 3-connected 8-manifold does not change $\sigma(W)$, α_W and the property that \overline{v} is 4-connected.

If A_0^2 is not divisible by 2, then by Lemmas 4.6 and 4.5 we may assume that $\sigma(W) = \alpha_W = sz^2 \cdot sz^2 = 0$.

If A_0^2 is divisible by 2, then $\rho_2(sz^2 \cdot sz^2) = \eta'_X = 0$, hence by Lemma 4.5 we may assume that $\sigma(W) = \alpha_W = sz^2 \cdot sz^2 = 0$.

Since $\eta_X = 0$, by Lemmas 4.6 and 4.5 we may assume that $\sigma(W) = \alpha_W = sz^2 \cdot sz^2 = \widehat{\eta}_{W,z,s} = 0$. Then we are done by the Bordism Theorem 4.3. \square

Diffeomorphism Theorem 4.7. *Let $X = (C_0, C_1, A_0, A_1, \varphi)$ be an admissible set such that $\pi_1(C_k) = H_3(C_k) = H_4(C_k, \partial) = 0$ and $H_2(\partial C_0)$ is free. Denote $\alpha_X := \rho_{28} \alpha_W \in \mathbb{Z}_{28}$ for some null-bordism (W, z) of X .³⁴ There is a diffeomorphism $C_0 \rightarrow C_1$ extending φ if and only if*

$$\alpha_X = 0, \quad \eta_X = 0 \quad \text{and, for } A_0^2 \text{ divisible by 2, } \eta'_X = 0.$$

The ‘only if’ part is simple (take $W = C_0 \times I \cup_{\overline{\varphi}} C_1$, where $\overline{\varphi} : C_0 \times 1 = C_0 \rightarrow C_1$ is given extension). We essentially proved the ‘if’ part in the course of the proof of the ‘if’ part of the Almost Diffeomorphism Theorem 2.6.

³²We can avoid using $(0, 0, 2, 0)$ by using the Framing Theorem 2.7(φ) and changing the structure of the proof of the injectivity of η_u .

³³An alternative proof is obtained by replacing \overline{E}_8 by a 3-connected 8-manifold $X \simeq S^4$ whose boundary is a homotopy sphere, $\sigma(X) = 1$ and $p_X = 6$ [Mi56].

³⁴The independence of α_X of W is essentially known. Note that α_X is also independent of φ because $\sigma(W) - 4p_W^2 = -2^7 \cdot 7s_1(N_{1,-1}) = 0$ in the notation of the subsection ‘Proof of the Framing Theorem 2.7’.

Conjecture 4.8. *Let (W, z) be a null-bordism of an admissible set $X = (C_0, C_1, A_0, A_1, \varphi)$ such that*

$$\pi_1(C_k) = H_3(C_k) = H_3(\partial C_0) = H_4(C_k, \partial) = p_1(C_k) = 0,$$

$H_2(\partial C_0)$ is free and the map $h_z : W \rightarrow \mathbb{C}P^\infty$ corresponding to z is 4-connected. Then (W, z) is spin bordant (relative to the boundary) to a product with the interval if and only if

$$\sigma(W) = \overline{p_W} \cap p_W = 0 \quad \text{and} \quad \overline{z^2} \cap p_W = \overline{z^2} \cap z^2 = 0 \in \mathbb{Z}_d.$$

Proof of Lemmas 4.5 and 4.6.

Proof of Lemma 4.5. Recall that $\sigma(\mathbb{H}P^2) = 1$ and $p_1^2(\mathbb{H}P^2) = 4$ [Hi53], cf. [Mi56, Lemmas 3 and 4]. So for $(\mathbb{H}P^2, 0)$ the quadruple is $(1, 0, 0, 0)$.

Take a 3-connected parallelizable 8-manifold \overline{E}_8 whose boundary is a homotopy sphere and whose signature is 8. Then $p_1(\overline{E}_8) = 0$. For $(28\overline{E}_8 \cup D^8, 0)$ the quadruple is $(28 \cdot 8, 28, 0, 0)$.

Take $(S^2)^4$ and the class z which is the sum of four summands, each represented by a product of three 2-spheres and a point. Then $z^4 = 24$. Denote $H_+ := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As a quadratic form $H_4((S^2)^4) \cong H_+ \oplus H_+ \oplus H_+$, so $\sigma((S^2)^4) = 0$. Since $(S^2)^4$ is almost parallelizable, we have $p_{(S^2)^4} = 0$. Thus for $((S^2)^4, z)$ the quadruple is $(0, 0, 24, 12)$.

By [KS91, Proposition 2.5] there is a closed spin 8-manifold W and $z \in H_6(W)$ such that $S_1 = S_2 = 0$ and $S_3 = 1$. In the notation of [KS91, spin case of (2.4)]

$$S_1 = \alpha_W/28, \quad S_2 = z^2(z^2 - \frac{1}{2}p_W)/12 \quad \text{and} \quad 2S_3 = 8S_2 + z^4.$$

Hence for (W, z) the quadruple is $(a, 0, 2, 0)$. \square

Lemma 4.9. *Assume that (W, z) is a null-bordism of an admissible set X .*

(p) $s'p_W = sp_W$ for each left inverses s, s' of j .

(z) *Suppose that $H_3(C_0) = H_3(W) = H_5(W, \partial) = 0$ and $H_2(\partial C_0)$ is free. For $x \in V_0$ there is a left inverse s' of j such that $s'z^2 = sz^2 + x$ if and only if x is divisible by $d := d(\partial_W z^2)$.*

Proof of (p). Denote by $\partial_0 : H_4(W, \partial) \rightarrow H_3(\partial W, C_0)$ the boundary homomorphism. The class $(\partial p_W) \cap C_0 = PDp_1(C_0) = 0$ goes to $\partial_0 p_W$ under the excision isomorphism $H_3(C_1, \partial) \rightarrow H_3(\partial W, C_0)$. Thus $\partial_0 p_W = 0$. Hence $p_W \in \text{im } j$ which implies (p). \square

Proof of (z). Since $H_3(C_0) = 0$, the map j_0 is onto, hence $\text{im } j = \text{im}(jj_0) = \ker \partial_W$. Since $H_2(\partial C_0)$ is free and $H_3(C_k) = 0$, by the Mayer-Vietoris sequence for $\partial W = C_0 \cup C_1$ we obtain that $H_3(\partial W)$ is free. This and $H_3(W) = H_5(W, \partial) = 0$ imply that $H_4(W, \partial) \cong V_0 \oplus H_3(\partial W)$. Identify these isomorphic groups by the isomorphism $j \oplus (\partial_W|_{\ker s})^{-1}$. Then z^2 is identified with $sz^2 \oplus \partial_W z^2$. The ‘only if’ part follows because $s'(sz^2 \oplus 0) = sz^2$, so $s'z^2 = sz^2 + s'\partial_W z^2$. The ‘if’ part follows because $\partial_W z^2/d \in H_3(\partial W) \subset H_4(W, \partial)$ is primitive, so for each $x_1 \in V_0$ there is a left inverse s' of j such that $s'(z^2/d) = s(z^2/d) + x_1$. \square

Proof of the Twisting Lemma 4.6. First we prove the second assertion. By [Mi56] there is a D^4 -bundle over S^4 whose Euler class is 0 and whose first Pontryagin class is 4. The double of this bundle is an S^4 -bundle $S^4 \widetilde{\times} S^4$ over S^4 whose first Pontryagin class is 4. We have $H_4(S^4 \widetilde{\times} S^4) \cong \mathbb{Z} \oplus \mathbb{Z}$ with evident basis. In this basis $p_{S^4 \widetilde{\times} S^4} = (4, 0)$ and the intersection form of $S^4 \widetilde{\times} S^4$ is H_+ .

Denote $W' := W \# S^4 \widetilde{\times} S^4$. Identify $H_6(W, \partial)$ with $H_6(W', \partial)$. Identify $H_4(W', C_0)$ with $V_0 \oplus H_+$ as groups with quadratic forms. Clearly,

$$\partial W' = \partial W, \quad \partial_{W'} z = \partial_W z \quad \text{and} \quad \widehat{\eta}_{W', z, s \oplus \text{id}} = \widehat{\eta}_{W, z, s}.$$

By (the ‘if’ part of) Lemma 4.9(z) there is a left inverse

$$s' : H_4(W', \partial) \rightarrow H_4(W', C_0) \quad \text{such that} \quad s'(z^2 \oplus (0, 0)) = sz^2 \oplus (0, d).$$

We have $p_{W'} = p_W \oplus (4, 0)$. By Lemma 4.9(p), $s'p_{W'} = (s \oplus \text{id})p_{W'} = sp_W \oplus (4, 0)$. So $sz^2 \cdot sz^2 = s'z^2 \cdot s'z^2$ and $\eta_{W', z, s'} - \eta_{W, z, s} = (0, d) \cap [(0, d) - (2, 0)] = (0, d) \cap (-2, d) = -2d$.

In this paragraph assume that d is even. We have $H_4(\mathbb{H}P^2 \# (-\mathbb{H}P^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$ with evident basis. In this basis $p_{\mathbb{H}P^2 \# (-\mathbb{H}P^2)} = (2, -2)$ and the intersection form of $\mathbb{H}P^2 \# (-\mathbb{H}P^2)$ is $\text{diag}(1, -1)$. Analogously to the above with $S^4 \widetilde{\times} S^4$ replaced by $\mathbb{H}P^2 \# (-\mathbb{H}P^2)$ we may change $\eta_{W, z, s}$ by

$$(0, d) \cap [(0, d) - (1, -1)] = (0, d) \cap (-1, d + 1) = -d^2 - d.$$

The difference $s'z^2 \cdot s'z^2 - sz^2 \cdot sz^2 = (0, d) \cap (0, d) = -d^2$ is divisible by 2. Hence we may change $\widehat{\eta}_{W, z, s}$ by $GCD(2d, d^2 + d) = d$ and preserve $\rho_2(sz^2 \cdot sz^2)$.

Now let us prove the first assertion. Since A_0^2 is not divisible by 2, d is odd. Hence in the above example change of $sz^2 \cdot sz^2$ is by an odd integer d^2 . \square

5. REMARKS (OMIT IN THE SUBMITTED VERSION)

The following properties from the definition of the admissibility are not necessary:

$H_3(\partial C_0) = 0$, $p_1(C_0) = p_1(C_1) = 0$ and $d(A_0^2) = d(A_1^2)$ for the Null-bordism Lemma 2.4,

$d(A_0^2) = d(A_1^2)$ for the definition of $\eta_{W, z}$,

$d(A_0^2) = d(A_1^2)$ and $p_1(C_0) = p_1(C_1) = 0$ for the definition of η'_X and the Bordism Theorem 4.3,

$p_1(C_0) = p_1(C_1) = 0$ for the Framing Theorem 2.7,

$d(A_0^2) = d(A_1^2)$ and $H_3(\partial C_0) = 0$ for Lemmas 4.6 and 4.9.

Remarks to the construction of a 1–1 correspondence between normal framings on an embedding $S^3 \rightarrow \mathbb{R}^7$ (up to homotopy) and $\mathbb{Z} \oplus \mathbb{Z}$. Surgery on a framed embedding $b : S^3 \times D^4 \rightarrow S^7$ gives a 8-manifold E_b which is the total space of a D^4 -bundle $E_b \rightarrow S^4$. The boundary ∂E_b is the total space of an S^3 -bundle $\xi_b : E_b \rightarrow S^4$. The map $b \mapsto \xi_b$ is a 1–1 correspondence [Wa62, Lemma 1]. Take the 1–1 correspondence between S^3 -bundles over S^4 and $\mathbb{Z} \oplus \mathbb{Z}$ constructed in [Mi56]. This gives an alternative construction of the above 1–1 correspondence.

The map assigning to b the diffeomorphism class of the total space E_b is a bijection. The inverse is given by $E \mapsto \left(\frac{2a_E \cap a_E - p_E \cap a_E}{4}, \frac{2a_E \cap a_E + p_E \cap a_E}{4} \right)$, where $a_E \in H_4(E)$ is the generator and we use the above 1–1 correspondence between the set of framings and $\mathbb{Z} \oplus \mathbb{Z}$.³⁵

³⁵The map assigning to b the diffeomorphism class of the total space ∂E_b is *not* a bijection (although the restriction of such a map gives a 1–1 correspondence between *unlinked* framed embeddings and diffeomorphism classes of total spaces of trivial Euler class bundles) [CE03].

Framed embeddings b corresponding to pairs $(a, -a)$ are characterized by being *unlinked* (i.e. such that the linking coefficient of $b(S^3 \times 0)$ and $b(S^3 \times x)$ is zero).

An isotopy F from an embedding $S^3 \rightarrow S^7$ to the standard embedding is not necessarily unique up to isotopy (of isotopies relative to the ends). So apriori we cannot just take as the ‘zero’ framing the image of the standard framing of the standard embedding under such an isotopy F . However, the above argument shows that we can.

An alternative proof of the Agreement Lemma.

The Agreement Lemma is an analogue of [Sk08', the Agreement Lemma]. For $H_1(N) \neq 0$ this analogue is more complicated because embeddings $N_0 \rightarrow \mathbb{R}^7$ are not necessarily isotopic.

A section $\xi : N_0 \rightarrow \partial C_f$ is called *faithful* if $\xi^! \partial A_f = 0$. When $H_2(N)$ has no torsion, this is equivalent to the triviality of the composition $H_2(N_0) \xrightarrow{\xi_*} H_2(\partial C_f) \xrightarrow{i_*} H_2(C_f)$.

Faithfulness is not equivalent to unlinkedness because in general $AD_{f|_{N_0}} \bar{\xi}_* \neq f|_{N_0}^! AD_{\bar{\xi}}$.

The Agreement Lemma is implied by the following result.

Faithful Section Lemma. (a) *A faithful section exists. It is unique on 2-skeleton of N up to fiberwise homotopy. [HH63, 4.3, BH70, Proposition 1.3].*

(b) *Under the assumptions of the Agreement Lemma φ maps a faithful section to a faithful section.*

Part (a) is implied by the following result.

Difference Lemma. $d(\xi, \eta) = \xi^! \partial A_f - \eta^! \partial A_f$.

This follows because

$$(\xi^! - \eta^!) \partial A_f = (\xi^! - \eta^!) \partial A_f \cap \nu^{-1} N_0 = (\xi^! - \eta^!) [\zeta N_0] = d(\xi, \zeta) - d(\eta, \zeta) = d(\xi, \eta).$$

Proof of the Faithful Section Lemma (b). Recall the equality on $\pm 2d(\xi, \eta)$ from the proof of the Agreement Lemma in §3. Let ζ be an unlinked section for f . Then for a faithful section ξ for f we have

$$PDe(\zeta^\perp) - PDe(\xi^\perp) = 2d(\zeta, \xi) = 2(\zeta^! - \xi^!) \partial A_f = 2\zeta^! \partial A_f = 2PDe(\zeta^\perp).$$

Here the first equality holds by the equality on $\pm 2d(\xi, \eta)$;

the second equality holds by the Difference Lemma,

the third equality holds because ξ is faithful,

the fourth equality holds by (the second equality of) the Section Lemma.

Since $H_2(N)$ has no 2-torsion, together with the equality on $\pm 2d(\xi, \eta)$ this implies that *a section $\xi : N_0 \rightarrow \partial C_f$ is faithful if and only if $PDe(\xi^\perp) = -PDe(\zeta^\perp)$.*

Now the lemma follows by the Section Lemma because $e((\varphi\xi)^\perp) = e(\xi^\perp)$. \square

We conjecture that $BH(f) - BH(f') = 2W_{f'}(f)$ for the *Whitney invariant* $W_{f'}(f)$ [Sk08, §2]. For simply-connected N the proof is analogous to [Sk08', §3].

The following assertion is proved analogously to [Sk08', the Difference Lemma (c)] (where A_0 is defined).

If $f = f'$ on N_0 and $\xi : N_0 \rightarrow \partial C_f$ is a section both for f and f' , then $W(f) - W(f') = A_0(\bar{\xi}_ - \bar{\xi}'_*)[N]$, where $\bar{\xi}'$ is constructed from ξ and f' .*

This assertion gives an alternative proof of the following statement used in the proof of the Agreement Lemma: *if $BH(f) = BH(f')$ and $H_1(N) = 0$, then any isomorphism maps an unlinked section of f to that of f' .*³⁶

³⁶If a section $\xi : N_0 \rightarrow \partial C_f$ is strongly unlinked, then it is faithful. If N is simply-connected, then the converse also holds because $N_0 \simeq \vee S_i^2$. If a section $\xi : N \rightarrow \partial C_f$ is strongly unlinked, then its restriction to N_0 is both faithful and unlinked, hence $BH(f) = 0$ by the italicized assertion in the proof of the Faithful Section Lemma (b). The same assertion implies that *for simply-connected N the existence of a strongly unlinked framing of ν_0 is equivalent to $BH(f) = 0$ (and hence to the compressibility of f).* Here the simply-connectedness assumption is essential: take an embedding $(S^1 \times S^3)_1 \# (S^1 \times S^3)_2$ such that $(x \times S^3)_1$ and $(x \times S^3)_2$ are linked, then for any section $\xi : N_0 \rightarrow \partial C_f$ we have $\xi^* in^* \neq 0 \in H^3(N_0)$. If ν is trivial, then the obstruction to extending a section $\xi : N_0 \rightarrow \partial C_f$ to N is $(\xi^! \partial A_f)^2 \in \mathbb{Z}$. Thus unlinked or faithful section on N_0 extends to N if and only if $BH(f) = 0$.

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