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# On the global isometric embedding of pseudo-Riemannian manifolds

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It is shown that any pseudo-Riemannian manifold has (in Nash's sense) a proper isometric embedding into a pseudo-Euclidean space, which can be made to be of arbitrarily high differentiability. The application of this to the positive definite case treated by Nash gives a new proof using a Euclidean space of substantially lower dimension. The general result is applied to the space-time of relativity, and the dimensions and signatures of the spaces needed to embed various cases are evaluated.

## 1. Definitions and classical results

Positive definite Riemannian manifolds have historically been approached from two viewpoints: either their properties were defined intrinsically, or they were regarded as subsets of a Euclidean space of higher dimension. Thanks to the work of Nash (1954, 1956) and Whitney (1936), it has been known for some time that these approaches were equivalent, in the sense that any intrinsically defined Riemannian manifold can be embedded, with appropriate differentiability, in a Euclidean space. The aim of this paper is to show that the same situation holds in the case of pseudo-Riemannian spaces, with metrics of indefinite signature.

We shall deal entirely with a pseudo-Riemannian, $m$-dimensional, $C^\infty$, Hausdorff, separable manifold (without boundary), $M$. A general pseudo-Euclidean space will be denoted by $E^{pq}$, defined as $R^q$ with a covariant metric $\eta$ whose components in the natural coordinates are

$$\eta_{\alpha\beta} = \begin{cases} \delta_{\alpha\beta} & (p+1 \leqslant \alpha \leqslant q), \\ -\delta_{\alpha\beta} & (1 \leqslant \alpha \leqslant p). \end{cases}$$

Throughout Greek indices between $\alpha$ and $\epsilon$ will run from 1 to $q$. In addition we write $E^q$ for $E^{0,q}$.

A map $f$: $M \to E^{pq}$ will be termed a $C^k$ *embedding* if

(i) $f$ is of differentiability class $C^k$ ($k \geqslant 1$) and $df$ has rank $m$ at all points in $M$;

(ii) $f$ is one-one;

(iii) the limit set of $f$ has a null intersection with $fM$. Conditions (ii) and (iii) ensure that the embedding is topological, (iii) defining what Nash termed a proper embedding. 'Metric' will mean '(pseudo)-Riemannian metric' throughout. Other notation is standard, and may be found in, for example, Kobayashi & Nomizu (1963).

Such an $f$ will induce a metric $g(f)$ on $M$ given by

$$g(f)(X, Y) = \eta(f_* X, f_* Y). \tag{1}$$

We shall need to consider sums and direct products of embeddings defined by

$$(f_1+f_2)(x) = f_1(x)+f_2(x) \tag{2a}$$

$$(f_1\times f_3)(x) = (f_1(x), f_3(x)), \tag{2b}$$

where
$$f_1, f_2\colon M \to E^{pq},$$
$$f_3\colon M \to E^{rs},$$

and the sum in (2 $a$) is taken by regarding $E^{pq}$ as a vector space. From (1) we have

$$g(f_1\times f_2) = g(f_1)+g(f_2). \tag{3}$$

Metrics on $M$ will be partially ordered by setting

$$g_1 < g_2 \quad \text{for} \quad g_1(X,X) < g_2(X,X),\ \forall X\in T(M).$$

On occasion 0 will be used to denote the zero metric.

We first turn to the problem of inducing by an embedding a given positive definite metric, which can be approached as follows.

Lemma 1. $M$ can be embedded in $R^{2m+1}$ by a map $f$ whose limit set is the origin.

*Proof.* A method can be found in Milnor (1958). There, $M$ is mapped differentiably into $R$, which we may do so as to yield the origin as limit set, and $R$ is then regarded as a subset of $R^{2m+1}$, when the map can be modified to an embedding. Our requirement for the limit set is easily accommodated.

Lemma 2. $M$ can be embedded in $R^{2m}$.

*Proof.* See Whitney (1944). The proof cannot usefully be abbreviated here.

Lemma 3. If $g$ is a positive definite metric on $M$ then there is an embedding $f'$ of $M$ into $E^{2m}$ for $M$ compact, or into $E^{2m+1}$ for $M$ non-compact, such that

$$g(f') < g.$$

*Proof.* (i) $M$ compact. Construct $f$ as in the previous lemma, then put $f' = \alpha f$, where $\alpha$ is a sufficiently small positive constant. (ii) $M$ non-compact. Construct an embedding $f$ as in lemma 1. Then the function

$$\rho(r) = \inf\left\{\frac{g(X,X)}{g(f)(X,X)}\middle|\, \exists x,\ X\in T_x(M),\ \|f(x)\| \geqslant r\right\} \tag{4}$$

is defined, since $\{x|\,\|f(x)\| \geqslant r\}$ is compact. If now we map $E^{2m+1}$ into itself by transforming the radial coordinate according to

$$r\mapsto \int_0^r \rho(r')\,\mathrm{d}r'$$

we obtain a so-called 'short embedding' satisfying the lemma.

This result is also a consequence of a theorem of Kuiper (1955) that any embedding can be transformed into an embedding into a space of one higher dimension which satisfies lemma 3.

We now define a second fundamental form for an embedding $f\colon M \to E^{pq}$ as follows. Let $\underset{\sigma}{X}$ be a normal basis of local cross sections of the normal bundle over $f(M)$, $\nu$ (a basis which exists if $g(f)$ is non-degenerate). Define $\Omega\colon T(M) \to \nu$ by

$$\Omega(A) = \sum_{\sigma} \underset{\sigma}{X}\, \eta(\underset{\sigma}{X}, \underset{\sigma}{X})\, \eta(\underset{\sigma}{X}, \nabla_{f_* A} f_* A), \tag{5}$$

where $\nabla$ denotes covariant differentiation in $E^{pq}$. If $\Omega$ is everywhere two–one we call $f$ *perturbable*; in coordinate form, taking coordinates $z^\alpha$ in $E^{pq}$ and local coordinates $x^a$ in an open set $U$ in $M$, a perturbable embedding is one such that the map $\omega$ from $\nu$ to the space of symmetric covariant tensors on $M$ given by

$$\omega\colon \zeta \mapsto -2\eta_{\alpha\beta} \frac{\partial^2 z^\alpha(f(x))}{\partial x^a\, \partial x^b} \zeta^\beta\, \mathrm{d}x^a\, \mathrm{d}x^b$$

is onto.

We now define a $C^k$-neighbourhood of $g$ to mean an open collection of metrics whose derivatives of orders from zero to $k$ are close to those of $g$. If $S$ denotes the bundle of symmetric covariant second-rank tensors over $M$, we recall that a metric may be regarded as a cross-section of (i.e. a map from $M$ into) $S$. There is a natural map $\theta$ from the vector space of $C^k$ cross-sections of $S$ into the space of $C^k$ cross-sections of the $k$-jet bundle over $S$. Then the *$C^k$-topology* on the space of metrics can be defined formally by requiring a neighbourhood of $g$ to be a set of metrics whose image under $\theta$ forms an open neighbourhood of $\theta g$.

In the appendix we prove the following lemma due, in a weaker form applying only to compact manifolds, to Nash (1956):

LEMMA 4. If $h\colon M \to E^{ps}$ is perturbable and is $C^{k+2}$, then there is a $C^k$-neighbourhood $A$ of $g(h)$ (providing $k \geqslant 3$) such that for any $g$ in $A$ there is a perturbable embedding $h'\colon M \to E^{ps}$ with $g(h') = g$.

LEMMA 5. If $g$ is a positive definite metric on $M$, then there exists an embedding $h\colon M \to E^q$, where $q = \frac{1}{2}m(n+S)$, such that $h$ is perturbable and $g(h) < g$.

*Proof*. For $M$ compact see Nash (1956). The outline of the argument as applied to the general case can be given. From the methods of lemmas 1 and 3 there is an immersion $f$ of $M$ in $E^{2m}$ such that $g(f) < g$. Let $v^i$ $(i = 1, 2, \ldots, 2m)$ denote coordinates in $E^{2m}$ and put

$$\omega^\sigma = D^\sigma_{ij} v^i v^j \quad (\sigma = 1, 2, \ldots, p),$$

where the $D^\sigma$ are constant symmetric $2m \times 2m$ matrices. We choose them so that they span a $p$-dimensional subspace $J$ of the set $N$ of all such matrices. For coordinates $x^a$ about a fixed point $x$ of $M$ there is a linear map $b_x\colon D^\sigma_{ij} \mapsto \partial^2\omega^\sigma/\partial x^a\, \partial x^b$ from $N$ into the space of symmetric tensors of rank two at $x$. Then $b_x J$ will be the whole of this space if the dimension of the intersection of $J$ with the kernel of $b_x$ is not greater than $p - \frac{1}{2}m(m+1)$, a condition which fails only on a set of $J$'s of dimension

$u = p(2m^2+m-p-1)+\frac{1}{2}m(m+1)-1$. If now we let the point $x$ range over all of $M$, so that each point of $M$ determines such a set, then the set of $J$'s for which there is a $b_x$ such that $b_x|J$ is not onto is of dimension not greater than $m+u$; we use the fact that $f$ is an immersion and $M$ is separable. If $p \geqslant \frac{1}{2}m(m+3)$, this set will be of smaller dimension than the set of all $J$, and so we can choose a $J$ on which $b_x$ is onto for all $x$ in $M$. We see that there is sufficient freedom in our choice of $J$ for us to be able to choose $\frac{1}{2}m(m+5)$ independent $D$'s such that any $\frac{1}{2}m(m+3)$ of them define a $J$ of this sort. It can now be verified that this means that any immersion of the form

$$z^\sigma = C^\sigma_i w^i + D^\sigma_{ij} w^i w^j \quad (\sigma = 1, 2, \ldots, \tfrac{1}{2}m(m+5))$$

(where the $D$'s are as discussed above, and the $C$'s are arbitrary) will be perturbable. It will also be seen that this result is valid for any $C^2$ manifold, whether compact or not, since the argument does not actually depend on any algebraic assumptions. Our lemma follows on taking $f$ to satisfy the intersection conditions for a 'regular' immersion (no triple or higher intersections, and the two tangent spaces intersecting in $\{\mathbf{0}\}$ at double intersections) as well as $g(f) < g$. Our new immersion will also satisfy the condition on the induced metric if we take $C$ and $D$ small enough, and we may then alter it by an arbitrarily small amount to make it an embedding while retaining perturbability.

Now Kuiper (1955) gives a construction whereby a $C^1$ isometric embedding can be derived from the $f'$ of lemma 3. Such an embedding is 'pathological' in that in general its second derivative exists nowhere; this is because the embedding is defined as the limit of a sequence of alterations of $f'$, changed in one coordinate patch at a time, which converges only in its first derivative. However, by terminating the process so that at any point of $M$ an embedding $f''$ is obtained from $f'$ by a finite number of alterations, we obtain a map that is still smooth. Thus we have

Lemma 6. If $g$ is a positive definite metric on $M$ and if $G$ is a $C^0$-neighbourhood of $g$, then there is a map $f''\colon M \to E^{2m}$ ($M$ compact) or $M \to E^{2m+1}$ ($M$ non-compact) such that $g(f'') \in G$.

Theorem 1. Any $C^\infty$ manifold $M$ with positive definite $C^k$-metric $g$ can be $C^k$-isometrically embedded in $E^q$ provided $k \geqslant 3$ and $q \geqslant \frac{1}{6}m(2m^2+37)+\frac{5}{2}m^2+1$.

*Proof.* We start with the embedding $h$ of lemma 5. If $k$ is any other differentiable map $M \to E^r$, and if we put $l = h \times k$, then $l$ will also be perturbable and so, by lemma 4, there will be a $C^k$-neighbourhood of $g(l)$, for any $k \geqslant 3$, which can be reached by isometric embeddings. If $g$ itself lies in this neighbourhood, we shall call $l$ *nearly isometric*. We proceed to construct $k$ as a product of mappings by triangulating $M$ and using induction to make our embeddings isometric on simplexes of successively higher dimension.

Proposition. Let $s$ be an (open) $r$-simplex of the triangulation and let $l_r = h_r \times k_r$ where $k_r\colon M \to E^{q_r}$ is a $C^k$ immersion and $h_r$ (which will be derived from $h$ by a succession of applications of lemma 4) is a $C^k$ perturbable embedding. Suppose that $g = g(l_r)$ on a neighbourhood $B$ of $\partial s$ and that $g(l_r) < g$ elsewhere on $s$. Let $A$ be any closed set disjoint from $s$. Then there exists a $C^\infty$ map $k_r'$: $M \to E^{q_r+r^2+2m-r}$ such that:

(i) $k_r \times 0 = k_r'$ on an open neighbourhood $K$ of $\partial s \cup A$ (where $k_r \times 0$ is simply $k_r$ regarded as an embedding into the first $q_r$ coordinates of $E^{q_2+r^2+2m-r}$);

(ii) $l_r' \equiv h_r' \times k_r'$, for some perturbation $h_r'$ of $h_r$, is nearly isometric on an open neighbourhood of $\bar{s}$.

(iii) $g(l_r') < g$ outside $A$ and this neighbourhood.

*Proof of proposition.* Choose Euclidean coordinates in an open set $U$ of $M$ containing $\bar{s}$ such that on $s$

$$x_p = 0,$$

and $$(g - g(l_r))_{ap} = \delta_{ap} \quad \text{for} \quad r < p \leqslant m$$

and $$1 \leqslant a, b \leqslant m.$$

The $\frac{1}{2}r(r+1)$ functions

$$\{v^\alpha \,|\, \alpha = 1, 2, \ldots, \tfrac{1}{2}r(r+1)\} = \{\tfrac{1}{2}(x^i + x^j) \,|\, 1 \leqslant i \leqslant j \leqslant r\}$$

are such that the tensors over $s$

$$K^\alpha = \frac{\partial v^\alpha}{\partial x^i} \frac{\partial v^\alpha}{\partial x^j} \mathrm{d}x^i \, \mathrm{d}x^j$$

form a basis for all symmetric tensors over $s$.

For small enough $a(x) \geqslant 0$ we will have

$$g - a(x) \sum_\alpha K^\alpha \geqslant g(l_r).$$

So, by lemma 6, we can alter $k_r$ to $k_r''$ so that

$$g - g(h) - g(k_r'') = \sum_\alpha a_\alpha(x) K^\alpha$$

for some non-negative $C^k$ functions $a_\alpha(x)$. No alteration of $k_r$ is necessary on $B$, where we shall have $a_\alpha = 0$. We extend the $a_\alpha$ to the whole of $U$ by applying the same procedure to each surface $x^r =$ const. as we have applied to $s$.

Now put $$\rho(x, y)^2 = (x^{r+1} - y^{r+1})^2 + \ldots + (x^m - y^m)^2,$$

and let $L(t)$ be the set $\{x | \rho(s, x) \leqslant t\}$. Choose $t_2$ and a neighbourhood $K$ of $A \cup \partial s$ so that $K \cap L(t_2) \subset B$, and then construct a $C^\infty$ function $\xi(x)$ so that

$$\xi(x) = 1 \quad x \in L(t_1) \backslash B,$$
$$\xi(x) = 0 \quad x \in K \backslash L(t_2),$$

for some $t_1$, $t_2 > t_1 > 0$. We can now make our embeddings nearly isometric on $\bar{s}$ by defining a map $f$: $M \to E^{r^2+2m-r}$ which makes up the deficit $g - g(h) - g(k_r'')$ on $\bar{s}$. $f$ takes $\{x^\alpha\}$ into a point with coordinates

$$y^a = \begin{cases} C(\xi a_a)^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \sin \lambda v^a & (1 \leqslant a \leqslant n \equiv \frac{1}{2}r(r+1)), \\ C(\xi a_{a-n})^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \cos \lambda v^{a-n} & (n+1 \leqslant a \leqslant 2n), \\ C(\xi)^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \sin \lambda x^{a-2n+r} & (2n+1 \leqslant a \leqslant 2n+m-r), \\ C(\xi)^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \cos \lambda x^{a-r^2-m+r} & (r^2+m \leqslant a \leqslant r^2+2m-r). \end{cases}$$

Then $h \times f \times k_r''$ induces the metric $g' + \bar{g}/\lambda^2$, where $\bar{g}$ is independent of $\lambda$ and $g' = g$ on $\bar{s}$, with $g'_{ij} = g_{ij}$ $(i,j = 1, 2, \ldots, r)$ on $L(t_1)$. We shall be able to make $\bar{g}/\lambda^2$ and any finite number of its derivatives arbitrarily small be taking large enough $\lambda$, and so obtain a nearly isometric embedding, eventually. We can be more specific about the perturbation of $h$ referred to above, on recalling that the $C^k$-topology for scalar or tensor functions on a compact domain is metric. Denote such a metric by $d(\,,\,)$. Then an embedding $l$ will be nearly isometric if

$$d(g(l), g) < \epsilon,$$

It is now possible to choose $C$ so that $g - g'$ is positive definite in a neighbourhood of $s$, say in $L(t_3)$, with

$$d(g', g) = \epsilon_3 < \epsilon.$$

Let $g^*$ denote the value of $g'$ with this $C$ and with $t_1 = \infty$ (i.e. with the factor $\xi$ omitted from the definition of $f$).

To make the embedding nearly isometric on $L(t_1)$ we now alter $h$ so as to change $g$ on a neighbourhood containing $L(t_1)$, using the methods of Janet (1926), Leichtweiss (1956), and Friedman (1961). We merely indicate the necessary modifications to these methods.

Let $S_u = \{x_p = 0 | p > u\}$. We proceed by induction defining a new embedding

$$z^\alpha = h_r^{\dagger\alpha} + \zeta^\alpha,$$

where $h_r^\dagger$ is an analytic approximation to $h_r$ with $d(h_r^\dagger, h_r) < \epsilon_2$ in a $C^{k+1}$ topological metric. The inductive step is to extend $\zeta$ from a submanifold $S'_u$ of $S_u$ to a submanifold of $S_{u+1}$, with $S'_r = s$. We try to achieve by the embedding $z^\alpha$ a metric $g^\dagger$ which can be chosen analytic, positive definite, and with

$$d(g^\dagger, g + g(h_r) - g^*) < \epsilon_1.$$

If we define

$$E_{ab} = g^\dagger_{ab} - z^\alpha{}_{,a} z_{\alpha,b},$$

then the equations to be satisfied, namely

$$E_{ij} = E_{(u+1)i} = E_{(u+1)(u+1)} = 0 \quad (i,j = 1, 2, \ldots, u)$$

can be written in the form of Janet's equations (2) to (7), where they are separated into a set of initial conditions and a set of differential equations for (in our case) the unknowns $\zeta$. Since $h^\dagger$ is perturbable (for small enough $\epsilon_2$) these can be solved by the methods of Leichtweiss, keeping the embedding $z^\alpha$ perturbable throughout each stage of the induction, including the last $(u = m)$.

We now have a map

$$h'_r = h_r + \zeta$$

defined in a neighbourhood $Z$ of $s$, which, provided we take $\epsilon_1$ and $\epsilon_2$ small enough, can be made to satisfy

$$d(g^* - g(h_r) + g(h'_r), g) < \epsilon_3$$

for any $\epsilon_3$. If now we restrict the domain of $\zeta$ to $L(\delta)$, then for small enough $\delta$ there will be an $\eta$ for which we can extend $\zeta$ over $M$ in a $C^\infty$ manner in such a way that $h_r + \zeta$ is still perturbable and that, for small enough $\epsilon_3$,

$$g^* - g(h_r) + g(h'_r) - g(f) < g$$

in $L(\eta)$. Further, we can arrange for $\eta < t_3$.

Now if we choose $t_2 < \delta$ and $t_1 < t_2$, so that the part $g(f)$ of our metric goes to zero while $\zeta$ is still giving the metric $g^\dagger$, we see that all the conditions of the proposition are satisfied by $h'_r$ as defined above and by $k'_r = k''_r \times f$, with $\lambda$ taken sufficiently large.

Theorem 1 now follows by induction over $r$. We start with $l_0$ as the product of the perturbable embedding $h$ of lemma 5 and any immersion into $E^{2m+1}$ (for $M$ non-compact; see lemma 3) such that

$$g(l_0) < g$$

everywhere. After $r$ steps of application of the proposition, we can make the embedding isometric in a neighbourhood of all the $(r-1)$-simplexes. The proposition can now be applied to each $r$-simplex in turn to get a nearly isometric embedding in a neighbourhood of $r$-simplexes: we then perturb $h$ by lemma 4 to get an isometric embedding in a neighbourhood of $r$-simplexes.

## 2. Relativity and other indefinite cases

Consider now the case of interest in general relativity, where the metric on a 4-manifold $M$ (space-time) has one negative and three positive eigenvalues, and may not unreasonably be taken as of differentiability class $C^k$ with $k = 3$. It has become clear from the work of Hawking (1967), Geroch (1967) and others that physically vital questions about the nature and existence of singularities in the universe cannot be separated from considerations of the global properties of such a manifold; the most important of these are causal properties, that is, those derived from the relation between points of 'connectability by a smooth timelike curve'. Now causal properties are related to the possibility of certain types of embedding: if an $M$ is smoothly embedded in a normally hyperbolic pseudo-Euclidean space $E$, then since $E$ can contain no closed time-like curves, neither can $M$. Further, it can easily be shown by the methods given below that such an $M$ is stably causal (defined later) and that, conversely, any stably causal space-time is conformally related to a space-time which can be so embedded. Thus it is likely that the investigation of embedding might shed some light on these 'global' features.

Our method will be to write

$$g = g_1 + g_2,$$

where $g_1$ is a positive definite and $g_2$ a negative semi-definite metric. Each $g_i$ will be induced by a separate map $f_i$, $f_1$ being given by theorem 1 of the previous section. $g$ is then induced by the embedding $f_1 \times f_2$, as explained in the proof of equation (3) above. We now proceed to determine the map $f_2$.

LEMMA 7. Let $M$ be a $C^k$ pseudo-Riemannian manifold whose metric $g$ has rank $m$ and signature $s$. Then there exists a map $f: M \to E^{pp}$, where $p = \frac{1}{2}(m-s)+1$, such that $g(f) < g$.

*Proof*. Consider first the case of general relativity, where $s = m-2$. Take an arbitrary map $\tilde{f}: M \to E^{11}$ of rank one. In a sufficiently small neighbourhood of any point $x$ of $M$ we can choose coordinates $(x^1, x^2, \ldots, x^m)$ with $\partial/\partial x^m$ time-like and $\tilde{f}_*(\partial/\partial x^m) \neq 0$. We now use Kuiper's (1955) method to modify this map to induce a metric less than $g$. Let $S_\lambda$ be the curve $\{(x,y) | y = \sin \lambda x\} \subset R^2$, and let $\chi: R^1 \to S_\lambda$ be the $C^\infty$ map for which

$$\chi(n\pi) = (n\pi, 0),$$
$$\|d\chi\| = \text{constant}.$$

Cover $M$ by neighbourhoods $\{U_i\}$ of the above form and let $\{\xi_i\}$ be a $C^\infty$ partition of unity subordinate to this covering. Write $\tilde{f}(x) \equiv \tilde{f}(x^a, x^m)$ $(a = 1, 2, \ldots, m-1)$ and consider the map

$$k(x) = \begin{cases} \tilde{f}(x) & x \notin U_1 \\ \tilde{f}(x^a, x^m(1-\xi_1) + \xi_1 \chi^1(x^m)) + (0,1)\chi^2(x^m)\xi_1 & \text{otherwise.} \end{cases}$$

where we have regarded $E^{11}$ as the 1-axis of $E^{22}$ and denoted components of $\chi$ by superscripts. We can verify that for large enough $\lambda$ this map induces a metric less than $g$ on $M$. Further, we can repeat the process on each domain $U_i$, using the altered normal vector to $\tilde{f}_*(\partial/\partial x^m)$ in place of $(0, 1)$. The argument can easily be repeated in the general case. Take an arbitrary map $\tilde{f}$ of maximal rank from $M$ to $E^{dd}$ where $d = \frac{1}{2}(m-s)$ is the number of negative eigenvalues of the metric of $M$. The kernel of $d\tilde{f}_x$ for any point $x$ of $M$ has dimension $m-d$ and so we can choose a $d$-dimensional purely time-like plane in $T_x(M)$, that is, one with negative definite metric, which is mapped topologically by $d\tilde{f}_x$. Thus, as before, we can choose coordinates about $x$, $(x^1, x^2, \ldots, x^m)$ with $\partial/\partial x^l$ $(l = 1, 2, \ldots, d)$ timelike and $\tilde{f}_*(\partial/\partial x^l) \neq 0$ Kuiper's process may now be applied to each timelike coordinate in the same way as for the case $d = 1$.

We can now give the main theorem of this paper.

THEOREM 2. Any $m$-manifold $M$ with $C^k$ pseudo-Riemannian metric $(k \geqslant 3)$ of rank $r$ and signature $s$ can be embedded in $E^{p, p+q}$ provided that

$$p \geqslant m - \tfrac{1}{2}(r+s) + 1 \quad \text{and} \quad q \geqslant \tfrac{1}{2}m(3m+11)$$

for compact $M$ or $q \geqslant \frac{1}{6}m(2m^2+37) + \frac{5}{2}m^2 + 1$ for non-compact $M$.

*Proof*. We can write $g = g_0 + g'$ where $g_0$ is of rank $m$ and signature $r+s-m$, while $g'$ is of rank $a \geqslant m-r$ and signature $a$. Let $f_1$ be the map given by lemma 7 such that $g(f_1) < g_0$. Then let $f_2$ be the embedding given by theorem 1 such that $g(f_2) = g_0 - g(f_1) + g'$, or by Nash (1956) for $M$ compact.

Then $f_1 \times f_2$ is the required embedding.

*Corollary*. The space-time of general relativity can be embedded isometrically in

$E^{2,q+2}$ (pseudo-Euclidean space of signature $q-2$) where $q = 46$ or $q = 87$ for compact or non-compact space-time, respectively.

REMARK. For the case of general relativity the number of negative eigenvalues in the embedding metric is the best possible, since we know that there are space-times that cannot be embedded in an $E^{1,q+1}$, as described earlier.

To achieve $p = 1$, an embedding into 'Minkowski space', we need to restrict $M$ further. A space-time is said to be globally hyperbolic if it is time-orientable and if, for all $x, z \in M$, $J^+(x) \cap J^-(z)$ is compact, where $J^+(x)$ (resp. $J^-(x)$) is the set comprising $x$ together with all points accessible from $x$ by a positively (resp. negatively) oriented non-space-like curve (that is, one whose tangent vector has non-positive length-squared). Such a space-time is *stably causal*: its metric has a $C^0$-neighbourhood in which all metrics are causal. Hence it has a global time function $\tau$ which increases on any timelike or null curve (Hawking 1969); this function has, for a globally hyperbolic space-time, the property that the sets $\tau^{-1}(t) \cap J^{\pm}(x)$ and $J^{\pm}(\tau^{-1}(t)) \cap J^{\pm}(x)$ are compact (Geroch 1969).

LEMMA 8. If $M$ is a globally hyperbolic space-time, then there exists a map $f: M \to E^{11}$ such that $g(f) < g$.

*Proof*. Scale $\tau$ so that $C = \tau^{-1}(0)$ exists. Put on $C$ a $C^\infty$ function $\sigma$ onto the positive reals so that the sets $\tau^{-1}([0,s])$ are compact, and extend $\sigma$ to $M$ by dragging along the trajectories of the vector field normal to the surfaces $\tau^{-1}(t)$. At any point with $\sigma \neq 0$, we may choose $m-2$ other coordinates with respect to which the metric may be written

$$g = -v^2\,\mathrm{d}\tau^2 + g_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \tag{6}$$

$A, B = 1, 2, \dots, m-1$; $x^1 \equiv \sigma$. If $h$ is the map into $E^2$ defined by $\tau, \sigma$ and $\phi$ is any map from $E^2$ to $E^{11}$, the map $f = \phi h$ will satisfy $g - g(f) > 0$ if

$$V^2\left[\left(\frac{\partial\phi}{\partial\sigma}\right)^2 + M^2\right] < M^2\left(\frac{\partial\phi}{\partial\tau}\right)^2, \tag{7}$$

where $M^{-2} = g^{11} = G(\mathrm{d}\sigma, \mathrm{d}\sigma)$, $G$ being the contravariant metric associated with $g$. Putting

$$A_{\pm} = MG(\mathrm{d}\tau) \pm VG(\mathrm{d}\sigma),$$

(7) can be written

$$(A_+\phi)\,(A_-\phi) > M^2V^2. \tag{8}$$

Rescale $\tau$ so that on $\sigma^{-1}([0,1]) \equiv Y$, say,

$$MV < 1. \tag{9}$$

Put

$$H^{\pm}(s,t) = J^{\pm}(\tau^{-1}(0)) \cap J^{\mp}(\tau^{-1}(t) \cap \sigma^{-1}([0,s])),$$

and define

$$f'(x) = \int_{H^+(\sigma(x),\,\tau(x))} \omega \quad \text{for} \quad \tau(x) > 0,$$

where $\omega$ is some positive volume element on $M$. It is easily seen that

$$A_+ f' > A_- f' > 1 \quad \text{outside} \quad Y, \tag{10}$$

provided that we choose $\omega$ big enough in $Y$. If now we also choose $\omega$ large enough on the rest of $M$ we can make $A_- f$ as large as we please and so satisfy (8) outside $Y$. In the same way we define $f'$ from $J^-(C)$ to the negative reals by considering $\int_{H^-(\sigma(x),\tau(x))} \omega$. Thus we have satisfied (8) for the whole of $M$ outside $C \cup Y$. Putting $f' = 0$ on $C \setminus Y$ makes $f'$ continuous there, and then we may smooth out $f'$ to a $C^\infty$ function $f$ while still satisfying (7). By our choice in (9) and (10) we can also extend $f$ in a $C^\infty$ way over $Y$ to give a function satisfying the lemma.

Corollary. A globally hyperbolic space-time can be embedded isometrically in $E^{1,q+1}$, provided $q$ is as in theorem 2.

*Proof.* As in theorem 2.

Finally, certain unsatisfactory features of the embeddings need to be pointed out. First, the embedding given by theorem 1 is only $C^k$, for finite $k$: there is no guarantee that, if $g$ is $C^\infty$, there exists a $C^\infty$ embedding. In a sense, we have only moved the 'pathology' of Kuiper's (1955) construction to a higher order of differentiability. Secondly, it is tempting to identify the limit set of the embedding, if any, with the 'singularity' of a geodesically incomplete space-time; if this were possible it would provide a useful way of looking at such objects. It will be seen, however, that the construction we have given produces a limit set of one point (see lemmas 1 and 3). This can always be avoided by suitable modifications of the embedding, but the possibility of such degenerate limit sets serves to emphasize the danger of this identification.

## Appendix. Proof of lemma 4

(*a*) *Smoothing.* Place on $M$ an arbitrary positive definite metric $h$ (to be restricted later). Let $\rho(z_1, z_2)$ be the infimum of the lengths of curves from $z_1$ to $z_2 \in M$, measured with $h$, and let $\alpha$ be a $C^\infty$ function from and into the positive reals such that

$$\alpha(x) = 0 \quad \text{for} \quad x > 1,$$

and

$$\int_0^1 \Omega_{m-1} r^{m-1} \alpha(r)\, \mathrm{d}r = 1,$$

where $\Omega_m$ is the area of the unit $m$-sphere. If $f$ is a real function on $M$, define a smoothed $f$ by

$$(S_\sigma f)(x) = \int f(y)\, \sigma^m \alpha(\sigma \,.\, \rho(x,y))\, \omega_h(y),$$

where $\omega_h$ is the volume element defined by $h$. The support of the integrand is the closure of

$$N_\sigma(x) = \{y \mid \rho(x,y) < 1/\sigma\}.$$

We choose $h$ so that this is always compact (assuming that $M$ is non-compact) by covering $M$ with a sequence of compact sets $M_i$ such that $M_i \subset \operatorname{int}(M_{i+1})$ and then scaling $h$ so that

$$N_1(M_i) \equiv \{y \mid \exists\, x \in M_i \quad \text{such that} \quad y \in N_1(x)\} \subset M_{i+1}.$$

Now take $\sigma$ to be a function of the integers increasing sufficiently rapidly to give

$$\bigcup_{r=1}^{\infty} N_r(N_{r-1}(\ldots(N_1(x))\ldots))$$

compact closure (where now we put $N_k$ for $N_{\sigma(k)}$ and similarly for $S$). Define also

$$(V_n^k f)(x) = \sup_{y \in N_n(x)} \max_{l \leqslant k} \{\langle \nabla^l{}_h f, \nabla^l{}_h f \rangle\}^{\frac{1}{2}}, \tag{11}$$

the pointed inner brackets denoting the usual contraction with metric tensors $h$, $\nabla_h$ being covariant differentiation with respect to $h$. We use $V$ to give orders of magnitude, so its exact definition is immaterial. Finally, we shall write $A(n) \lesssim B(n)$ in place of the usual '$O$' notation.

By analogy with the relation for the derivative of a convolution we obtain

$$\left| \frac{\partial}{\partial x^p} S_n f - S_n \frac{\partial}{\partial x^p} f \right| \lesssim \frac{B^p(x)}{\sigma(n)} V_n^{|p|-1} f + C^p(x) V_n^{|p|-2} f \tag{12}$$

for any multi-index $p$ and for any coordinate system about $x$ (the functions on the right depending on the coordinates). If $j$, $l$, and $n$ are such that $N_j(N_n(x)) \subset N_l(x)$ then it follows from the definitions that

$$(V_j^k S_n f)(x) \lesssim A^{k-h}(x) [\sigma(n)]^{k-h} (V_l^h f)(x) \tag{13}$$

and, for the same $j$, $l$, $n$,

$$(V_j^k (S_n f - f))(x) \lesssim \begin{cases} \dfrac{D^k(x)}{\sigma(n)} (V_l^{k+1} f)(x) + C^k(x) (V_l^{k-2} f)(x) & (k \geqslant 2) \\ \dfrac{D^k(x)}{\sigma(n)} (V_l^{k+1} f)(x) & (k < 2) \end{cases} \tag{14}$$

from (12).

(*b*) Now let $H: M \to E^{pq}$ be a perturbable embedding inducing a non-degenerate metric, so that the normal space at $x$ intersects the tangent space in $\{\mathbf{0}_x\}$. Let $z^\alpha$ denote coordinates in $E^{pq}$ and $x^a$ local coordinates in $M$. Then if we make the change $z^\alpha \to z^\alpha + \delta z^\alpha$ subject to

$$\eta_{\alpha\beta}\, \delta z^\alpha \frac{\partial z^\beta}{\partial x^a} = 0 \quad (\forall a), \tag{15}$$

the metric is changed by an amount

$$\delta g_{ab} = -2\eta_{\alpha\beta} \frac{\partial^2 z^\alpha}{\partial x^a\, \partial x^b} \delta z^\beta. \tag{16}$$

We shall now use iteration. At the $n$th stage, having reached an embedding $H_n$ with coordinates $z_n$ we put $g_n$ for $g(H_n)$ and set

$$\delta g_n = g - g_n. \tag{17}$$

This $\delta g$ determines a $\delta z$ by equations (15) and (16); it will be fixed uniquely if we also require, as in Nash's method,

$$\|\delta z_n\| \quad \text{is a minimum,} \tag{18}$$

where $\| \ \|$ is any $C^\infty$ norm in the normal bundle. The snag is that the change in the embedding that is required at any stage is determined through the second derivatives of the embedding. Thus changes will only converge if their second derivatives do. We circumvent this (still following Nash) by using relations (12) to (14) applied to a smoothed version of $H_n$. The actual change we make from $H_n$ to $H_{n+1}$ will be given by

$$z_{n+1} = z_n + S_{n+1}\,\delta z_n. \tag{19}$$

Now equation (18) does not alter the linearity of the system: it merely restricts $\delta z_n$ to an orthogonal complement of the null-space of the linear operator

$$L_n\colon X \mapsto -2\eta_{\alpha\beta}(\partial^2 z_n^\alpha/\partial x^a\,\partial x^b)\,X^\beta\,\mathrm{d}x^a\,\mathrm{d}x^b.$$

The condition of perturbability (that this operator is onto) implies that, after this restriction by (18), $L_n$ has a well-defined inverse. Thus we can replace (15), (16) and (18) by

$$\delta z_n = L_n^{-1}\,\delta g_n. \tag{20}$$

From (19) we compute the value of $\delta g_{n+1}$ in terms of $\delta z_n$; from this (20) gives us $\delta z_{n+1}$ in terms of $\delta z_n$. If we take $\sigma(n) = n^2$ then $N_{n+1}(N_{n+1}(x)) \subset N_n(x)$ so that we can apply (13), which, together with (14) gives expressions for the $V_n^r(\delta z_n)$ $(r = 1, 2, \dots, k)$ in terms of the $V_{n-1}^s(\delta z_{n+1})$. We can show that these expressions give an appropriately convergent process by using (13) to show that if the following two relations

$$(1)\quad \sum_{i=1}^{n} V_i^k(\delta z_i) < A,$$

$$(2)\quad V_n^r(\delta z_n) < H^r/n^2 \quad \text{for} \quad r < k$$

hold for $n \geqslant M$, where $M$ is a large enough integer, then they hold for $n = M+1$. (The constants $A$, $H$ and $M$ depend only on the point in the manifold at which the relations are evaluated.) The lemma now follows on observing that we can always make these relations hold as required by taking initial values of $\delta z$ and its derivatives sufficiently small.

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