FRIEDMANN-TYPE COSMOLOGICAL SOLUTIONS IN WESSON'S 5D SPACE-TIME-MASS THEORY OF GRAVITY

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Friedmann-type cosmological solutions in a 5D space–time–mass gravity proposed recently by Wesson are obtained for a zero-pressure perfect fluid. The behaviour of the solutions is discussed for three cases in which $k=+1, 0, -1$, respectively. It is found that the "mass scale-factor" $\mu(t)$, which characterizes the rest mass of a typical particle, is evolving with cosmic time just as the spatial scale-factor $a(t)$.

1. A number of researchers believe that the strength of gravity may change with time by some theoretical consideration (for example, Mach's principle, Dirac's large number hypothesis [1]). This has led to a series of attempts to construct variable-gravity theories. These theories were reviewed in 1978 by Wesson [2]. They are usually based on the idea that Newton's gravitational constant G changes with time. Recently, Wesson proposed an interesting 5D space–time–mass gravity in which the rest mass of a typical particle may change with time. Using dimensional analysis, Wesson introduced the fifth coordinate $x^4=(G/c^2)m$ (c is velocity of light and m is the rest mass) besides the 4D space–time coordinates, and extended Einstein's general relativity from the 4D space–time to the 5D space–time–mass directly. Thus, matter itself is brought into a geometrical formalism, and Wesson's theory contains Einstein's theory embedded within it.

It is useful to find and investigate solutions of the field equations in the 5D gravity theory to understand the meaning of the fifth dimension subspace and provide predictions which can be used to test the theory itself. Several authors have given some cosmological solutions in vacuum [6–9]. In this Letter, we will find Friedmann-type cosmological solutions for a zero-pressure perfect fluid and discuss them.

2. The concepts of coordinates and line-element are quite different, and a quantity having real geometrical meaning is just the latter. The line-element of the fifth dimension subspace is $\sqrt{-g_{44}}\,\mathrm{d}x^4$ (the signature is taken as $(-\ +\ +\ +\ -)$). It always has the dimension of length. However, x^4 can have arbitrary dimension because a coordinate transformation can always be made. This leads us to think that the rest mass should not be regarded as the fifth coordinate itself, but should be the "length" of the fifth dimension subspace. Thus, we can define

$$m=\frac{c^2}{G}\int_{x_4}^{x^4+\Delta x^4}\sqrt{-g_{44}}\,\mathrm{d}x^4\,, \tag{1}$$

without deviation from the spirit of Wesson's ideas. Eq. (1) means that a body occupies a finite "length" in the fifth dimension subspace, and the change of its rest mass is depicted completely by g_{44}. The rest mass of a body can be called "proper mass" when the gravitational field is absent, and it can be written as

$$m_0=\frac{c^2}{G}\Delta l_0\,. \tag{2}$$

In such a view, the behaviour of the rest mass of a typical particle in a gravitational field is very analogous to that of the length of a standard ruler in a gravitational field.

For some interesting problems, for example, the homogeneous and isotropic cosmological model which we will discuss in the following, it could be assumed that the fifth dimension subspace is homogeneous and isotropic. In this case, to choose a "mass comoving coordinate" with $u^4 = dx^4/d\tau = 0$ is possible, and is also convenient.

3. We assume that the cosmological principle could be extended to the 5D space–time–mass and choose comoving coordinates with $u_0 = 1$ and $u^\mu = 0$ ($\mu = 1$, 2, 3, 4) for $u^i = dx^i/d\tau$. In this way, we take

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2\right) - \mu^2(t)\,d\psi^2 \tag{3}$$

as the line-element of our model universe, where $k = +1, 0, -1$, and units are chosen such that $c = 1$. It is well known that $a(t)$ is a spatial scale-factor. Analogously, $\mu(t)$ may be called the mass scale-factor. The energy–momentum tensor for a perfect fluid is taken in the form suggested by Grøn [10],

$$T^i{}_j = \mathrm{diag}(-\rho, p, p, p, -p_4)\,, \tag{4}$$

with $p_4 = 0$. We restrict ourselves to the case $p = 0$. The 5D gravitational field equations $G_{ij} = -8\pi G T_{ij}$ read

$$3\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2} + 2\frac{\dot a\dot\mu}{a\mu}\right) = 8\pi G\rho\,, \tag{5}$$

$$2\frac{\ddot a}{a} + \frac{\ddot\mu}{\mu} + \frac{\dot a^2}{a^2} + \frac{k}{a^2} + 2\frac{\dot a\dot\mu}{a\mu} = 0\,, \tag{6}$$

$$\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2} = 0\,. \tag{7}$$

The covariant energy conservation law $T^{ij}{}_{;j} = 0$ gives the equation

$$\dot\rho + \rho\left(3\frac{\dot a}{a} + \frac{\dot\mu}{\mu}\right) = 0\,, \tag{8}$$

which can also be derived from eqs. (6)–(8).

Let us use t_0 to express the present time and denote $a_0 = a(t_0)$, $\mu_0 = \mu(t_0)$, $\rho_0 = \rho(t_0)$, and so on. It is well known that the present value of the Hubble constant H_0 and the decelarating parameter q_0 are defined by $H_0 = \dot a_0/a_0$ and $q_0 = -a_0\ddot a_0/\dot a_0^2$ respectively. Analogously, we introduce two new parameters h_0 and Q_0 by the definitions

$$h_0 = \dot\mu_0/\mu_0\,,\quad Q_0 = -\mu_0\ddot\mu_0/\dot\mu_0^2\,. \tag{9}$$

The field equations give the relationships between them as follows,

$$\frac{k}{a_0^2} = (q_0 - 1)H_0^2\,, \tag{10}$$

$$\rho_0 = \frac{3}{8\pi G}(q_0 H_0 + h_0)H_0\,, \tag{11}$$

$$q_0 H_0^2 + Q_0 h_0^2 - 2H_0 h_0 = 0\,. \tag{12}$$

The solutions of eqs. (6)–(8) are

$$a = [t(2C_1 - kt)]^{1/2}\,, \tag{13}$$

$$\mu = (C_2 t + C_3)a^{-1}\,, \tag{14}$$

$$\rho = \frac{3}{8\pi G}(C_1 C_2 + C_3 k)(C_2 t + C_3)^{-1}a^{-2}\,, \tag{15}$$

where C_1, C_2 and C_3 are integration constants. Substitution of the present values of the above quantities into the solutions gives

$$C_1 = a_0^2 q_0^{1/2} H_0\,,\quad C_2 = a_0\mu_0(H_0 + h_0)\,,$$

$$C_3 = a_0\mu_0\frac{q_0^{1/2}H_0 - h_0}{(q_0^{1/2}+1)H_0} \tag{16}$$

and

$$t_0 = (q_0^{1/2}+1)^{-1}H_0^{-1}\,. \tag{17}$$

Straightforward calculation gives

$$\dot a = (C_1 - kt)a^{-1}\,,\quad \ddot a = -C_1^2 a^{-3}\,,$$

$$\dot\mu = [(C_1C_2 + C_3 k)t - C_1 C_3]a^{-3}\,. \tag{18}$$

4. Now discuss the behaviour of the solutions for the three cases $k = +1, 0, -1$.

(i) $k = +1$ ($q_0 > 1$). In this case we have

$$a_0^2 = (q_0 - 1)^{-1}H_0^{-1}\,, \tag{19}$$

$$C_1 = q_0^{1/2}(q_0 - 1)^{-1}H_0^{-1}\,,$$

$$C_2 = \mu_0(q_0 - 1)^{-1/2}H_0^{-1}(H_0 + h_0)\,,$$

$$C_3 = \mu_0(q_0^{1/2}H_0 - h_0)(q_0 - 1)^{-1/2} \times (q_0^{1/2}+1)^{-1}H_0^{-2}\,. \tag{16'}$$

The universe is spatial-closed. Its radius $a(t)$ will reach its maximum

$$a_m = q_0^{1/2}(q_0-1)^{-1}H_0^{-1}$$

at

$$t=t_m = q_0^{1/2}(q_0-1)^{-1}H_0^{-1}\,.$$

The behaviour of the mass subspace depends on the value of h_0. If $h_0=-H_0$, then $C_2=0$ and $\mu(t)$ contracts when $a(t)$ expands, and vice versa. If $h_0=q_0^{1/2}H_0$, then $C_3=0$ and $\mu(t)$ increases monotonously with time. If $|h_0|<H_0$, $\mu(t)$ decreases with time from $t=0$ to

$$t=t_T=\frac{C_1C_3}{C_1C_2+C_3}\,,$$

and then increases with time.

(ii) $k=0$ ($q_0=1$). We obtain

$$C_1=a_0^2H_0\,,\quad C_2=a_0\mu_0(H_0+h_0)\,,$$

$$C_3=a_0\mu_0\frac{H_0-h_0}{2H_0}\,. \qquad (16'')$$

The universe spatially expands infinitely. For the mass subspace, $\mu(t)=\mu_0\sqrt{2H_0t}$ if $h_0=H_0$. If $|h_0|<H_0$, $\mu(t)$ decreases with time from $t=0$ to

$$t=t_T=\tfrac{1}{2}(H_0-h_0)(H_0+h_0)^{-1}H_0^{-1}\,,$$

and then increases with time. If we accept the result observed about the rate of change of Newton's gravitational constant, $|\dot{G}_0/G_0|<0.6\times10^{-11}$/yr [11], and take $|h_0|=\frac{1}{2}|\dot{G}_0/G_0|<0.03H_0$, we find $0.94t_0<t_T<1.06t_0$.

(iii) $k=-1$ ($q_0<1$). We have

$$C_1=\frac{q_0^{1/2}}{(1-q_0)H_0},\quad C_2=\frac{\mu_0(H_0+h_0)}{\sqrt{1-q_0}\,H_0}\,,$$

$$C_3=\frac{\mu_0(q_0^{1/2}H_0-h_0)}{\sqrt{1-q_0}\,(\sqrt{q_0}+1)}H_0^{-2}\,. \qquad (16''')$$

The conditions $C_2>0$, $C_3>0$, and $C_1C_2>C_3$, i.e. $q_0H_0>h_0$ must be satisfied because $\rho(t)$ must be positive. The behaviour of the solutions is in analogy to that in case (ii).

In brief, in our model universe the rest mass of a typical particle is evolving as well as the spatial space. Its evolution depends on the value of h_0. Besides, the observation of the mass decelarating parameter Q_0 could provide some useful information.

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