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For a 5 dimensional approach to Deformed Special Relativity

Florian Girelli

Perimeter Institute, 31 Caroline Street North Waterloo, Ontario, Canada N2L2Y5

E-mail: Tfgirelli@perimeterinstitute.ca

Abstract. DSR is a theory supposed to describe in an effective way the quantum gravitational fluctuations around flat space-time. I will first recall quickly why and then will argue that a consistent treatment of DSR should be done from a 5 dimensional perspective. I will provide arguments coming from a dynamical point of view, then from considering multiparticles states.

Introduction

The semi classical limit in Quantum Gravity (QG) is a hard task to achieve, but of course extremely important. There have been different proposals [2] but actually no effective progress since these proposals. It is however important to be able to define such limit as now some experiments will be given results in a very close future. This is the case for example for the GLAST and AUGER experiments. Hopefully QG will become real physics where one can do predictions and actually falsify or confirm them. QG phenomenology is of interest right now, and we need to have some models to bring in front of the experimentalists when they will present their results.

As deriving the QG semi classical limit is a hard task, we can proceed in a different way, that is construct it in an effective way by introducing some QG features in Special Relativity. The idea is to introduce one (or some) of the Planckian scales[1]: l_P, M_P, E_P in such a way that this is compatible with the symmetries. The clash is of course with the Lorentz symmetries, so it will be natural to modify or deform these latter[1]. We then obtain Deformed Special Relativity (DSR), that has been argued now in a number of different ways to be a natural semi classical limit of QG [3]. One then tries to understand this new DSR physics. There are actually a couple of problems that arise, in particular how to describe many particles. This problem can be solved when actually considering an extra dimension which meaning is to be determined. In fact from a 5d perspective, the different deformations also appear in a nice unified way.

In the first section I will recall the definition of DSR and its main problems. In the second section, I will show how the 5d approach solve the different problems. I will also argue about the meaning of this 5th dimension.

1. Definition of DSR

In general DSR is defined from a momentum perspective. The most natural point of view to see how DSR arises is from a non linear realization of the Lorentz symmetries, as introduced by

[1] Note that for M_P, we don't need to deform the symmetries [8].

[4]. One wants to introduce some momentum coordinates that are for example energy bounded by $E_P = \kappa$. These coordinates p can be obtained from usual coordinates π (that transform linearly) by a non linear map U_κ. For example we can take $p^\mu = \frac{\pi^\mu}{1+\frac{\pi_0}{\kappa}}$, or conversely $\pi^\mu = \frac{p^\mu}{1-\frac{p_0}{\kappa}}$. To proceed to do a deformed Lorentz deformation we undeform to get to the linear coordinates then proceed to the linear Lorentz transformation then deform back: $p' = U(\Lambda U^{-1}(p))$. If the deformation is nice enough we can modify the boosts generators in a very compact way, like here, we have simply $N_i \to N_i + \frac{p_0}{\kappa} x^\mu p_\mu$. Because of the deformation, we obtain a modified dispersion relation: $\pi_\mu \pi^\mu = m^2 = \frac{p_\mu p^\mu}{(1-\frac{p_0}{\kappa})^2}$.

This method doesn't tell us directly how space-time should be reconstructed. In [5] was proposed two ways: either send the deformation as such on space time to obtain an energy dependent metric, or deform adequately space-time.

Physics depends here on the choice of the deformation, as different deformations imply different dispersion relations, I shall come back on this point in the next section. Kowalski-Glikman nicely discovered that in fact the different approaches of DSR could be nicely unified from a geometric perspective [7]. The deformed space of momentum could be seen as different embedding of $\mathbf{R}^4$. This unified in the same way apparently different approaches, like Snyder's, but also gave a nice geometric meaning to the proposed addition of momenta.

The de Sitter space is given by $P_A P^A = -\kappa^2$, with the 5d metric $+----$. The different embeddings are then (respectively Snyder [8], Bicrossproduct basis [9], Magueijo-Smolin (MS))

$$P_\mu = U_\kappa(p_\mu) = \left(1 - p^2/\kappa^2\right)^{-\frac{1}{2}} p_\mu, \;\; P_4 = \left(1 - p^2/\kappa^2\right)^{-\frac{1}{2}} \tag{1}$$

$$P_0 = -\kappa \sinh \frac{p_0}{\kappa} - \frac{p^2}{2\kappa} \exp \frac{p_0}{\kappa}, \;\; P_i = -p_i \exp \frac{p_0}{\kappa}, \;\; P_4 = \kappa \cosh \frac{p_0}{\kappa} - \frac{p^2}{2\kappa} \exp \frac{p_0}{\kappa} \tag{2}$$

$$P_\mu = \frac{p_\mu}{1 - \kappa p_0}. \tag{3}$$

These 3 deformations reproduce the 3 different kinds of bounds one can put in momentum space: rest mass, 3d momentum, energy. The Casimir of the deformed symmetries can be read from the P_4 coordinates. Note that in the Snyder case there is no modification of the latter (that was precisely Snyder's goal)...

This geometric approach also allows to define space-time in a nice way: it is the tangent space of the de Sitter space. Depending on the different embedding we have different expressions for the vectors in the tangent space, vectors that are identified with the space-time coordinates. These vectors are of course some Lie algebra elements or combinations of then as de Sitter space can be seen as the coset $SO(4,1)/SO(3,1)$. In general there are then non commutative.

$$\textbf{Snyder}: \;\; x_\mu \equiv \frac{1}{\kappa} J_{\mu 4} = \frac{1}{\kappa}(X_\mu P_4 - X_4 P_\mu), \;\;\; \textbf{Bicrossproduct}: x_0 \equiv \frac{1}{\kappa} J_{04}, \;\;\; x_i \equiv J_{i4} + J_{i0} \tag{4}$$

Note that in the MS case, this approach doesn't reproduce the Rainbow space-time metric proposal, but in the bicrossproduct basis case, it does reproduce the (non commutative) space-time as constructed from the Heisenberg double [7].

After having introduced in a symmetry consistent way a fundamental constant, and defined the associated space-time, we need to see what happens when considering many particles. Following the geometrical approach we see that a momentum is given by a group element in the coset $SO(4,1)/SO(3,1)$, to add 2 momenta we can use the present group structure. This is very similar to the Special Relativity case where one uses the coset structure of the hyperboloid to define the relativistic addition of the speeds. Due to the coset structure, this addition is in general non commutative but also non associative. The bicrossproduct basis is actually "the" one where the addition is associative, but still non commutative. On the algebraic level this is equivalent to the existence of a coassociative coproduct and a quantum group structure.

$$\textbf{Bic.}: p_1^0 \oplus p_2^0 = p_1^0 + p_2^0,\ p_1^i \oplus p_2^i = p_1^i + e^{-\frac{p_1^0}{\kappa}} p_2^i;\ \textbf{Snyder}: e^{ip_1^\mu J_{4\mu}} e^{ip_2^\mu J_{4\mu}} = e^{\theta^{\mu\nu}(p_1,p_2) J_{\mu\nu}} e^{i(p_1 \oplus p_2)^\mu J_{4\mu}}. \tag{5}$$

For the Snyder case, the addition is non commutative and non associative, this being encoded in a Lorentz transformation (which is similar to the Thomas precession). Momenta addition being non commutative and even non associative is a problem: we need to label particles in order to consider the multiparticles state, so that it seems that we are loosing the permutation symmetry between the particles. This is an annoying problem but even more annoying is the following problem: we embedded the momentum space in the de Sitter space to input the universal bound, when defining the addition we stayed on this de Sitter space which means that the resulting momentum is still bounded by the same quantity. This is contradicting with everydays life: we expect large composite bodies to have mass or energies bigger than the Planckian scales: this is the *soccer ball problem* or saturation problem.

2. 5d approach to DSR

In this section I would like to describe how a 5d approach can solve the different problems. First I would like to argue how on the dynamical level a 5d point of view can be natural, then I will show how actually the 5d approach really solves the problems, and finally I would like to give a physical interpretation to this fifth dimension.

2.1. 5d from dynamics

We have seen that the different DSR arise by considering the de Sitter space. In fact this space can be seen as embedded in a 5d Minkowski space of coordinates P_A by $P_A P^A = -\kappa^2$. It requires few imagination to see that this could some kind of mass shell condition. Let us take this idea seriously, we consider therefore a 10 dimensional phase space (X^A, P^A) with the usual symplectic structure $\{X_A, P_B\} = \eta_{AB}$. We have a DSR particle given by the action

$$\mathcal{S}_{DSR} = \int dX^A P_A - \lambda_1 (P_A P^A + \kappa^2) - \lambda_2 (P_4 - m^2). \tag{6}$$

λ_i is the Lagrange multiplier implementing the constraint. The second constraint is to encode that we are dealing with a particle which mass will be a function of the parameter M, as we saw that P_4 was actually containing the information on the particle mass. In usual Special Relativity we have the action

$$\mathcal{S}_{SR} = \int dx^\mu p_\mu - \lambda (p_\mu p^\mu + \kappa^2). \tag{7}$$

The physics is then given when doing different gaugefixings. A gaugefixing is defined as an extra constraint $\mathcal{C}$, such that $\{\mathcal{C}, \mathcal{H}\} \neq 0$. For example we can introduce the extra constraint $x_0 = t$, which encodes the fact that we choose x_0 as the time. In this gauge we recover the hamiltonian $p_0 = \pm\sqrt{m^2 + \overrightarrow{p}^2}$, which in the limit $c \to \infty$ is the usual galilean hamiltonian. There are many other choices of gaugefixing which can be physically interpreted as a different clocks. In any case the general observables of the theory are given by the combinations of the Dirac observables $(p_\mu, j_{\mu\nu})$, that is functions that commute with the constraints.

We can see that in 5d we can have the same kind of structure, different kinds of gaugefixing provide the different kinds of DSR that one can have. Indeed after gaugefixing one looks at the Dirac bracket that describes the symplectic structure on the reduced phase space. The Dirac bracket is then defined as

$$\{\phi, \psi\}_D = \{\phi, \psi\} - \{\phi, \mathcal{C}\} \left(\frac{1}{\{\mathcal{H}, \mathcal{C}\}}\right) \{\mathcal{H}, \psi\} - \{\phi, \mathcal{H}\} \left(\frac{-1}{\{\mathcal{H}, \mathcal{C}\}}\right) \{\mathcal{C}, \psi\}. \tag{8}$$

The gaugefixing we are considering actually fixes only the 5d mass shell condition, we leave the other one untouched. A possible gaugefixing is $\mathcal{C}_1 = X^A P_A - T = 0$ and the resulting Dirac bracket encode provides the Snyder symplectic form, as it can be easily calculated. A choice of proper coordinates on the reduced phase space is (x_μ, p_μ) that commute with both $\mathcal{H}_{5d}$ and $\mathcal{C}$: $\{x_\mu, \mathcal{H}_{5d}\} = \{x_\mu, \mathcal{C}_1\} = \{p_\mu, \mathcal{H}_{5d}\} = \{p_\mu, \mathcal{C}_1\} = 0$. Their Dirac bracket with any phase space function is exactly equal to their Poisson bracket with that same function. It is easy to see that the coordinates x_μ, p_μ as defined previously in the Snyder deformation actually commute with the constraint and the gaugefixing and are therefore a right choice. They are Dirac observables of the 5d theory (we didn't touch yet the extra constraint fixing the mass). We can then reduce the action to the 8 dimensional space and study the new action for the relativistic particle leaving in a Snyder like space and look at the modified physics.

$$\dot{X}_A P^A \to \dot{x}_\mu p^\mu - p_\mu \frac{x \cdot p}{\kappa^2 - p^2} \dot{p}^\mu. \tag{9}$$

In fact to have a good action we need to have the correct canonical variables. We can do a change of variable $x_\mu \to x'_\mu = x_\mu + p_\mu \frac{x \cdot p}{\kappa^2 - p^2}$, such that at the same time the symplectic form is canonical but also the action get reduced to the trivial action $\dot{x}'_\mu p^\mu$.

We can proceed in the same way for the bicrossproduct basis. In this case the gaugefixing is given by $\mathcal{C}_2 = \frac{X_0 - X_4}{P_0 - P_4} - T$. Once again the coordinates (x_μ, p_μ) that commute with both $\mathcal{H}_{5d}$ and $\mathcal{C}$ are the ones previously given in the previous section. There are once again Dirac observables and we can reduce the action to

$$\dot{X}_A P^A \to p_\mu \, \dot{x}_\mu + p_i \, x_i \, \dot{p}_0. \tag{10}$$

Again we can also find a change of variables that identify the canonical variables but also trivialize the action.

To examine the physics, we need to consider the second constraint $P_4 - M$, where P_4 would have been expressed in each case in terms of the p. We leave the further details of this analysis in [10]. In both cases, Snyder and bicrossproduct basis, the physical meaning of these trivializing coordinates is to be determined. It might happen that there are merely convenient tools but that physics is truly defined in terms of the non commutative ones [10].

We have seen here that the different DSR (except the MS one as space-time there is a different construction) arise from a 5d action. We shall see now how this 5d point of view naturally solve the soccer ball problem.

2.2. 5d as solving the problems

Before talking about the main problem of DSR, let me recall what happens in Special Relativity. A (spinless) particle can be seen as a representation of the Poincaré group, through the Casimir $p^2 = m^2$. When considering two particles, one has the tensor product of two representations and therefore the mass of the two particle state is given by a representation consistent with the two initial ones, just like for example a spin 1 is a possible outcome of having two spin 1/2. At the same time the momentum of the two particle state is simply given by the trivial sum $p^\mu_{tot} = p^\mu_1 + p^\mu_2$.

Now following the point of view that a particle in DSR should be seen from a 5d perspective, this means that we are not considering a particle as a representation of the Poincaré group $ISO(3,1)$ anymore but as a representation of the Poincaré de Sitter group, $ISO(4,1)$. This means that a particle is not described by the mass m, but by its cutoff κ.

When considering a two particles state, we need to look at tensor product of representations and we obtain therefore a new cutoff associated with the two particles state. On the other hand, "scattering" or momenta addition is not described in terms of the modified sum in (5), but by the usual simple addition [11]

$$P^A_{tot} = P^A_1 + P^A_2. \tag{11}$$

This should be compared once again to the Special Relativity case: one deforms the speed addition but the relativistic speeds add in a trivial way and are the physical objects carrying a representation of the Poincaré group. The deformed addition corresponds to a change of reference frame. So in DSR when moving to the 5d approach, we are actually adding some extra symmetry transformations due to the extra dimension. The momenta addition that we defined earlier on the de Sitter space is the analog of the modified speed addition: it is not a scattering (which is described in (11)) but a change of reference frame. We don't need then to worry about the non associativity or non commutativity. In fact the scattering (11) can be also seen from a 4d perspective. We can define the new addition by undeforming the p's to P do the addition of these linear quantities, then transform back: $p^\mu_{tot} = U_{2\kappa}(P^\mu_{tot}) = U_{2\kappa}(U^{-1}_\kappa(p^\mu_1) + U^{-1}_\kappa(p^\mu_2))$.Note that the embedding U_κ depends on the representation of the particle (κ), so that when deforming back $P_{tot} = P_1 + P_2$, we need to take into account the new representation (e.g. 2κ), and have $U_{2\kappa}$. This addition (including the κ rescaling) was in fact proposed by hand by Magueijo and Smolin in [4], but the from the 5d perspective is rigorously justified. Note that now this addition is also commutative and associative! We killed two birds with one stone.

We have obtained a rescaling of the cutoff when considering many bodies, and this is what was expected from the soccer ball problem: the cutoff is of gravitational origin, it is the Schwarzschild mass associated with the given size of the system (we assume it spherical symmetric and spinless, neutral). We can consider objects that are Planck like size but when considering a bunch of them, they form a composite object which maximum mass has to be renormalized roughly like the number of particles, since the Schwarzschild mass behaves linearly with the size of the system.

2.3. Fifth dimension as mass

As we have introduced an extra dimension, we need to provide an physical interpretation to it. We have seen that P_4 is basically giving the mass, so it is natural to say that this extra dimension should be related to the mass. This is our suggestion: the fifth dimension is the mass. The idea is not new [12]: one can construct some Kaluza-Klein theory where the 5d dimension is not compactified, but is the mass. When considering GR on this space we can naturally construct in an intrinsic way both matter and gravitational degrees of freedom [12]. In fact we can deduce this from a hand-wavy argument analog to the derivation of space-time. In this case we have $v^2 = \frac{dL^2}{dT^2} \leq c^2 \rightarrow ds^2 = c^2 dT^2 - dL^2$. Consider now the Schwarzschild ratio that tells us that $\frac{dM^2}{dL^2} \leq \frac{c^2}{G^2} = g^2 \rightarrow dS^2 = dL^2 - g^2 dM^2$. If the mass is considered as an extra dimension ($X_4 = M$) it is natural to have it changing, which might be at first a disturbing feature. It is not so much in fact when considering the effective rest mass (identified with the energy) of a Schwarzschild black hole: Brown and York calculated the effective energy contained in a shell of size R [13]:

$$M(R) = R\left(1 - \sqrt{1 - \frac{2m}{R}}\right), \tag{12}$$

where m is the ADM mass. We see that due to gravity, there is a renormalization of the mass. Usually we neglect the feedback of the particle on space-time, but here we are taking into account in an effective way these effects. To be careful at the two particles state, one should actually look at how two Schwarzschild -like metrics merge in one representing the two particles as one coarse-grained object. DSR from this point of view gives a first effective approximation of this phenomenon.

One should also notice that in fact quantum gravitational effects could be seen in a variation of the mass. Indeed just consider the Newtonian approximation, we have a renormalization of the Newton constant $G(k)$ in terms of the scale one is working. Instead of putting this renormalization into the Newton constant we can put it into the mass, which becomes then a variable: $\frac{G(k)m}{r^2} \rightarrow \frac{Gm(k)}{r^2}$. Finally we should also notice that we can also make a gaugefixing in the 5d action, illustrating this mass-variable physics. For this one chooses $\mathcal{C}_3 = X_4 - T$. We have then the dynamics (without considering the 4d mass shell condition) $g\frac{1}{\sqrt{1-\left(g\frac{dX_4}{ds}\right)^2}}\frac{dX^4}{ds} = P^4$, $\frac{1}{\sqrt{1-\left(g\frac{dX_4}{ds}\right)^2}}\frac{dX^\mu}{ds} = P^\mu$. If there is no change in rest mass, we recover the usual relativistic speed, but if there are changes in X_4, we obtain corrections. Note that as a final comment that the fluctuations of the mass would slightly violate the equivalence principle, as the existence of a 5d force would have a non trivial effect at the 4d level. If this can be more rigorously related to DSR, we could see here some new phenomenology to DSR.

Conclusion

I have presented the general ideas behind DSR and its general definition. If it used to have some problems, the 5d perspective shed a lot of light on how to solve these problems. We have now a general framework that at least is not right away contradicting with the common sense as the soccer ball problem was. We need to push the link with the notion of space-time mass to get a better understanding of the involved physics. Note that this might enlarge a lot the application of DSR as the space-time-mass approach can be seen as related to the Randall-Sundrum model [14], but also to a unified point of view of the renormalization group and effective description of quantum gravity [15]. We leave this for further studies but certainly the following developments will be very exciting!

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