# Five-dimensional Gravity and the Pioneer Effect 

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#### Abstract

In induced gravity theory the solution of the dynamics equations for the test particle on null path leads to additional force in four-dimensional space-time. We find such force from five-dimensional geodesic line equations and try to apply this approach to analysis of additional acceleration of Pioneer 10/11, using properties of the asymmetrically warped space-time.


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Recently studies [1, 2, 3, 4] of five-dimensional space-time theory analyze consequent departure from 4D geodesic motion by geometric force. In these papers through 5D null geodesic motion analysis in induced matter Kaluza-Klein (IM-KK) gravity theory it was found out appearance extra forces in 4D, when the scalar potential depends from coordinates of 4 D space-time, and cylinder conditions failed i.e. metric coefficients depend on the fifth coordinate. Energy-momentum tensor on 4D sheet is induced through metric dependence on the extra coordinate. This approach requires 5D vacuum field equations $\hat{R_{i j}}=0$, where $\hat{R_{i j}}$ is 5D Ricci tensor. Soliton metric [5, 6, 7], which is generalization of the standard 4D Schwarzschild solution, was obtained in this frame. More complicated equations are used for description of an inflationary universe in the Sitter space in Ref. [8] and a spherical source with radiation in Ref. [9]. In brane-world theory $[10,11]$ equations in the brane, contained specific energy-momentum tensor, should be solved with equations in the bulk, which are assumed to be 5D Einstein equations with negative cosmological constant. The brane-world gravity, described by the bulk metric with warp factor depended from extra coordinate, is considered, for example, in Refs. [12] and [13].

In this paper extra forces in 4D space-time, embedded in 5D space-time, are obtained from 5D null geodesic line equations. We analyze examples of the soliton metric, which describe 5D space-time with zero curvature, including case of asymmetrically warped space-time, meaning that the space and time coordinates have different warp factors. We study a possibility of explanation why the variable component of acceleration, formed the Pioneer effect, exists. We consider one as consequence of the

[^0]geometric force (also called fifth force), which is determined through an explicit metric dependence on extra coordinate of asymmetrically warped space-time.

Space-time-matter $[1,2,7,11]$ theory, interpreted fifth coordinate as the rest mass of particles, and brane [10, 13, 14] theory have different physical motivations for the introduction of a large extra dimension. Though they have same typical scenario and without electromagnetic potentials one works [2] for manifold of metrics

$$
\begin{equation*}
d S^{2}=d s^{2} \sigma \Phi^{2}\left(x^{m}, y\right) d y^{2} \tag{1}
\end{equation*}
$$

where $d s$ is 4D line element, $\Phi$ is scalar potential depended from 4D coordinates $x^{m}$ and extra dimension $y$, also $\sigma=1$. 4D line element is taken in form

$$
\begin{equation*}
d s^{2}=g_{i j}\left(x^{m}, y\right) d x^{i} d x^{j}, \tag{2}
\end{equation*}
$$

where $g_{i j}$ is metric tensor.
In IM-KK theory massive particles in 4D have not mass in 5D and move on null path i. e. $d S=0$. Therefore, 5 D particle dynamics equations are found for null geodesic line just as in 4D by extrimizing [15] function

$$
\begin{equation*}
I=\int_{a}^{b} d \lambda\left\{g_{i j} \frac{d x^{i}}{d \lambda} \frac{d x^{j}}{d \lambda}-\sigma \Phi^{2} \frac{d y^{2}}{d \lambda^{2}}\right\} \equiv \int_{a}^{b} h d \lambda, \tag{3}
\end{equation*}
$$

where $\lambda$ is affine parameter along the path of the particle terminated at the points $a$, $b$. Null geodesic line equations are given by

$$
\begin{equation*}
\frac{d^{2} X^{A}}{d \lambda^{2}}+\Gamma_{B C}^{A} \frac{d X^{B}}{d \lambda} \frac{d X^{C}}{d \lambda}=0 \tag{4}
\end{equation*}
$$

where $X^{A}$ are coordinates of 5 D space-time, and $\Gamma_{B C}^{A}$ are appropriate defined Christoffel symbols. It should be noticed that if we extremized action with Lagrangian $L \equiv h^{1 / 2}$ for particle moving on null path in 5D we would obtain division by zero, since this movement assigns $h=0$. Generally choice of parameter $\lambda$ is not arbitrary, and turned to differentiation with respect to $s$ in Eq. (4)we obtain

$$
\begin{equation*}
\frac{d^{2} X^{A}}{d s^{2}}+\Gamma_{B C}^{A} \frac{d X^{B}}{d s} \frac{d X^{C}}{d s}=-\omega \frac{d X^{A}}{d s} \tag{5}
\end{equation*}
$$

where $\omega=\frac{d^{2} X^{A}}{d \lambda^{2}} /\left(\frac{d s}{d \lambda}\right)^{2}$ and interval $d s$ is assumed to be timelike.
The first four components of Eq. (5), corresponded to the motion in 4D spacetime, are transformed to

$$
\begin{equation*}
\frac{D u^{i}}{d s} \equiv \frac{d^{2} x^{i}}{d s^{2}}+\Gamma_{j k}^{i} \frac{d x^{j}}{d s} \frac{d x^{k}}{d s}=f^{i} \tag{6}
\end{equation*}
$$

where $f^{i}$ is component of the "extra" force (per unit mass). Eq. (1) does not set sign of scalar potential and for null geodesic with spacelike extra coordinate ( $\sigma=1$ ) yields:

$$
\begin{equation*}
\frac{d y}{d s}=\frac{1}{\Phi} \tag{7}
\end{equation*}
$$

With this condition for metric (1) with 4D line element (2) fifth force is written as

$$
\begin{equation*}
f^{i}=-\frac{g^{i k}}{\Phi}\left(\frac{\partial \Phi}{\partial x^{k}}+\frac{\partial g_{k j}}{\partial y} u^{j}\right)-\omega u^{i} \tag{8}
\end{equation*}
$$

where $u^{j}$ is 4 -velocity. The fifth component of Eq. (6) takes the following form:

$$
\begin{equation*}
\frac{d^{2} y}{d s^{2}}+\frac{1}{\Phi^{2}}\left(\frac{\partial g_{i j}}{\partial y} u^{i} u^{j}+2 \frac{\partial \Phi}{\partial x^{i}} u^{i}+\frac{1}{\Phi} \frac{\partial \Phi}{\partial y}\right)+\omega \frac{d y}{d s}=0 \tag{9}
\end{equation*}
$$

By substitution (7) in (9) we obtain

$$
\begin{equation*}
\omega=-\frac{1}{\Phi}\left(\frac{\partial g_{i j}}{\partial y} u^{i} u^{j}+\frac{\partial \Phi}{\partial x^{i}} u^{i}\right) . \tag{10}
\end{equation*}
$$

Then equations (8) are rewritten as

$$
\begin{equation*}
f^{i}=-\left(g^{i k}-u^{i} u^{k}\right) \frac{1}{\Phi}\left(\frac{\partial \Phi}{\partial x^{k}}+\frac{\partial g_{k j}}{\partial y} u^{j}\right) . \tag{11}
\end{equation*}
$$

When $\Phi=1$, and metric (2) is orthogonal and conforms to asymmetrically warped space-time:

$$
\begin{equation*}
d s^{2}=M(y) \tilde{g}_{00}\left(x^{m}\right) d x^{02}+N(y) \tilde{g}_{i i}\left(x^{m}\right) d x^{i 2} \tag{12}
\end{equation*}
$$

where $M, N$ are functions of extra coordinate, components of fifth force (11) are following:

$$
\begin{gather*}
f^{0}=\left(\frac{M^{\prime}}{M}-\frac{N^{\prime}}{N}\right)\left(M \tilde{g}_{00} u^{02}-1\right) u^{0} \\
f^{i}=\left(\frac{M^{\prime}}{M}-\frac{N^{\prime}}{N}\right) M \tilde{g}_{00} u^{02} u^{i} \tag{13}
\end{gather*}
$$

where $\left({ }^{\prime}\right)$ denotes derivative with respect to $y$, and in second equation velocities $u^{i}$ conform to the spacelike coordinates.

In 5D empty space-time, including 4D sheet with spherical coordinate system $x^{i}=$ $(t, r, \varphi, \theta)$, gravity is described by soliton metric [7], whose line element may be written in form

$$
\begin{equation*}
d S^{2}=H^{a} d t^{2}-H^{-a-b} d r^{2}-H^{1-a-b} d \Omega^{2}-H^{b} d y^{2} \tag{14}
\end{equation*}
$$

where $d \Omega^{2}=r^{2} d \varphi^{2}+r^{2} \sin ^{2} \varphi d \theta^{2}, H(r)=1-2 m / r, m$ is mass parameter, and constants $a, b$ satisfy to relation

$$
\begin{equation*}
a^{2}+a b+b^{2}=1 \tag{15}
\end{equation*}
$$

Let us analyze modifications of this metric, having parameters conformed to (15), with coefficients being under constraint of zero Ricci scalar $\hat{R}$. That allows to consider one in frame of the induced matter scenario. In first example metric with coefficients, which agree to cylinder conditions, is chosen [16] in form

$$
\begin{equation*}
d S^{2}=H^{a} d t^{2}-H^{-a-b} d r^{2}-H^{1-a-b} d \Omega^{2}-H^{b} Q d y^{2} \tag{16}
\end{equation*}
$$

where $Q(t)=(1+p t)^{2}$, and $p$ is constant. This space-time has singularity with $t=1 / p$. In case $a=1, b=0$ we have Schwarzschild limit for metric of embedded 4D space-time. Then non-vanishing component of the Ricci tensor is following:

$$
\begin{equation*}
\hat{R}_{01}=\frac{p m}{r(r-2 m)(1+p t)} \tag{17}
\end{equation*}
$$

We note that condition $\hat{R}=0$ is fulfilled with every $a, b$ corresponding equation (15). Taken into account (11) in considered case Eqs. (6) yield

$$
\begin{equation*}
\frac{D u^{0}}{d s}=-\left(\frac{1}{H}-u^{02}\right) \frac{p}{1+p t} \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{D u^{i}}{d s}=u^{i} u^{0} \frac{p}{1+p t} \tag{19}
\end{equation*}
$$

In these equations $D u^{j} / d s$ correspond to standard Schwarzschild metric.
The next example is metric described asymmetrically warped space-time:

$$
\begin{equation*}
d S^{2}=H^{a} y^{2} d t^{2}-V\left(H^{-a-b} d r^{2}+H^{1-a-b} d \Omega^{2}\right)-l^{2} H^{b} d y^{2} \tag{20}
\end{equation*}
$$

where $V(y)=1+q \ln y$, and $q, l$ are constants. Here we have singularity with $y=e^{-1 / q}$ and

$$
\begin{gather*}
\hat{R}_{00}=\frac{3 q H^{a-b}}{2 l^{2} V}, \quad \hat{R}_{11}=-\frac{q^{2} H^{-a-2 b}}{4 l^{2} y^{2} V}, \quad \hat{R}_{14}=\frac{m[2(b-a) V+(a+2 b) q]}{2(r-2 m) r y V} \\
\hat{R}_{22} \sin ^{2} \varphi=\hat{R}_{33}=-\frac{q^{2} r^{2} H^{1-a-2 b}}{4 l^{2} y^{2} V} \sin ^{2} \varphi, \quad \hat{R}_{44}=\frac{3 q(q+2 V)}{4 y^{2} V^{2}} \tag{21}
\end{gather*}
$$

Expression for the geometric force acting on massive particles can be written as

$$
\begin{gather*}
f^{0}=l^{-1} H^{-b / 2}\left[\left(\frac{2}{y}-\frac{q}{y V}\right)\left(y^{2} H^{a} u^{02}-1\right)+\frac{b l m}{r^{2}} H^{b / 2-1} u^{1}\right] u^{0}, \\
f^{1}=l^{-1} H^{-b / 2}\left[\left(2-\frac{q}{V}\right) y H^{a} u^{02} u^{1}+\frac{b l m}{r^{2}} H^{b / 2-1}\left(\frac{H^{a+b}}{V}+u^{12}\right)\right], \\
f^{2,3}=l^{-1} H^{-b / 2}\left[\left(2-\frac{q}{V}\right) y H^{a} u^{02}+\frac{b l m}{r^{2}} H^{b / 2-1} u^{1}\right] u^{2,3} . \tag{22}
\end{gather*}
$$

Solution of Eq. (7) in case of invariable $r, \varphi, \theta$ gives

$$
\begin{equation*}
y=K \exp \left(\frac{t}{l} H^{(a-b) / 2}\right) \tag{23}
\end{equation*}
$$

where $K$ is constant. Assuming that $y=1$ corresponds to the background with infinite $r$ and time $t_{0}$, we obtain

$$
\begin{equation*}
K=\exp \left(-t_{0} / l\right) \tag{24}
\end{equation*}
$$

Then the proper time is brought in form

$$
\begin{equation*}
\tau=l \exp \left(\frac{t-t_{0}}{l} H^{a-b / 2}\right)-l \exp \left(\frac{-t_{0}}{l} H^{a-b / 2}\right) \tag{25}
\end{equation*}
$$

For finding of suggestion on physical interpretation of this approach we apply these results to analysis of the Pioneer effect. One consists in additional acceleration of the spacecrafts $[17,18,19]$ towards the Sun. It follows from Ref. [18] that force given the Pioneer effect has two components, namely, constant in considerable area of the solar system and sinusoid faded with increase distance. Possible explanation of existence of the constant component is in Ref. [19]. In Earth's center coordinate system a velocity, related to the spacecraft, approximately expressed as

$$
\begin{equation*}
u^{1}=u_{\mathrm{p}}^{1}+u_{\mathrm{E}}^{1} \cos \left(\nu t+\phi_{0}\right) \tag{26}
\end{equation*}
$$

where $u_{\mathrm{p}}^{1} \approx 4.13 \cdot 10^{-5}$ conforms to the radial velocity of the spacecraft [19] in relation to the Sun, $u_{\mathrm{E}}^{1}=0.997 \cdot 10^{-4}$ conforms to the orbital velocity of the Earth, $\nu=2 \pi \mathrm{y}^{-1}$ is frequency of the Earth rotation around the Sun, and $\phi_{0}$ is constant. The best fit of the amplitude of the spacecraft acceleration periodical component $\tilde{a}_{\mathrm{p}}[18]$ by means of function $L / r$ gives $L=1.28 \cdot 10^{-4} \mathrm{sm}^{2} / \mathrm{s}^{2}$.

When parameters of metric (20) are $a=1, b=0$, substitution of fifth force (22) in appropriate equations (6) with extra coordinate determined by (23) and (24) gives

$$
\begin{array}{r}
\frac{d u^{1}}{d s}+\frac{m H y^{2}}{r^{2} V} u^{02}-\frac{m H^{-1}}{r^{2}} u^{12}+(2 m-r)\left(u^{22}+\sin ^{2} \varphi u^{32}\right)= \\
{\left[\frac{(2-q) l+2\left(t-t_{0}\right)+2 t m / r}{(1+q \ln y) l^{2}}\right] y H u^{02} u^{1}} \tag{27}
\end{array}
$$

for the first component and

$$
\begin{array}{r}
\frac{D u^{0}}{d s}=l^{-1}\left[\left(\frac{2}{y}-\frac{q}{y V}\right)\left(y^{2} H u^{02}-1\right)\right] u^{0}, \\
\frac{D u^{2,3}}{d s}=l^{-1}\left[\left(2-\frac{q}{V}\right) y H u^{02}\right] u^{2,3} \tag{28}
\end{array}
$$

for the other components with $D u^{i} / d s$ corresponded to standard Schwarzschild metric. Let us determine constraints on the parameters of Eq. (27) in case of radial movement. Assuming that in the solar system area

$$
\begin{equation*}
\left|(2-q) l+2 c\left(t-t_{0}\right)\right| \ll \frac{2 t G m}{c r} \tag{29}
\end{equation*}
$$

where $t_{0}$ is time of the signal receiving, $t$ is time of the signal sending, $m$ is Sun's mass, $c, G$ are light velocity and gravitational constant, we obtain $t / l^{2} \approx 1.6 \cdot 10^{-27} \mathrm{~s} / \mathrm{sm}^{2}$ in this area. Hence proposed condition also requires $t_{0} \gg 4 \cdot 10^{15} \mathrm{~s}$ and $|l| \gg 1.6 \cdot 10^{21} \mathrm{sm}$. Then constant $q$ will be close to 2 and we must take into account singularity when $y \approx e^{-1 / 2}$.

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## References

[1] A. P. Billyard and W. N. Sajko, Gen. Rel. Grav. 33, 1929 (2001), gr-qc/0105074.
[2] J. Ponce de Leon, Phys. Lett. B523, 311 (2001), gr-qc/0110063.
[3] D. Youm, Phys. Rev. D62, 084002 (2000), hep-th/0004144.
[4] D. Youm, Mod. Phys. Lett. A16, 2371 (2001), hep-th/0110013.
[5] D. J. Cross and M. J. Perry, Nucl. Phys. B226, 23 (1983).
[6] R. D. Sorkin, Phys. Rev. Lett. 51, 87 (1983).
[7] J. M. Overduin and P.S. Wesson, Phys. Rept. 283, 303 (1997), gr-qc/9805018.
[8] L.-X. Li and J. R. Gott, III, Phys. Rev. D58, 103513 (1998), astro-ph/9804311.
[9] H. Liu and P. S. Wesson, J. Math. Phys. 42, 4963 (2001), gr-qc/0104009.
[10] T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D62, 024012 (2000), grqc/9910076.
[11] J. Ponce de Leon, Mod. Phys. Lett. A16, 2291 (2001), gr-qc/0111011.
[12] A. Chamblin et al., Phys. Rev. D63, 064015 (2001), hep-th/0008177.
[13] P. D. Mannheim, Phys. Rev. D64, 068501 (2001), hep-th/0101047.
[14] L. Randal and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999), hep-th/9906064.
[15] G. C. McVittie, General Relativity and Cosmology (Chapman and Hall Ltd., 1956).
[16] These results are derived by using computer package of P. Musgrave, D. Pollney and K. Lake GRTensorM 1.2 for Mathemathika, 1998.
[17] J. D. Anderson et al., Phys. Rev. Lett. 81, 2858 (1998), gr-qc/9808081.
[18] S. G. Turushev et al., XXXIV-th Rencontres de Morion Meeting on Gravitational Waves and Experimental Gravity 1999, Les Arcs, Savoi, France, gr-qc/9903024.
[19] J. D. Anderson et al., Phys. Rev. D65, 082004 (2002), gr-qc/0104064.


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