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**The embedding of the spacetime in five dimensions: an extension  
of Campbell-Magaard theorem**

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arXiv:gr-qc/0109076v2 22 Sep 2001

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# Abstract

We extend Campbell-Magaard embedding theorem by proving that any  $n$ -dimensional semi-Riemannian manifold can be locally embedded in an  $(n+1)$ -dimensional Einstein space. We work out some examples of application of the theorem and discuss its relevance in the context of modern higher-dimensional spacetime theories.

## I. INTRODUCTION

The old idea that our universe is fundamentally higher-dimensional with  $n = 4 + d$  space-time dimensions seems to be gaining grounds very rapidly in recent years. Mathematical schemes by which our ordinary spacetime is viewed as a hypersurface embedded in a higher dimensional space have been considered in several different contexts, such as strings [1], D-branes [2], Randall-Sundrum [3] models and non-compactified versions of Kaluza-Klein theories [4].

On the other hand, local isometric embeddings of Riemannian manifolds have long been studied in differential geometry. Of particular interest is a well known theorem which states that if the embedding space is flat, then the minimum number of extra dimensions needed to analytically embed a  $n$ -dimensional Riemannian manifold is  $d$ , with  $0 \leq d \leq n(n-1)/2$  [5].

It turns out, however, that if the embedding space is allowed to be Ricci-flat then the minimum number of extra dimensions that are necessary for the embedding falls dramatically to  $d = 1$ . This is the content of a little known but powerful theorem due to Campbell [6], the proof of which was given by Magaard [7]. Campbell-Magaard's result has acquired fundamental relevance for granting the mathematical consistency of five-dimensional embedding theories and also has been applied to investigate how lower-dimensional theories could be related to (3+1)-dimensional vacuum Einstein gravity [8].

The increasing attention given to the Randall-Sundrum model [3] in which the embedding space, i.e., the bulk, corresponds to an Einstein space, rather than a Ricci-flat one, has led us to wonder whether Campbell-Magaard theorem could be generalized and what sort of generalization could be done. Research in this direction, where a scheme for extending Campbell-Magaard theorem for embedding spaces with a non-null Ricci tensor has been put forward quite recently by Anderson and Lidsey [9].

The purpose of the present paper is to prove that Campbell-Magaard theorem can, indeed, be extended to include Einstein spaces. Our proof is entirely inspired in Magaard's reasoning although some adaptations to the more general semi-Riemannian character of the spaces had to be made.

The paper is organized as follows. Section 2 is devoted to state and prove an extension of Campbell-Magaard theorem, in which the embedding manifold is an Einstein space. The proof is rather involved and resorts to auxiliary theorems and lemmas. In section 3 we apply the general result to some examples and, finally, Section 4 contains our conclusion.

## II. ISOMETRIC EMBEDDING IN AN EINSTEIN SPACE

Campbell-Magaard theorem for local isometric embedding in Ricci-flat space refers to Riemannian manifolds, i.e., those endowed with positive-definite metrics. It turns out that for our purposes of generalization this restrictive condition is not essential, so in what follows we shall consider semi-Riemannian manifold with metrics of indefinite signature instead. First let us introduce some definitions, set the notation and present preliminary theorems and lemmas.

A  $n$ -dimensional manifold  $M^n$  is termed semi-Riemannian if it is endowed with a metric, i.e., a symmetric and non-degenerated second-rank tensor field of arbitrary signature. (In this paper we are considering only manifolds and metrics which are analytic).

**Definition 1** *Consider a differential map  $\Phi : U \subset M^n \rightarrow N^{n+k}$ , where  $U$  is an open subset of  $M^n$ , and  $N^{n+k}$  ( $k \geq 0$ ) is a manifold of dimension  $n+k$ . Then  $\Phi$  is called a local isometric*

embedding if the following conditions hold:

- i)  $d\Phi_p : T_p M^n \rightarrow T_{\Phi(p)} N^{n+k}$  is injective for all  $p \in U$ ;
- ii)  $g_p(v, w) = \tilde{g}_{\Phi(p)}(d\Phi(v), d\Phi(w))$  for all  $v, w \in T_p M^n$ , where  $\tilde{g}$  denotes the metric of  $N^{n+k}$ .
- iii)  $\Phi$  is a homeomorphism onto its image in the induced topology.

If  $\Phi$  is of class  $C^k$  (analytic) then the embedding is said to be of class  $C^k$  (analytic). Naturally, a local isometric embedding may be characterized in terms of coordinates. For instance, let  $\mathbf{x} = \{x^1, \dots, x^n\}$  and  $\mathbf{y} = \{y^1, \dots, y^{n+k}\}$  denote coordinate patches for  $U \subset M^n$  and  $V \subset N^{n+k}$ , respectively, with  $\Phi(p) \in V$ . The embedding  $\Phi$  determines a relation between the coordinates denoted by

$$y^\alpha = \sigma^\alpha(x^1, \dots, x^n), \quad (1)$$

where  $\sigma = \mathbf{y} \circ \Phi \circ \mathbf{x}^{-1} : \mathbf{x}(U) \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$ . (Throughout Latin and Greek indices will run from 1 to  $n$  and 1 to  $n+k$ , respectively)

In this way the isometric condition leads to

$$g_{ij} = \frac{\partial \sigma^\alpha}{\partial x^i} \frac{\partial \sigma^\beta}{\partial x^j} \tilde{g}_{\alpha\beta} \quad (2)$$

where the functions  $g_{ij}$  and  $\tilde{g}_{\alpha\beta}$  are the components of metric with respect to the coordinate bases  $\{\partial_{x^i}\}$  and  $\{\partial_{y^\alpha}\}$ , respectively. We, therefore, say that  $M^n$  can be local and isometrically embedded in  $N^{n+k}$  if there exist  $n+k$  differentiable functions  $y^\alpha = \sigma^\alpha(x^1, \dots, x^n)$ , (embedding functions) such that the Jacobian matrix  $\frac{\partial \sigma^\alpha}{\partial x^i}$  has rank  $n$  and (2) holds.

Given two arbitrary semi-Riemannian manifolds  $M^n$  and  $N^{n+k}$  it may happen that there exists no isometric embedding between them. Thus, it is of interest to find out conditions assuring or not the existence of embedding, in particular, if the embedding space  $N^{n+k}$  is not specified except that it belongs to a collection of manifolds,  $\mathcal{M}_\pi$ , whose members, say, share a certain geometrical property. For example,  $\pi$  may express a restriction of the following kinds: to be flat, to have constant curvature, to be Ricci-flat, to be an Einstein space, and so forth. This way of formulating the problem motivates the definition below.

**Definition 2** We say that a semi-Riemannian manifold  $M^n$  has an embedding in the set  $\mathcal{M}_\pi$ , if there is at least a member of  $\mathcal{M}_\pi$ , say  $N^{n+k}$ , in which  $M^n$  is embeddable.

The following is a theorem which establishes necessary and sufficient conditions for the existence of local isometric embedding of a  $n$ -dimensional semi-Riemannian space  $(M^n, g)$  in the set  $\mathcal{M}_\pi^{n+1}$  of  $(n+1)$ -dimensional semi-Riemannian spaces  $(N^{n+1}, \tilde{g})$  that satisfy the (nonspecified) property  $\pi$ . The original version of this theorem is due to Magaard [7] and is restricted to the Riemannian case, though the extension to semi-Riemannian manifold is straightforward.

**Theorem 1.** Let  $(M^n, g)$  constitute a semi-Riemannian manifold,  $\mathcal{M}_\pi^{n+1} = \{(N^{n+1}, \tilde{g}),$  which satisfy the property  $\pi\}$  and  $\mathbf{x} = \{x^1, \dots, x^n\}$  a coordinate system covering a neighborhood  $U$  of  $p \in M^n$ . A necessary and sufficient condition for a local analytical embedding of  $M^n$  at  $p$ , with line element

$$ds^2 = g_{ik} dx^i dx^k, \quad (3)$$

in  $\mathcal{M}_\pi^{n+1}$  is that

i) there exist analytic functions

$$\bar{g}_{ik} = \bar{g}_{ik}(x^1, \dots, x^n, x^{n+1}) \quad (4)$$

$$\bar{\phi} = \bar{\phi}(x^1, \dots, x^n, x^{n+1}) \quad (5)$$

defined in some open set  $D \subset \mathbf{x}(U) \times \mathbb{R}$  containing the point  $(x_p^1, \dots, x_p^n, 0)$  and satisfying the conditions

$$\bar{g}_{ik}(x^1, \dots, x^n, 0) = g_{ik}(x^1, \dots, x^n) \quad (6)$$

in an open set  $A \subset \mathbf{x}(U)$ ;

$$\bar{g}_{ik} = \bar{g}_{ki} \quad (7)$$

$$|\bar{g}_{ik}| \neq 0 \quad (8)$$

$$\bar{\phi} \neq 0 \quad (9)$$

in  $D$ . ( $|\bar{g}_{ik}|$  denotes the determinant of  $\bar{g}_{ik}$ );

ii) and also that

$$ds^2 = \bar{g}_{ik} dx^i dx^k + \varepsilon \bar{\phi}^2 dx^{n+1} dx^{n+1}, \quad (10)$$

where  $\varepsilon^2 = +1$ , be a line element in a certain coordinate neighborhood  $V$  of some manifold  $N^{n+1} \in \mathcal{M}_\pi^{n+1}$ .

The essential idea of this theorem is that there exists a coordinate system “adapted” to the embedding in such a manner that the image of the embedding coincide with the hypersurface  $x^{n+1} = 0$  of the embedding space and the condition of isometry reduces to (5).

While the sufficient condition is easily demonstrated, the proof of the necessary condition of this theorem is very long and will be omitted [7,10].

We now consider a  $(n+1)$ -dimensional semi-Riemannian manifold  $(N^{n+1}, \tilde{g})$  and  $\{y^1, \dots, y^{n+1}\}$  a coordinate system defined in an open set of  $V \subset N^{n+1}$ . Let  $\tilde{g}_{\alpha\beta}$  and  $\tilde{R}_{\alpha\beta}$  denote the components of the metric and Ricci tensor, respectively. The manifold  $(N^{n+1}, \tilde{g})$  is called an Einstein space if

$$\tilde{R}_{\alpha\beta} = \frac{2\Lambda}{1-n} \tilde{g}_{\alpha\beta}, \quad (11)$$

where  $n \geq 2$  and  $\Lambda$  is a constant. Of course (11) is equivalent to  $\tilde{G}_{\alpha\beta} = \Lambda \tilde{g}_{\alpha\beta}$ ,  $\tilde{G}_{\alpha\beta}$  being the components of the Einstein tensor, and for this reason  $\Lambda$  will be occasionally referred to as the cosmological constant.

As is well known, at each point of an arbitrary semi-Riemannian space  $N^{n+1}$  there exists a coordinate neighborhood in which the metric has the form

$$ds^2 = \bar{g}_{ik} dy^i dy^k + \varepsilon \bar{\phi}^2 (dy^{n+1})^2. \quad (12)$$

Let us now consider the inclusion map  $\iota(y^1, \dots, y^n) = (y^1, \dots, y^n, 0)$ . This map determines an embedding of the hypersurface  $\Sigma_0$ , defined by  $y^{n+1} = 0$ , in  $N^{n+1}$ . This hypersurface endowed with the induced metric by the inclusion map, which is given by

$$g_{ik}(y^1, \dots, y^n) = \frac{\partial t^\alpha}{\partial y^i} \frac{\partial t^\beta}{\partial y^k} \tilde{g}_{\alpha\beta} = \bar{g}_{ik}(y^1, \dots, y^n, 0) \quad (13)$$

constitutes a semi-Riemannian space. The intrinsic curvature of  $\Sigma_0$  and the curvature of  $N^{n+1}$  calculated at  $\Sigma_0$  are related by the Gauss-Codazzi equations. In the coordinates (12), the Gauss-Codazzi relations can be written in the form [5]:

$$R_{mkij} = \tilde{R}_{mkij} + \varepsilon (\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im}), \quad (14)$$

$$\nabla_j \Omega_{ik} - \nabla_i \Omega_{jk} = \frac{1}{\phi} \tilde{R}_{(n+1)kij}, \quad (15)$$

where  $R_{mkij}$  and  $\tilde{R}_{mkij}$ <sup>1</sup> are the components of the curvature tensor of  $\Sigma_0$  and of  $N^{n+1}$ , respectively,  $\nabla_i$  denoting the covariant derivative with respect to the metric  $g_{ik}$ , and  $\Omega_{ik}$  (the covariant components of the extrinsic curvature tensor of  $\Sigma_0$ ) are given by

$$\Omega_{ik} = -\frac{1}{2\bar{\phi}} \frac{\partial \bar{g}_{ik}}{\partial y^{n+1}}, \quad (16)$$

in the coordinates (12). We are now interested in obtaining the components of the Ricci tensor  $R_{\alpha\beta} = g^{\delta\gamma} R_{\delta\alpha\gamma\beta}$  in the coordinates (12). With the help of the Gauss-Codazzi equations and from the expression

$$\tilde{R}_{i(n+1)k}^{(n+1)} = \left( -\frac{\varepsilon}{\phi} \frac{\partial \Omega_{ik}}{\partial y^{n+1}} + \frac{1}{\phi} \nabla_i \nabla_k \phi - \varepsilon g^{jm} \Omega_{jk} \Omega_{im} \right),$$

which can be obtained by a straightforward calculation, we are left with

$$\tilde{R}_{ik} = R_{ik} + \varepsilon g^{jm} (\Omega_{ik}\Omega_{jm} - 2\Omega_{jk}\Omega_{im}) - \frac{\varepsilon}{\phi} \frac{\partial \Omega_{ik}}{\partial y^{n+1}} + \frac{1}{\phi} \nabla_i \nabla_k \phi \quad (17)$$

$$\tilde{R}_{i(n+1)} = \phi g^{jk} (\nabla_j \Omega_{ik} - \nabla_i \Omega_{jk}) \quad (18)$$

$$\tilde{R}_{(n+1)(n+1)} = \varepsilon \phi^2 g^{ik} \left( -\frac{\varepsilon}{\phi} \frac{\partial \Omega_{ik}}{\partial y^{n+1}} + \frac{1}{\phi} \nabla_i \nabla_k \phi - \varepsilon g^{jm} \Omega_{jk} \Omega_{im} \right). \quad (19)$$

At this stage, it should be clear that the results just obtained may be carried over to any hypersurface  $\Sigma_c$  defined by  $y^{n+1} = c = \text{const}$ . Owing to this we shall henceforth introduce a slight different notation: the metric induced on any  $\Sigma_c$ , i.e.,  $\bar{g}_{ik}(y^1, \dots, y^n, c)$ , will be denoted simply by  $\bar{g}_{ik}$ ; likewise all quantities associated with  $\bar{g}_{ik}$  will be marked with a bar

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<sup>1</sup> $R_{kij}^l = \Gamma_{ik,j}^l - \Gamma_{jk,i}^l + \Gamma_{jm}^l \Gamma_{ik}^m - \Gamma_{im}^l \Gamma_{jk}^m$



sign. However, for convenience, only the metric induced on  $\Sigma_0$  may also be denoted by  $g_{ik}$  (without the bar).

Now let us turn our attention to the case when the embedding manifold  $N^{n+1}$  is an Einstein space. Since we can assume that any point  $q \in N^{n+1}$  lies in some hypersurface  $\Sigma_c$ , we can use the equations above for decomposing the Ricci tensor of the embedding manifold in terms of its intrinsic and extrinsic parts with respect to any hypersurface  $\Sigma_c$ , with  $c = y_q^{n+1}$ . If  $N^{n+1}$  is an Einstein space, then it follows from (10) that

$$\tilde{R}_{ik} = \bar{R}_{ik} + \varepsilon \bar{g}^{jm} (\bar{\Omega}_{ik} \bar{\Omega}_{jm} - 2\bar{\Omega}_{jk} \bar{\Omega}_{im}) - \frac{\varepsilon}{\bar{\phi}} \frac{\partial \bar{\Omega}_{ik}}{\partial y^{n+1}} + \frac{1}{\bar{\phi}} \bar{\nabla}_i \bar{\nabla}_k \bar{\phi} = \frac{2\Lambda}{1-n} \bar{g}_{ik} \quad (20)$$

$$\tilde{R}_{i(n+1)} = \bar{\phi} \bar{g}^{jk} (\bar{\nabla}_j \bar{\Omega}_{ik} - \bar{\nabla}_i \bar{\Omega}_{jk}) = 0 \quad (21)$$

$$\tilde{G}_{(n+1)}^{(n+1)} = -\frac{1}{2} \bar{g}^{ik} \bar{g}^{jm} (\bar{R}_{ijkm} + \varepsilon (\bar{\Omega}_{ik} \bar{\Omega}_{jm} - \bar{\Omega}_{jk} \bar{\Omega}_{im})) = \Lambda \quad (22)$$

where in the last equation, from the definition of the Einstein tensor,  $\tilde{G}_{(n+1)}^{(n+1)} = \tilde{R}_{(n+1)}^{(n+1)} - \frac{1}{2} \tilde{R}$ ,  $\tilde{R}$  being the curvature scalar  $\tilde{R} = \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta}$ .

After writing the equations above let us concentrate on the problem of embedding a semi-Riemannian manifold in an Einstein space. The equations (20), (21) and (22) may be looked upon as a set of partial differential equations for  $\bar{g}_{ik}$  and  $\bar{\phi}$ . At this point, our strategy is to show that if the components  $g_{ik}(x^1, \dots, x^n)$  of a metric of  $M^n$  with respect to some coordinate system are given, then there exists an open set of  $\mathbb{R}^{n+1}$  where the above-mentioned equations admit a solution  $\bar{g}_{ik}(y^1, \dots, y^n, y^{n+1})$  and  $\bar{\phi}(y^1, \dots, y^n, y^{n+1})$  which satisfies the initial condition  $\bar{g}_{ik} = g_{ik}$  at  $\Sigma_0$ . Moreover, it will be shown that the functions  $\bar{g}_{ik}$  and  $\bar{\phi}$  possess all properties which are necessary to constitute a line element of a  $(n+1)$ -dimensional semi-Riemannian manifold. Then, as  $\bar{g}_{ik}$  and  $\bar{\phi}$  satisfy the Eqs. (20), (21) and (22) the metric originated by them represents that of an Einstein space. Then, by virtue of Theorem 1, the existence of the embedding will be guaranteed. However, before proceeding to the final demonstration we shall make use of two lemmas. Let us consider the first one.

**Lemma 1.** *Let the functions  $\bar{g}_{ik}$  and  $\bar{\phi}$  be analytic at  $(0, \dots, 0) \in \Sigma_0 \subset \mathbb{R}^{n+1}$  and satisfy the conditions (7), (8), (9), and the equation (20) in an open set of  $\mathbb{R}^{n+1}$  which contains  $(0, \dots, 0, 0) \in \mathbb{R}^{n+1}$ . If, in addition,  $\bar{g}_{ik}$  and  $\bar{\phi}$  satisfy (21) and (22) at  $\Sigma_0$ , then  $\bar{g}_{ik}$  e  $\bar{\phi}$  also*

satisfy (21) and (22) in some open set of  $\mathbb{R}^{n+1}$  containing  $(0, \dots, 0, 0)$ .

*Proof.* By assumption  $\bar{g}_{ik}$  and  $\bar{\phi}$  satisfy (7), (8) and (9), hence the functions  $\tilde{g}_{\alpha\beta}$  defined by  $\tilde{g}_{ik} = \bar{g}_{ik}$ ,  $\tilde{g}_{n+1n+1} = \bar{\phi}^2$  and  $\tilde{g}_{in+1} = 0$  for  $i, k = 1, \dots, n$ , may be considered as the components of a  $(n+1)$ -dimensional metric tensor  $\tilde{g}$ . The coefficients of the connection and the components of the curvature tensor associated to the metric  $\tilde{g}_{\alpha\beta}$  can be calculated as usual. Let us now define the tensor  $\tilde{F}_{\alpha\beta} = \tilde{G}_{\alpha\beta} - \Lambda\tilde{g}_{\alpha\beta}$ , where  $\tilde{G}_{\alpha\beta}$  is the Einstein tensor. Then, as a consequence of Bianchi identities for the curvature tensor, it follows that  $\tilde{F}_{\alpha\beta}$  is divergenceless, i.e.,

$$\tilde{\nabla}_\alpha \tilde{F}_\beta^\alpha = 0. \quad (23)$$

This equation can be rewritten as

$$\frac{\partial \tilde{F}_\beta^{n+1}}{\partial y^{n+1}} = -\frac{\partial \tilde{F}_\beta^i}{\partial y^i} - \tilde{\Gamma}_{\mu\lambda}^\mu \tilde{F}_\beta^\lambda + \tilde{\Gamma}_{\lambda\beta}^\mu \tilde{F}_\mu^\lambda. \quad (24)$$

We have assumed that  $\bar{g}_{ik}$  and  $\bar{\phi}$  satisfy the equations (20), (21) and (22) at  $\Sigma_0$ , hence  $\tilde{F}_\beta^\alpha = 0$  at  $\Sigma_0$ . Moreover,  $\left. \frac{\partial \tilde{F}_\beta^i}{\partial y^i} \right|_{y^{n+1}=0} = \left. \frac{\partial}{\partial y^i} \left( \tilde{F}_\beta^i \right) \right|_{y^{n+1}=0} = 0$ . Therefore, we conclude from (24) that

$$\left. \frac{\partial \tilde{F}_\beta^{n+1}}{\partial y^{n+1}} \right|_{y^{n+1}=0} = 0. \quad (25)$$

Let us look into Eq. (24) separately for  $\beta = n+1$  and  $\beta = i$ . Taking first  $\beta = n+1$  gives

$$\frac{\partial \tilde{F}_{n+1}^{n+1}}{\partial y^{n+1}} = -\frac{\partial \tilde{F}_{n+1}^i}{\partial y^i} - \tilde{\Gamma}_{\mu\lambda}^\mu \tilde{F}_{n+1}^\lambda + \tilde{\Gamma}_{\lambda n+1}^{n+1} \tilde{F}_{n+1}^\lambda + \tilde{\Gamma}_{n+1 n+1}^i \tilde{F}_i^{n+1} + \tilde{\Gamma}_{k n+1}^i \tilde{F}_i^k \quad (26)$$

From the definition of the Einstein tensor we can write  $\tilde{G}_j^i = \tilde{R}_j^i - \delta_j^i \left( \tilde{R}_k^k + \tilde{G}_{n+1}^{n+1} \right)$ . By assumption,  $\tilde{R}_{ij} = \frac{2\Lambda}{1-n} \bar{g}_{ij}$  not only at the hypersurface  $y^{n+1} = 0$ , but also for some open set  $V \subset \mathbb{R}^{n+1}$ , with  $0 \in U$ . Thus, the equality  $\tilde{F}_i^k = -\delta_i^k \tilde{F}_{n+1}^{n+1}$  holds in  $V$ . This implies that

$$\frac{\partial \tilde{F}_{n+1}^{n+1}}{\partial y^{n+1}} = -\frac{\partial \tilde{F}_{n+1}^i}{\partial y^i} - \tilde{\Gamma}_{\mu\lambda}^\mu \tilde{F}_{n+1}^\lambda + \tilde{\Gamma}_{\lambda n+1}^{n+1} \tilde{F}_{n+1}^\lambda + \tilde{\Gamma}_{n+1 n+1}^i \tilde{F}_i^{n+1} - \tilde{\Gamma}_{i n+1}^i \tilde{F}_{n+1}^{n+1} \quad (27)$$

In terms of the components of  $\tilde{F}_{n+1}^{n+1}$  and  $\tilde{F}_i^{n+1}$  the equation above may be written as

$$\frac{\partial \tilde{F}_{n+1}^{n+1}}{\partial y^{n+1}} = -\varepsilon \bar{\phi}^2 \bar{g}^{ij} \frac{\partial \tilde{F}_i^{n+1}}{\partial y^j} - 2\tilde{\Gamma}_{in+1}^i \tilde{F}_{n+1}^{n+1} + \left( -\varepsilon \frac{\partial (\bar{\phi}^2 \bar{g}^{ij})}{\partial y^j} - \varepsilon \bar{\phi}^2 \bar{g}^{ij} \tilde{\Gamma}_{kj}^k + \tilde{\Gamma}_{n+1n+1}^i \right) \tilde{F}_i^{n+1} \quad (28)$$

Analogously for  $\beta = i$  we obtain

$$\frac{\partial \tilde{F}_i^{n+1}}{\partial y^{n+1}} = \frac{\partial \tilde{F}_{n+1}^{n+1}}{\partial y^i} + 2\tilde{\Gamma}_{n+1i}^{n+1} \tilde{F}_{n+1}^{n+1} + \left( \tilde{\Gamma}_{n+1i}^k + \varepsilon \bar{\phi}^2 \bar{g}^{kj} \tilde{\Gamma}_{ij}^{n+1} - \tilde{\Gamma}_{n+1\mu}^\mu \delta_i^k \right) \tilde{F}_k^{n+1} \quad (29)$$

By taking into account (28) and (29) it can be easily shown by mathematical induction that

$$\left. \frac{\partial^r \tilde{F}_\beta^{n+1}}{\partial (y^{n+1})^r} \right|_{y^{n+1}=0} = 0 \quad (30)$$

for any integer  $r \geq 0$ . As a consequence we conclude that  $\tilde{F}_\beta^{n+1} = 0$  in a neighborhood of the origin. Indeed, as  $\bar{g}_{ik}$  and  $\bar{\phi}$  are analytic at  $0 \in \mathbb{R}^{n+1}$ , then there exists such a neighborhood in which  $\tilde{F}_\beta^{n+1}$  can be expressed as a Taylor series about  $0 \in \mathbb{R}^{n+1}$ , which each term of this series being null. Since the equation  $\tilde{F}_\beta^{n+1} = 0$  is equivalent to the equations (21) and (22), then the lemma is proved.

### A. The Cauchy-Kowalewski theorem and the existence of the embedding.

Although essential to the main result to be presented later, Lemma 1 says nothing about the existence of the solutions  $\bar{g}_{ik}$  and  $\bar{\phi}$ . Thus we have to resort to the following theorem.

**Theorem (Cauchy-Kowalewski).** *Let us consider the set of partial differential equations:*

$$\frac{\partial^2 u^A}{\partial (y^{n+1})^2} = F^A \left( y^\alpha, u^B, \frac{\partial u^B}{\partial y^\alpha}, \frac{\partial^2 u^B}{\partial y^\alpha \partial y^i}, \right), \quad A = 1, \dots, m \quad (31)$$

where  $u^1, \dots, u^m$  are  $m$  unknown functions of the  $n+1$  variables  $y^1, \dots, y^n, y^{n+1}$ ,  $\alpha = 1, \dots, n+1$ ,  $i = 1, \dots, n$ ,  $B = 1, \dots, m$ . Also, let  $\xi^1, \dots, \xi^m, \eta^1, \dots, \eta^m$ , functions of the variables  $y^1, \dots, y^n$ , be analytic at  $0 \in \mathbb{R}^n$ . If the functions  $F^A$  are analytic with respect to each of their arguments around the values evaluated at the point  $y^1 = \dots = y^n = 0$ , then there exists a unique solution of equations (31) which is analytic at  $0 \in \mathbb{R}^{n+1}$  and that satisfies the initial condition

$$u^A (y^1, \dots, y^n, 0) = \xi^A (y^1, \dots, y^n) \quad (32)$$

$$\frac{\partial u^A}{\partial y^{n+1}} (y^1, \dots, y^n, 0) = \eta^A (y^1, \dots, y^n), \quad A = 1, \dots, m. \quad (33)$$

By using (16), we can rewrite (20) as

$$\begin{aligned} \frac{\partial^2 \bar{g}_{ik}}{\partial (y^{n+1})^2} = & \varepsilon \frac{4\Lambda}{1-n} \bar{\phi}^2 \bar{g}_{ik} + \frac{1}{\bar{\phi}} \frac{\partial \bar{\phi}}{\partial y^{n+1}} \frac{\partial \bar{g}_{ik}}{\partial y^{n+1}} - \frac{1}{2} \bar{g}^{jm} \left( \frac{\partial \bar{g}_{ik}}{\partial y^{n+1}} \frac{\partial \bar{g}_{jm}}{\partial y^{n+1}} - 2 \frac{\partial \bar{g}_{im}}{\partial y^{n+1}} \frac{\partial \bar{g}_{jk}}{\partial y^{n+1}} \right) \\ & - 2\varepsilon \bar{\phi} \left( \frac{\partial^2 \bar{\phi}}{\partial y^i \partial y^k} - \frac{\partial \bar{\phi}}{\partial y^j} \bar{\Gamma}_{ik}^j \right) - 2\varepsilon \bar{\phi}^2 \bar{R}_{ik}. \end{aligned} \quad (34)$$

Owing to the symmetry condition  $\bar{g}_{ik} = \bar{g}_{ki}$ , we can express Eq. (34) in terms of the functions  $\bar{g}_{ik}$  with  $i \leq k$ . If  $\bar{\phi}$  is regarded as a known function, then (34) becomes a set of partial differential equations for the  $m = \frac{n(n+1)}{2}$  unknown function  $\bar{g}_{ik}$  ( $i \leq k$ ). This set of equations has the same form as (31). We also note that the right-hand side of (34) consists of rational functions of the coordinates  $y$ , the functions  $\bar{g}_{ik}$  ( $i \leq k$ ) and their derivatives (up to first order with respect to  $y^{n+1}$  and up to second order relatively to the other coordinates).

Therefore, if we take  $\bar{\phi} \neq 0$  analytic at  $0 \in \mathbb{R}^{n+1}$  and if the initial conditions

$$\bar{g}_{ik} (y^1, \dots, y^n, 0) = g_{ik} (y^1, \dots, y^n) \quad (35)$$

$$\frac{\partial \bar{g}_{ik}}{\partial y^{n+1}} (y^1, \dots, y^n, 0) = -2\bar{\phi} (y^1, \dots, y^n, 0) \Omega_{ik} (y^1, \dots, y^n), \quad (36)$$

hold in some neighborhood of the point  $0 \in \mathbb{R}^n$ , where  $g_{ik}$  and  $\Omega_{ik}$  are arbitrary analytic functions with  $|g_{ik}| \neq 0$  at the origin, then the right-hand side of (34) will also be analytic at

$$y^1 = 0 \dots y^{n+1} = 0; \bar{g}_{ik} \Big|_0; \frac{\partial \bar{g}_{ik}}{\partial y^1} \Big|_0 \dots \frac{\partial \bar{g}_{ik}}{\partial y^{n+1}} \Big|_0; \frac{\partial^2 \bar{g}_{ik}}{\partial y^j \partial y^m} \Big|_0 \quad (37)$$

We conclude, then, from the Cauchy-Kowalewski theorem, that Eq. (34) admits a unique solution  $\bar{g}_{ik} (y^1, \dots, y^{n+1})$  which is analytic at  $0 \in \mathbb{R}^{n+1}$  and also satisfies the given initial conditions. It should be noted that the determinant  $|\bar{g}_{ik}|$  (which due to the initial conditions is non-null at the origin) remains non null in some open set of  $\mathbb{R}^{n+1}$  as a consequence of the continuity of the solution. These results may be summed up in the following lemma.

**Lemma 2.** Let  $g_{ik}(y^1, \dots, y^n)$  and  $\Omega_{ik}(y^1, \dots, y^n)$ , for  $i, k = 1, \dots, n$ , and  $\bar{\phi}(y^1, \dots, y^n, y^{n+1})$ , be arbitrary functions which are analytic at  $0 \in \mathbb{R}^n$  and  $0 \in \mathbb{R}^{n+1}$ , respectively, with  $g_{ik} = g_{ki}$ ,  $|g_{ik}| \neq 0$ ,  $\Omega_{ik} = \Omega_{ki}$  in some open set of  $\mathbb{R}^n$  containing  $0 \in \mathbb{R}^n$ , and  $\bar{\phi} \neq 0$  in some open set of  $\mathbb{R}^{n+1}$  containing  $0 \in \mathbb{R}^{n+1}$ . Then there exists a unique set of functions  $\bar{g}_{ik}(y^1, \dots, y^n, y^{n+1})$ , which are analytic at  $0 \in \mathbb{R}^{n+1}$ , that satisfy: i) the conditions (7), (8) and the equation (20) in a neighborhood of  $0 \in \mathbb{R}^{n+1}$ ; and ii) the initial conditions (6) and (36).

Now if we identify the functions  $g_{ik}$  of the initial conditions with the components of a semi-Riemannian manifold  $M^n$ , then from lemma 1, lemma 2 and theorem 1 we can proof the following theorem:

**Theorem 2.** Let  $M^n$  be a  $n$ -dimensional semi-Riemannian manifold with metric given by

$$ds^2 = g_{ik}dx^i dx^j,$$

in coordinate system  $\{x^i\}$  of  $M^n$ . Let  $p \in M^n$  have coordinates  $x_p^1 = \dots = x_p^n = 0$ . Then,  $M^n$  has a local isometric and analytic embedding (at the point  $p$ ) in a  $(n+1)$ -dimensional Einstein space with cosmological constant  $\Lambda$  if and only if there exist functions  $\Omega_{ik}(x^1, \dots, x^n)$ , ( $i, k = 1, \dots, n$ ), that are analytic at  $0 \in \mathbb{R}^n$  and such that

$$\Omega_{ik} = \Omega_{ki} \tag{38}$$

$$g^{jk}(\nabla_j \Omega_{ik} - \nabla_i \Omega_{jk}) = 0 \tag{39}$$

$$g^{ik}g^{jm}(R_{ijkm} + \varepsilon(\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im})) = -2\Lambda. \tag{40}$$

*Proof.* Let  $\mathcal{M}_\Lambda^{n+1}$  be the collection of all  $(n+1)$ -dimensional Einstein spaces with cosmological constant  $\Lambda$ . If  $M^n$  has a local and analytic embedding in  $\mathcal{M}_\Lambda^{n+1}$ , at the point  $p$ , then in accordance with theorem 1, there exist functions  $\bar{g}_{ik}(x^1, \dots, x^{n+1})$ , satisfying (6), and  $\bar{\phi}(x^1, \dots, x^{n+1})$  that are analytic at  $0 \in \mathbb{R}^{n+1}$  such that

$$ds^2 = \bar{g}_{ik}dx^i dx^k + \varepsilon \bar{\phi}^2 dx^{n+1} dx^{n+1} \tag{41}$$

is the line element of some member of  $\mathcal{M}_\Lambda^{n+1}$  expressed in a conveniently chosen coordinate system. Therefore, this metric satisfies the equations (20), (21) and (22) in a neighborhood

of  $0 \in \mathbb{R}^{n+1}$ . In particular, this is true for points lying on the hypersurface  $x^{n+1} = 0$ , where  $\bar{g}_{ik} = g_{ik}$ , from (6). Thus, if we define  $\Omega_{ik}(x^1, \dots, x^n) = \bar{\Omega}_{ik}(x^1, \dots, x^n, 0)$ , then, the functions  $\Omega_{ik}$  necessarily satisfy (38), (39) and (40).

Let us consider the sufficient condition. First, we choose  $\bar{\phi}(x^1, \dots, x^{n+1}) \neq 0$  and analytic at  $0 \in \mathbb{R}^{n+1}$ . According to lemma 2, there exists a unique set of functions  $\bar{g}_{ik}(x^1, \dots, x^{n+1})$  satisfying (6), (7), (8), (16), (20) and the condition  $\bar{\Omega}_{ik}(x^1, \dots, x^n, 0) = \Omega_{ik}(x^1, \dots, x^n)$ . If  $\Omega_{ik}$  and  $g_{ik}$  satisfy (39) and (40), then, from lemma 1 the functions  $\bar{g}_{ik}(x^1, \dots, x^{n+1})$  satisfy (20), (21) and (22) in a neighborhood of  $0 \in \mathbb{R}^{n+1}$ , which in turn implies that the line element formed with  $\bar{g}_{ik}$  and  $\bar{\phi}$  is that of an Einstein space with cosmological constant  $\Lambda$ . Then, theorem 1 tell us that  $M^n$  has a local isometric and analytical embedding in  $\mathcal{M}_\Lambda^{n+1}$ .  $\square$

We now want to show that once the functions  $g_{ik}$  are given, the equations (38), (39) and (40) always admit a solution for  $\Omega_{ik}$ . These equations constitute a set of  $n$  partial differential equations (Eq. (39)) plus a constraint equation (Eq.(40)) for  $\frac{n(n+1)}{2}$  independent functions  $\Omega_{ik}$ . Except for  $n = 1$ , the number of unknown functions is greater than (or equal to ( $n = 2$ )) the number of equations. Out of the set of functions  $\Omega_{ik}$  we pick  $n$  functions to be regarded as the unknowns. We proceed to put (39) in the form required for application of the Cauchy-Kowalewski (first-order derivative version) theorem to assure the existence of the solution. The detailed proof is a bit laborious, so we shall omit some of its parts.

For the sake of the argument and with no loss of generality we assume that we are using a coordinate system in which  $g_{11} \neq 0$  and  $g_{1k} = 0$ ,  $k = 2, \dots, n$ . Thus,  $g^{11} = \frac{1}{g_{11}}$  and  $g^{1k} = 0$ . Eq. (39) can be written as

$$g^{rs} (\Omega_{sk,r} - \Omega_{rs,k} + \Omega_{rt} \Gamma_{sk}^t - \Omega_{kt} \Gamma_{sr}^t) = 0. \quad (42)$$

Recalling that  $\Omega_{ik} = \Omega_{ki}$ , it is not difficult to see that (42) may be put in the form

$$g^{rs} \left( \Omega_{sk,r} + \Omega_{ks,r} - 2 \Omega_{rs,k} \right) - g^{rr} \Omega_{rr,k} + g^{rs} \left( \Omega_{tr} \Gamma_{sk}^t + \Omega_{rt} \Gamma_{sk}^t - \Omega_{tk} \Gamma_{sr}^t - \Omega_{kt} \Gamma_{sr}^t \right) = 0. \quad (43)$$

Likewise and taking advantage of the special form of the metric we can write the Eq. (40) as

$$\begin{aligned}
& 2g^{11}\Omega_{11} g^{rs} \left( \begin{matrix} \Omega_{rs} + \Omega_{sr} \\ r \leq s \quad s < r \end{matrix} \right) - 2g^{11} g^{rs} \Omega_{1r}\Omega_{1s} \quad + \quad (44) \\
& + g^{rs} g^{tu} \left[ \left( \begin{matrix} \Omega_{rs} + \Omega_{sr} \\ r \leq s \quad s < r \end{matrix} \right) \left( \begin{matrix} \Omega_{tu} + \Omega_{ut} \\ t \leq u \quad u < t \end{matrix} \right) - \left( \begin{matrix} \Omega_{ru} + \Omega_{ur} \\ r \leq u \quad u < r \end{matrix} \right) \left( \begin{matrix} \Omega_{st} + \Omega_{ts} \\ s \leq t \quad t < s \end{matrix} \right) \right] + \varepsilon R = -\varepsilon \Lambda
\end{aligned}$$

Our next step is to identify in the Eq. (43) the terms containing derivatives of  $\Omega_{ik}$  with respect to the coordinate  $x^1$ . Let us consider first the case  $k = 1$ . Thus, we have

$$g^{rs} \left( \begin{matrix} \Omega_{1s,r} - 2 \Omega_{rs,1} \\ r,s > 1 \quad 1 < r < s \end{matrix} \right) - g^{rr} \Omega_{rr,1} + g^{rs} \left( \begin{matrix} \Omega_{tr} \Gamma_{s1}^t + \Omega_{rt} \Gamma_{s1}^t - \Omega_{11} \Gamma_{sr}^1 - \Omega_{1t} \Gamma_{sr}^t \\ r,s > 1 \quad t \leq r \quad r < t \end{matrix} \right) = 0 \quad (45)$$

In order to write (45) in the form specified by Cauchy-Kowalewski (in its first-derivative version) we should decide what among the functions  $\Omega_{ik}$  ( $i \leq k$ ) are to be chosen as unknowns. We also note that since  $|g_{ik}| \neq 0$  there exists at least an index  $r' > 1$  such that  $g^{r'n} \neq 0$ . We choose  $\Omega_{r'n}$  as one of the unknown functions and solve (45) for  $\frac{\partial \Omega_{r'n}}{\partial x^1}$ . Thus, it is possible to put (45) in the form

$$\begin{aligned}
\frac{\partial \Omega_{r'n}}{\partial x^1} = \frac{1}{g^{r'n} (\delta_{r'n} - 2)} & \left[ - g^{rs} \Omega_{1s,r} + 2 g^{rs} \Omega_{rs,1} + g^{rr} \Omega_{rr,1} + \right. \quad (46) \\
& \left. + g^{r'r'} \Omega_{r'r',1} (1 - \delta_{r'n}) - g^{rs} \left( \begin{matrix} \Omega_{tr} \Gamma_{s1}^t + \Omega_{rt} \Gamma_{s1}^t - \Omega_{11} \Gamma_{sr}^1 - \Omega_{1t} \Gamma_{sr}^t \\ r,s > 1 \quad t \leq r \quad r < t \end{matrix} \right) \right],
\end{aligned}$$

where no sum over  $r'$  is implied.

For  $k \geq 2$  we have

$$\begin{aligned}
\frac{\partial \Omega_{1k}}{\partial x^1} = g_{11} & \left[ - g^{rs} \left( \begin{matrix} \Omega_{sk,r} + \Omega_{ks,r} - 2 \Omega_{rs,k} \\ r,s > 1 \quad s \leq k \quad k < s \quad r < s \end{matrix} \right) - g^{11} \Omega_{11,k} - g^{rr} \Omega_{rr,k} \right. \quad (47) \\
& \left. - g^{rs} \left( \begin{matrix} \Omega_{tr} \Gamma_{sk}^t + \Omega_{rt} \Gamma_{sk}^t - \Omega_{tk} \Gamma_{sr}^t - \Omega_{kt} \Gamma_{sr}^t \\ t \leq r \quad r < t \quad t \leq k \quad k < t \end{matrix} \right) \right], \quad k \geq 2.
\end{aligned}$$

From (44) we can express  $\Omega_{11}$  in terms of the other  $\Omega_{ik}$ . Thus

$$\begin{aligned}
\Omega_{11} = \frac{1}{2g^{11} g^{rs} \left( \begin{matrix} \Omega_{rs} + \Omega_{sr} \\ r,s > 1 \quad r \leq s \quad s < r \end{matrix} \right)} & \left[ 2g^{11} g^{rs} \Omega_{1r}\Omega_{1s} \right. \quad (48) \\
& \left. - g^{rs} g^{tu} \left[ \left( \begin{matrix} \Omega_{rs} + \Omega_{sr} \\ r \leq s \quad s < r \end{matrix} \right) \left( \begin{matrix} \Omega_{tu} + \Omega_{ut} \\ t \leq u \quad u < t \end{matrix} \right) - \left( \begin{matrix} \Omega_{ru} + \Omega_{ur} \\ r \leq u \quad u < r \end{matrix} \right) \left( \begin{matrix} \Omega_{st} + \Omega_{ts} \\ s \leq t \quad t < s \end{matrix} \right) \right] - \varepsilon (R + \Lambda) \right]
\end{aligned}$$

Finally, substituting  $\Omega_{11}$  from (48) into (46) and (47) we obtain a set of partial differential equations for the functions  $\Omega_{ik}$ . If we regard the functions  $\Omega_{ik}$  ( $i \leq k$ ) with  $i > 1$  and  $(i, k) \neq (r', n)$  as analytic functions already known, then we apply the Cauchy-Kowalewski theorem (first derivative version) to this set of differential equations considering  $\Omega_{1k}$  ( $k > 1$ ) and  $\Omega_{r'n}$  as the unknown functions. We, then, choose  $\Omega_{ik}(x^1, \dots, x^n)$  ( $i \leq k, i > 1, (i, k) \neq (r', n)$ ) and the initial conditions  $\Omega_{1k}(0, x^2, \dots, x^n) = f_k(x^2, \dots, x^n)$  ( $k > 1$ ) and  $\Omega_{r'n}(0, x^2, \dots, x^n) = f_1(x^2, \dots, x^n)$ . Of course the chosen functions must be analytic at  $0 \in \mathbb{R}^n$  and satisfy the condition

$$g^{rs} \left( \Omega_{rs} + \Omega_{sr} \right) \Big|_0 \neq 0. \quad (49)$$

It should be noted that it is always possible to have functions  $\Omega_{ik}$  that satisfy the condition above. For instance, if we take  $\Omega_{ik} = 0$  ( $i \leq k, i > 1, (i, k) \neq (r', n)$ ) this condition reduces to  $g^{r'n} \Omega_{r'n} \Big|_0 \neq 0$ . Hence we just choose  $\Omega_{r'n} \neq 0$ .

Once we have specified the functions  $\Omega_{ik}(x^1, \dots, x^n)$  ( $i \leq k, i > 1, (i, k) \neq (r', n)$ ) the right-hand side of (46) and (47) becomes function of the arguments

$$x^1, \dots, x^n; \Omega_{1k}, \Omega_{r'n}; \Omega_{1k,j}, \Omega_{r'n,j} \quad (50)$$

$k > 1$                        $k, j > 1$                        $j > 1$

which is analytic at

$$x^1 = 0, \dots, x^n = 0; \Omega_{1k} \Big|_0, \Omega_{r'n} \Big|_0; \Omega_{1k,j} \Big|_0, \Omega_{r'n,j} \Big|_0. \quad (51)$$

$k > 1$                        $k, j > 1$                        $j > 1$

Therefore, the Cauchy-Kowalewski theorem asserts that there exists a unique set of functions  $\Omega_{1k}(x^1, x^2, \dots, x^n)$  ( $k > 1$ ) and  $\Omega_{r'n}(x^1, x^2, \dots, x^n)$ , analytic at  $0 \in \mathbb{R}^{n+1}$  which satisfy the equations (46) e (47). We determine  $\Omega_{11}$  from (48) by taking the chosen functions  $\Omega_{ik}(x^1, \dots, x^n)$  ( $i \leq k, i > 1, (i, k) \neq (r', n)$ ) and the solutions of the system of equations. From (49) and due to the analyticity of  $g_{ik}$  and of the solution we conclude that  $\Omega_{11}$  is analytic at the origin. Therefore, the existence of analytic functions  $\Omega_{ik}$  satisfying (38), (39) and (40) is demonstrated. The above may be summarized in the following lemma:

**Lemma 3.** *Let  $g_{ik}(x^1, \dots, x^n)$  and  $\Omega_{ik}(x^1, \dots, x^n)$  ( $i \leq k, i > 1, (i, k) \neq (r', n)$ ) be analytic functions at the origin  $0 \in \mathbb{R}^n$ , with  $\Omega_{ik}$  satisfying the initial conditions  $\Omega_{1k}(0, x^2, \dots, x^n) =$*



$f_k(x^2, \dots, x^n)$  ( $k > 1$ ) and  $\Omega_{r'n}(0, x^2, \dots, x^n) = f_1(x^2, \dots, x^n)$ , where  $f_k$  are analytic at  $0 \in \mathbb{R}^n$ . If, in addition, the condition (49) is fulfilled then there exists a unique set of functions  $\Omega_{ik}(x^1, \dots, x^n)$  ( $i, k = 1, \dots, n$ ), analytic at  $0 \in \mathbb{R}^{n-1}$ , which satisfy the equations (38), (39) and (40).

Therefore, according to the lemma above, if we are given a set of analytic functions  $g_{ik}$ , then the existence of analytic functions  $\Omega_{ik}$  which satisfy the equations (38), (39) and (40) is assured. In this way lemma 3 tell us that the sufficient conditions of theorem (2) are satisfied, so we can state the final theorem.

**Theorem 3.** *Let  $M^n$  ( $n > 1$ ) be a semi-Riemannian space with line element*

$$ds^2 = g_{ik}dx^i dx^k,$$

*expressed in a coordinate system which covers a neighborhood of a point  $p \in M^n$  whose coordinates are  $x_p^1 = \dots = x_p^n = 0$ . If  $g_{ik}$  are analytic functions at  $0 \in \mathbb{R}^n$ , then  $M^n$  can be embedded at  $p$  in some  $(n + 1)$ -dimensional Einstein space  $N^{n+1} \in \mathcal{M}_\Lambda$ .*

Two comments are in order. First, if the  $\frac{n(n-1)}{2} - 1$  specified arbitrary functions  $\Omega_{ik}(x^1, \dots, x^n)$  ( $i \leq k, i > 1, (i, k) \neq (r', n)$ ) obey the conditions

- i) the functions  $\Omega_{ik}$  ( $i \leq k, i > 1, (i, k) \neq (r', n)$ ) are analytic at  $0 \in \mathbb{R}^n$ ;
- ii) the  $n$  functions  $\Omega_{1k}(0, x^2, \dots, x^n) = f_k(x^2, \dots, x^n)$  ( $k > 1$ ) and  $\Omega_{r'n}(0, x^2, \dots, x^n) = f_1(x^2, \dots, x^n)$  are analytic at  $0 \in \mathbb{R}^{n-1}$ , with  $g^{rs} \left( \Omega_{rs} + \Omega_{sr} \right) \Big|_0 \neq 0$ ;
- iii) a function  $\bar{\phi}(x^1, \dots, x^{n+1}) \neq 0$ , analytic at  $0 \in \mathbb{R}^{n+1}$ , is chosen;

then the line element of the embedding space as referred to in theorem 1 is unique..

Second, if we consider the case  $\Lambda = 0$ , then clearly this theorem reduces to Campbell-Magaard theorem, which establishes the existence of local analytic embedding of any Riemannian manifold in the set of Ricci-flat spaces. In this sense, theorem (3) is a generalization of the Campbell-Magaard theorem.

### III. APPLICATIONS OF THE EXTENDED CAMPBELL-MAGAARD THEOREM

Let us consider some cases where  $(M^n, g)$  is a Lorentzian manifold of dimension  $n = 4$ . We know from theorem (3) that there exists at least one Einstein space of dimension  $n = 5$  in which  $(M^4, g)$  can be embedded. In this section, we shall discuss the embedding of Minkowski and Schwarzschild space-time and exhibit explicitly the five-dimensional embedding Einstein spaces.

It is interesting to observe that the very proof of theorem (3) suggests a procedure for obtaining the embedding. Indeed, let  $g_{ik}$  be the components of the metric of  $M^n$  in a coordinate system and let  $g_{ik}$  be analytic at the origin of  $\mathbb{R}^n$ . We first take

$$\bar{g}_{ik}(x^1, \dots, x^n, 0) = g_{ik}(x^1, \dots, x^n), \quad (52)$$

and then look for functions  $\Omega_{ik}$  which satisfy the equations (38), (39) and (40) at  $\Sigma_0$ . Given that the function  $\bar{\phi}$  is arbitrary we can take  $\bar{\phi} = 1$  for simplicity. Then, the derivative of  $\bar{g}_{ik}$  with respect to the extra coordinate at  $\Sigma_0$  will be given by

$$\frac{\partial \bar{g}_{ik}}{\partial x^{n+1}}(x^1, \dots, x^n, 0) = -2\Omega_{ik}(x^1, \dots, x^n). \quad (53)$$

From  $\bar{g}_{ik}$  and the derivative  $\frac{\partial \bar{g}_{ik}}{\partial x^{n+1}}$  we now calculate  $\frac{\partial^2 \bar{g}_{ik}}{\partial (x^{n+1})^2}$  at  $\Sigma_0$  by using (34):

$$\frac{\partial^2 \bar{g}_{ik}}{\partial (x^{n+1})^2}(x^1, \dots, x^n, 0) = \varepsilon \frac{4\Lambda}{1-n} g_{ik} - 2g^{jm}(\Omega_{ik}\Omega_{jm} - 2\Omega_{im}\Omega_{jk}) - 2\varepsilon R_{ik}. \quad (54)$$

The equation (34) is supposed to hold in an open set of  $\mathbb{R}^{n+1}$ , hence it can be differentiated with respect to  $x^{n+1}$ . Doing this, we are able to put the third derivative of  $\bar{g}_{ik}$  as a function of terms which with respect to the coordinate  $x^{n+1}$  contain at most second derivatives. In this manner we can obtain  $\frac{\partial^3 \bar{g}_{ik}}{\partial (x^{n+1})^3}(x^1, \dots, x^n, 0)$  from the derivatives  $\frac{\partial \bar{g}_{ik}}{\partial x^{n+1}}(x^1, \dots, x^n, 0)$  and  $\frac{\partial^2 \bar{g}_{ik}}{\partial (x^{n+1})^2}(x^1, \dots, x^n, 0)$  already calculated. This process can go on indefinitely by successively differentiating Eq. (34) with respect  $x^{n+1}$  and such a procedure will give us the derivatives of all orders calculated at  $\Sigma_0$ . By collecting all these terms we obtain the expression of  $\bar{g}_{ik}$  into a power series about the origin  $0 \in \mathbb{R}^{n+1}$ , which by the Cauchy-Kowalewski theorem,

is convergent in a neighborhood of  $0 \in \mathbb{R}^{n+1}$ . Thus, the functions  $\bar{g}_{ik}$  and  $\bar{\phi} = 1$  together constitute the components of the sought embedding Einstein space.

Let us apply these ideas by first considering the embedding of Minkowski space-time. In Cartesian coordinates the metric of Minkowski spacetime has the form

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (55)$$

hence the components  $g_{ik} = \eta_{ik} = \text{diag}(-1, 1, 1, 1)$  are analytic, and, by theorem (3), it can be embedded in a five-dimensional Einstein space with arbitrary cosmological constant  $\Lambda$ . The first task is to find functions  $\Omega_{ik}$  which satisfy the equations (38), (39) and (40). As Minkowski spacetime has no curvature the equation (40) yields

$$\eta^{ik}\eta^{jm} (\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im}) = -2\varepsilon\Lambda, \quad (56)$$

It is easy to verify that the choice  $\Omega_{ik} = -\frac{c}{2}\eta_{ik}$ , where  $c$  is a constant, solves the above equation if we take  $c = \sqrt{-\frac{2\varepsilon\Lambda}{3}}$ . Also, this choice satisfies (39). On the other hand, choosing  $\bar{\phi} = 1$ , equation (36) becomes

$$\left. \frac{\partial \bar{g}_{ik}}{\partial u} \right|_{u=0} = c\eta_{ik}. \quad (57)$$

Then, from (34) we obtain

$$\begin{aligned} \left. \frac{\partial^2 \bar{g}_{ik}}{\partial u^2} \right|_{u=0} &= \left[ \varepsilon \frac{4\Lambda}{1-4} \bar{g}_{ik} - \frac{1}{2} \bar{g}^{jm} \left( \frac{\partial \bar{g}_{ik}}{\partial u} \frac{\partial \bar{g}_{jm}}{\partial u} - 2 \frac{\partial \bar{g}_{im}}{\partial u} \frac{\partial \bar{g}_{jk}}{\partial u} \right) \right] \Big|_{u=0} = \\ &= -\varepsilon \frac{4\Lambda}{3} \eta_{ik} - \frac{c^2}{2} \eta^{jm} (\eta_{ik}\eta_{jm} - 2\eta_{im}\eta_{jk}) = c^2 \eta_{ik}. \end{aligned} \quad (58)$$

Following the procedure outlined formerly the higher-order derivatives can be easily calculated and we finally obtain

$$\bar{g}_{ik} = \sum_{p=0}^{\infty} \frac{c^p}{p!} u^p \eta_{ik} = e^{cu} \eta_{ik} \quad (59)$$

Therefore, Minkowski spacetime can be embedded in the Einstein space whose metric is given by

$$ds^2 = e^{\sqrt{-\frac{2\varepsilon\Lambda}{3}}u} (-dt^2 + dx^2 + dy^2 + dz^2) + \varepsilon du^2. \quad (60)$$

and  $u = 0$  corresponds to the sought embedding. Of course the choice of  $\varepsilon$  depends upon the given  $\Lambda$ , since we must have  $\varepsilon\Lambda < 0$ . If want  $\varepsilon$  and  $\Lambda$  to have the same sign, then another choice of  $\Omega_{ik}$  has to be made. If  $\Lambda$  is negative, then the embedding space is closely related to the so-called bulk, in the Randall-Sundrum braneworld scenario [3].

As a second example let us consider the embedding of Schwarzschild spacetime whose geometry can be described by the line element

$$ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \quad (61)$$

where  $m$  is a constant. Except for  $r = 0$  and  $r = 2m$  the metric components are analytic in the subset of  $\mathbb{R}^4$  corresponding to the range of the coordinates. Though theorem (3) seems to single out the point  $(0, 0, 0, 0)$  as far as the embedding is concerned it should be clear that there is nothing special about this point and the embedding can also be achieved at any other point where the metric components are analytic.

As Schwarzschild spacetime is Ricci-flat the equation (40) yields

$$g^{ik} g^{jm} (\Omega_{ik} \Omega_{jm} - \Omega_{jk} \Omega_{im}) = -2\varepsilon\Lambda. \quad (62)$$

A solution of this equation is given by

$$\Omega_{ik} = -\frac{c}{2} g_{ik}, \quad (63)$$

where  $g_{ik}$  are the metric components of Schwarzschild spacetime and  $c = \sqrt{-\frac{2\varepsilon\Lambda}{3}}$ . Due to the compatibility condition  $\nabla_j g_{ik} = 0$ , it is immediately seen that the equation (39) is satisfied. Again, let us choose  $\bar{\phi} = 1$ . At this point we could repeat the iterative procedure employed in the previous example and obtain the metric components as a power series. Though this method is quite direct, sometimes it becomes a bit laborious. Here, let us follow a different route. We shall assume the *ansatz*

$$\bar{\Omega}_{ik}(t, r, \theta, \varphi, u) = -\frac{c(u)}{2} g_{ik}(t, r, \theta, \varphi). \quad (64)$$

For  $u = 0$  the equations (39) e (40) will hold provided that  $c(0) = \sqrt{-\frac{2\varepsilon\Lambda}{3}}$ . Now from the

definition of  $\bar{\Omega}_{ik}$ , with  $\bar{\phi} = 1$ , we have

$$\frac{\partial \bar{g}_{ik}}{\partial u} = c(u) g_{ik}. \quad (65)$$

After integrating the equation above and taking into account the initial conditions imposed on  $\bar{g}_{ik}$  we have  $\bar{g}_{ik}(t, r, \theta, \varphi, u) = f(u) g_{ik}$ , where  $f(u) = \int c(w) dw$ , with  $f(0) = 1$ . On the other hand the Ricci tensor  $\bar{R}_{ik}$  associated with  $\bar{g}_{ik}$  vanishes everywhere, so (34) yields the ordinary differential equation for  $f(u)$ :

$$f'' + \frac{f'^2}{f} + \frac{4\varepsilon\Lambda}{3} f = 0. \quad (66)$$

The solution of (66) is given by

$$f(u) = e^{\sqrt{-\frac{2\varepsilon\Lambda}{3}}u}. \quad (67)$$

Hence the conditions  $c(0) = \sqrt{-\frac{2\varepsilon\Lambda}{3}}$  and  $f(0) = 1$  are satisfied. We conclude then that Schwarzschild spacetime can be embedded in the Einstein space:

$$ds^2 = e^{\sqrt{-\frac{2\varepsilon\Lambda}{3}}u} \left[ - \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right] + \varepsilon du^2, \quad (68)$$

the embedding taking place for  $u = 0$ .

A third application of theorem 3 may be illustrated by turning our attention to a more general situation. Suppose  $(M^4, g)$  is an Einstein space, that is

$$R_{ik} = -\lambda g_{ik}. \quad (69)$$

Our aim is to find a five-dimensional  $(M^5, \tilde{g})$  with a given arbitrary cosmological constant  $\Lambda$  in which  $(M^4, g)$  can be embedded.

As in the former example let us assume that

$$\bar{g}_{ik} = f(u) g_{ik}. \quad (70)$$

with  $f(0) = 1$ . Since we now have  $R = -4\lambda$ , the equation (40) can be satisfied only if  $f'(0) = \sqrt{-\frac{2\varepsilon\Lambda}{3} + \frac{4\varepsilon\lambda}{3}}$ .

Owing to the fact that  $\bar{g}_{ik}$  and  $g_{ik}$  are related by a conformal transformation which depends on the extra coordinate  $u$  only, it follows that  $\bar{R}_{ik} = R_{ik} = -\lambda g_{ik}$ . Thus, Eq. (34) is equivalent to

$$f'' + \frac{f'^2}{f} + \frac{4\varepsilon\Lambda}{3}f - 2\varepsilon\lambda = 0. \quad (71)$$

It is not difficult to show that the solution which satisfies the initial conditions imposed on  $f$  is given by [9]

$$f(u) = \left[ \cosh \left( \sqrt{-\frac{\varepsilon\Lambda}{6}}u \right) + \left( 1 - 2\frac{\lambda}{\Lambda} \right) \sinh \left( \sqrt{-\frac{\varepsilon\Lambda}{6}}u \right) \right]^2. \quad (72)$$

In this way we conclude that the Einstein space  $(M^4, g)$  can be embedded in the Einstein space

$$ds^2 = f(u)g_{ik}dx^i dx^j + \varepsilon du^2, \quad (73)$$

with  $f(u)$  being given by (72).

#### IV. CONCLUSION

The recent appearance of physical models which regard the ordinary spacetime as a hypersurface embedded in a five-dimensional manifold has naturally raised the question of what kind of mathematical conditions both the embedded and the spaces are subject to. An answer to this question necessarily involves a careful account of the mathematical theory of embedding. Particularly useful and clarifying are the Campbell-Magaard theorem and its extension to the case in which the embedding manifold is an Einstein space. Belonging to the latter kind is the embedding considered in the Randall-Sundrum braneworld. On the other hand embeddings in Ricci-flat five-dimensional manifolds are crucial for the non-compactified approach to Kaluza-Klein gravity [4]. Of course, if according to some new physical model the five-dimensional surrounding manifolds should obey some field equations, e.g., Einstein field equations, it would be of importance to investigate whether further extensions of Campbell-Magaard theorem could be achieved in order to accommodate these models. We are currently doing some research in this direction.

## V. ACKNOWLEDGMENTS

The authors thank Dr. E. M. Monte for helpful discussions. This work was supported financially by CNPq.

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