does not "soft land," that is, we assume that $y'(T(\theta)) < 0$, where $T(\theta)$ is the impact time. From (2) and the definitions of $f_2$ and $f_4$, we have

$$y'(t)=v \sin \theta f_2'(t)-gf_4'(t)=e^{-f_1(t)}(v \sin \theta-gf_3(t))$$

and hence the impact assumption is

$$f_3(T(\theta))>\frac{v}{g} \sin \theta. \tag{9}$$

In terms of the function

$$\rho(\theta)=\frac{R(\theta)}{v \cos \theta}$$

we have by (1),

$$R(\theta)=x(T(\theta))=v \cos \theta f_2(T(\theta))$$

and hence

$$T(\theta)=f_2^{-1}(\rho(\theta)).$$

The impact assumption (9) is therefore equivalent to

$$f_3(f_2^{-1}(\rho(\theta)))>\frac{v}{g} \sin \theta. \tag{10}$$

Now, $R(\theta)$ is differentiable if and only if $\rho(\theta)$ is differentiable. By (3), $\rho(\theta)$ is defined by $P(\rho(\theta),\ \theta)=0$, where

$$P(\rho,\theta)=v \sin \theta\rho-gf_4(f_2^{-1}(\rho)).$$

Finally, at $\rho = \rho(\theta)$,

$$\frac{\partial P}{\partial \rho}=v \sin \theta-gf_4'(f_2^{-1}(\rho))f_2^{-1\prime}(\rho)$$
$$=v \sin \theta-gf_3(f_2^{-1}(\rho))<0$$

by (10), and hence $\rho(\theta)$ is differentiable by the Implicit Function Theorem.[7]


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# Minkowski diagrams in momentum space


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## I. INTRODUCTION

Minkowski diagrams in configuration space, with points representing events, are often used in undergraduate courses on special relativity. Similar diagrams in momentum space are seldom shown, and the object of this note is to demonstrate their pedagogical usefulness in discussing particle interactions. In configuration space each point has coordinates $(t,\mathbf{x})$; in momentum space the coordinates are $(E,\mathbf{p})$. Two examples should be sufficient to show how such diagrams can be used.

## II. EXAMPLES

### A. Fission

In this example there is just one space dimension: Minkowski space is two dimensional. A particle of mass $m$ is represented by its mass shell, a hyperbola opening in the positive $E$ direction, given by

$$\left(\frac{E}{c}\right)^2-p^2=(mc)^2.$$

Figure 1 shows two such mass shells belonging to masses $m$ and $M>m$, each labeled by its mass. The scale on the energy axis is chosen as $E/c$ rather than $E$, so the two mass shells cross the $E/c$ axis at $mc$ and $Mc$, respectively. Each point on an $m$ mass shell represents a state of a particle of mass $m$, i.e., possible values of its energy and momentum. A vector from the origin to such a point represents the energy–momentum $(E-p)$ vector of that state.

Consider a particle of mass $M$ at rest, say a uranium nucleus, that undergoes fission to two particles of equal mass $m$. The vertical arrow in Fig. 1 represents the original uranium $E-p$ vector. $E-p$ conservation implies that the $E-p$ vectors of the two fission fragments add up to the original one, and since the total momentum is zero, the momenta of the two fission fragments must be negatives: their $E-p$ vectors have opposite $p$ components. Symmetry of the $m$ mass shell about the $E/c$ axis then implies that their $E/c$ components are equal, and conservation then implies than each $E/c$ component is equal to $Mc/2$. It is clear from the dia-

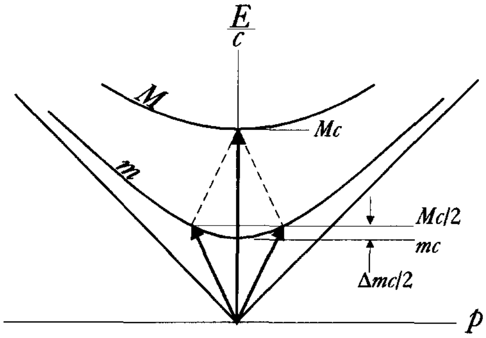


Fig. 1. Fission.

gram that each $E/c$ component is higher than the point at which the $m$ mass shell crosses the $E/c$ axis, i.e., greater than $mc$, so $m<\frac{1}{2}M$,

$$Mc-2mc\equiv\Delta mc>0.$$

As the fission fragments interact with their surroundings, they slow down and eventually come to rest. Then their total $E/c$ is $2mc$, so the energy they give up to their surroundings is just $\Delta E=\Delta mc^2$. This is the real content of the famous equation $E=mc^2$, involving measurable energy changes rather than absolute values relative to some more or less arbitrarily chosen zero of energy. Note that the mass of the fission fragments is not determined. But because their energies are both $\frac{1}{2}Mc^2$, the mass $m$ and momentum $p$ are related by

$$\left(\tfrac{1}{2}Mc\right)^2-p^2=(mc)^2.$$

The logical order in which to present this in class is first to draw the $M$ mass shell, then the two $E-p$ vectors of the fission fragments, and only then to draw in the $m$ mass shell.

This example is easily generalized to fission fragments of unequal masses. Also, a similar diagram can be used to illustrate fusion or the binding energy of the deuteron. Then $M$ is less than $2m$, and the $M$ mass shell crosses the $E/c$ axis below $2mc$.

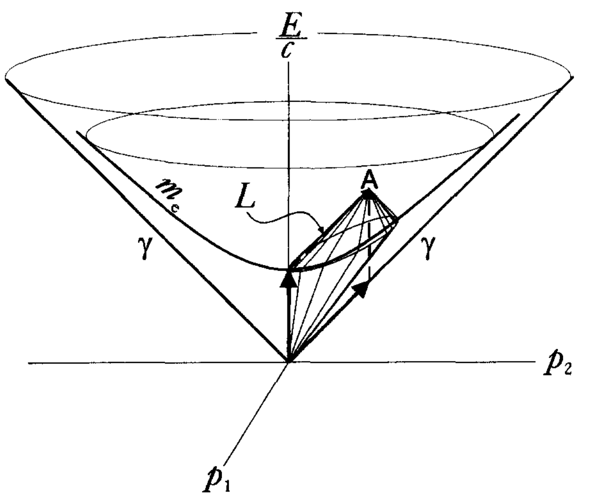


Fig. 2. Compton scattering.

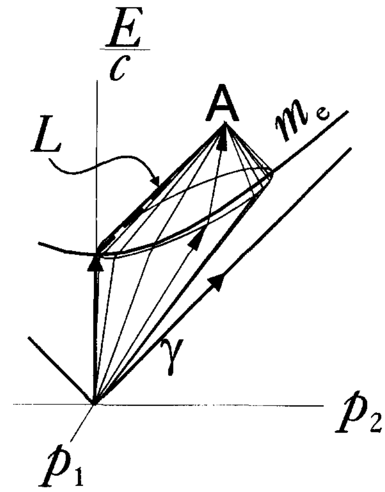


Fig. 3. Compton scattering (detail).

### B. Compton scattering

Now take Minkowski space to be three dimensional, as in Fig. 2. The mass shell is now a hyperboloid of revolution. In the figure the intersection of the $(E/c,p_2)$ plane with the electron mass shell is the hyperbola labeled $m_e$, and the intersection with the light cone consists of the two lines labeled $\gamma$. The light cone is the mass shell of the photon, whose equation is

$$\left(\frac{E}{c}\right)^2-|\mathbf{p}|^2=0.$$

The vertical arrow in Fig. 2 is the $E-p$ vector of an electron at rest, and the other arrow represents an incident photon. The system's total $E-p$ vector is represented by the point labeled A (the vector to A is not drawn to avoid confusion). After scattering, the electron $E-p$ vector (again on the electron mass shell) plus the scattered photon $E-p$ vector (again on the light cone) must add up to A. A way to draw this is to construct an inverted light cone $L$ with its vertex at A. The $E-p$ vectors of all possible scattered photons arrive at A from the closed curve, almost a circle, at which $L$ intersects $m_e$ in this three-dimensional space–time (in four dimensions this would be a closed surface, almost a sphere).

Figure 3 is an enlargement of part of Fig. 2. One possible combination of scattered electron and photon $E-p$ vectors is indicated with arrows. The direction of the scattered photon is obtained by projecting its $E-p$ vector onto the $(p_1,p_2)$ plane, so the different lines on the cone represent photons moving in different directions. It is immediately evident that the photon energy $E$, and hence its frequency $\nu$ and wavelength $\lambda$, are determined by its direction.

## III. CONCLUSION

Other particle interactions can also be visualized on similar Minkowski diagrams. The goal of this note is to show how the dynamics can be visualized, not to perform the calculations. The equations of the mass shells can be used, however, as a starting point for going on to the calculations.