Derivation of relativistic force transformation equations from Lorentz force law

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Relativistic transformation equations for the 3-vector force are derived from the Lorentz force law by using the well-known transformation equations for electromagnetic fields and velocity. The derivation is a simple alternative to the conventional derivation based on relativistic expressions for energy and momentum. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

Conventional derivations of relativistic force transformation equations involve rather lengthy and fairly complicated operations with relativistic expressions for mechanical energy and momentum.[1–3] However, the historical and logical foundation of the relativity theory is electrodynamics rather than mechanics, and the basic equations of the relativity theory (not counting the Lorentz transformation relations for space and time) are the relativistic expressions for electric and magnetic fields. But electric and magnetic fields are force fields. It may be expected, therefore, that relativistic force transformation equations could be derived simply and directly from relativistic transformation equations for electric and magnetic fields without any recourse to mechanics. Such a derivation is presented in this paper.

It should be noted that the phrase "relativistic force transformation equations" is somewhat ambiguous, because in the standard presentations of the special relativity theory two different "relativistic" forces are used. One of them is the 3-vector force defined by

$$\mathbf{F}=\frac{d}{dt}\left(\frac{m_0\mathbf{v}}{(1-v^2/c^2)^{1/2}}\right), \tag{1}$$

the other is the 4-vector force, also known as the *Minkowski* force, defined by

$$F_\mu=\frac{dp_\mu}{d\tau}, \tag{2}$$

where p_μ is the 4-vector momentum, and τ is the "proper" time. Only the latter force is Lorentz invariant and therefore represents the true relativistic generalization of Newton's second law.

The transformations that we shall derive are for the 3-vector force given by Eq. (1). Equation (1) is basically an empirical relation and reflects the fact that the equation

$$\frac{d}{dt}\left(\frac{m_0\mathbf{v}}{(1-v^2/c^2)^{1/2}}\right)=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{3}$$

is the correct equation of motion for charged particles in the presence of electric and magnetic fields, although it is also a plausible extension of Newton's second law to charged particles moving at relativistic velocities.[4]

Naturally, the 3-vector force, Eq. (1), and the 4-vector force, Eq. (2), are closely related. In fact, Eq. (2) is frequently obtained ("constructed") as a Lorentz-invariant generalization of Eq. (1).[5] As a result, the spatial components of the 4-vector force are proportional to the corresponding components of the 3-vector force [the proportionality factor is $1/(1-v^2/c^2)^{1/2}$]. But even when Eq. (2) is constructed without the use of Eq. (1), care is taken to demonstrate the compatibility of Eq. (2) with the equation of motion, Eq. (3), as an indirect proof of the viability of the 4-vector force. The 4-vector force is essentially significant only for theoretical investigations, while for practical calculations the 3-vector force is used.[6]

One customarily refers to the 3-vector force simply as the "force." Accordingly, in the derivation and discussion that follows, we shall use force in lieu of "3-vector force."

II. THE DERIVATION

The force experienced by a point charge q moving with velocity $\mathbf{u}$ in the presence of an electric field $\mathbf{E}$ and a magnetic flux density field $\mathbf{B}$ is given by the Lorentz force law

$$\mathbf{F}=q(\mathbf{E}+\mathbf{u}\times\mathbf{B}). \tag{4}$$

This law does not depend on the inertial reference frame in which q, $\mathbf{u}$, $\mathbf{E}$, and $\mathbf{B}$ are measured (in fact, it is frequently used as the *definition* of $\mathbf{E}$ and $\mathbf{B}$). Therefore in an inertial reference frame Σ' moving with velocity v relative to the laboratory (reference frame Σ) in the direction of their common x axis, Lorentz force law can be written as

$$\mathbf{F}'=q(\mathbf{E}'+\mathbf{u}'\times\mathbf{B}'), \tag{5}$$

where the primes are used to indicate quantities measured in the moving reference frame (there is no prime on q because the charge does not depend on the velocity with which it moves). All we need to do to obtain an equation transforming $\mathbf{F}'$ to $\mathbf{F}$ is to express $\mathbf{E}$, $\mathbf{u}$, and $\mathbf{B}$ in Eq. (4) in terms of primed quantities and to group the latter together in the form of Eq. (5).

However, when dealing with the relativistic transformation, it is usually much simpler to write the transformation equations in terms of the Cartesian components of the vectors involved rather than in terms of the vectors themselves. In terms of the components, Eqs. (4) and (5) are

$$F_x=q(E_x+u_yB_z-u_zB_y), \tag{6}$$

$$F_y=q(E_y+u_zB_x-u_xB_z), \tag{7}$$

$$F_z=q(E_z+u_xB_y-u_yB_x), \tag{8}$$

and

$$F'_x=q(E'_x+u'_yB'_z-u'_zB'_y), \tag{9}$$

$$F'_y=q(E'_y+u'_zB'_x-u'_xB'_z), \tag{10}$$

$$F'_z=q(E'_z+u'_xB'_y-u'_yB'_x). \tag{11}$$

The transformation equations for electric and magnetic fields that we need are[7]

$$E_x=E'_x, \tag{12}$$

$$E_y = \gamma(E'_y + vB'_z), \tag{13}$$

$$E_z = \gamma(E'_z - vB'_y), \tag{14}$$

and

$$B_x = B'_x, \tag{15}$$

$$B_y = \gamma(B'_y - vE'_z/c^2), \tag{16}$$

$$B_z = \gamma(B'_z + vE'_y/c^2), \tag{17}$$

where

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}. \tag{18}$$

We also need the following velocity transformation equations.[8]

$$u_x = \frac{(u'_x + v)}{1 + vu'_x/c^2}, \tag{19}$$

$$u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)}, \tag{20}$$

and

$$u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)}. \tag{21}$$

A. Transformation equation for obtaining the x component of F

Substituting Eqs. (12), (20), (17), (21), and (16) into Eq. (6) and canceling γ, we have

$$F_x = q\left[E'_x + \frac{u'_y}{1 + vu'_x/c^2}\left(B'_z + \frac{vE'_y}{c^2}\right) - \frac{u'_z}{1 + vu'_x/c^2} \times \left(B'_y - \frac{vE'_z}{c^2}\right)\right]. \tag{22}$$

Adding and subtracting

$$\frac{vu'_y u'_z B'_x}{c^2(1 + vu'_x/c^2)},$$

we obtain

$$\begin{aligned} F_x &= q\left[E'_x + \frac{u'_y}{1 + vu'_x/c^2}\left(B'_z + \frac{vE'_y}{c^2}\right) - \frac{u'_z}{1 + vu'_x/c^2} \times \left(B'_y - \frac{vE'_z}{c^2}\right)\right] + \frac{vu'_y u'_z B'_x}{c^2(1 + vu'_x/c^2)} - \frac{vu'_y u'_z B'_x}{c^2(1 + vu'_x/c^2)} \\ &= q\left[E'_x + \frac{u'_y}{1 + vu'_x/c^2}\left(B'_z + \frac{vE'_y}{c^2} + \frac{vu'_z B'_x}{c^2}\right) - \frac{u'_z}{1 + vu'_x/c^2}\left(B'_y - \frac{vE'_z}{c^2} + \frac{vu'_y B'_x}{c^2}\right)\right] \\ &= q\left[E'_x + \frac{vu'_y}{c^2(1 + vu'_x/c^2)}\left(\frac{c^2}{v}B'_z + E'_y + u'_z B'_x\right) - \frac{vu'_z}{c^2(1 + vu'_x/c^2)}\left(\frac{c^2}{v}B'_y - E'_z + u'_y B'_x\right)\right]. \end{aligned} \tag{23}$$

Adding and subtracting $u'_x B'_z$ inside the parentheses of the first term and $u'_x B'_y$ inside the parentheses of the second term, we then have

$$\begin{aligned} F_x = q\Bigg[E'_x &+ \frac{vu'_y}{c^2(1 + vu'_x/c^2)} \times \left(\frac{c^2}{v}B'_z + u'_x B'_z - u'_x B'_z + E'_y + u'_z B'_x\right) \\ &- \frac{vu'_z}{c^2(1 + vu'_x/c^2)} \times \left(\frac{c^2}{v}B'_y + u'_x B'_y - u'_x B'_y - E'_z + u'_y B'_x\right)\Bigg]. \end{aligned} \tag{24}$$

Simplifying Eq. (24), we obtain

$$\begin{aligned} F_x = q\Bigg[E'_x &+ \frac{vu'_y}{c^2(1 + vu'_x/c^2)}\left(\frac{c^2(1 + vu'_x/c^2)}{v}B'_z - u'_x B'_z + E'_y + u'_z B'_x\right) - \frac{vu'_z}{c^2(1 + vu'_x/c^2)} \\ &\times \left(\frac{c^2(1 + vu'_x/c^2)}{v}B'_y - u'_x B'_y - E'_z + u'_y B'_x\right)\Bigg], \end{aligned} \tag{25}$$

or

$$\begin{aligned} F_x = q\Bigg[E'_x &+ u'_y B'_z - u'_z B'_y + \frac{vu'_y}{c^2(1 + vu'_x/c^2)} \times (E'_y + u'_z B'_x - u'_x B'_z) \\ &+ \frac{vu'_z}{c^2(1 + vu'_x/c^2)}(E'_z + u'_x B'_y - u'_y B'_x)\Bigg]. \end{aligned} \tag{26}$$

Comparing Eq. (26) with Eqs. (9), (10), and (11), we recognize that Eq. (26) can be written as

$$F_x = F'_x + \frac{vu'_y}{c^2(1 + vu'_x/c^2)}F'_y + \frac{vu'_z}{c^2(1 + vu'_x/c^2)}F'_z, \tag{27}$$

which is the transformation equation for obtaining the x component of the force measured in the laboratory system from the x, y, and z components of the force measured in the moving system.

B. Transformation equation for obtaining the y component of F

Substituting Eqs. (13), (21), (15), (19), and (17) into Eq. (7), we have

$$F_y = q\left[\gamma(E'_y + vB'_z) + \frac{u'_z}{\gamma(1 + vu'_x/c^2)}B'_x - \frac{u'_x + v}{1 + vu'_x/c^2}\gamma\left(B'_z + \frac{v}{c^2}E'_y\right)\right]. \tag{28}$$

Factoring out

$$\frac{\gamma}{1 + vu'_x/c^2},$$

simplifying, and rearranging, we obtain

$$F_y = \frac{q\gamma}{1+vu_x'/c^2}\left[(E_y'+vB_z')\left(1+\frac{vu_x'}{c^2}\right) + u_z'\left(1-\frac{v^2}{c^2}\right)B_x' - (u_x'+v)\left(B_z'+\frac{v}{c^2}E_y'\right)\right]$$

$$= \frac{q\gamma}{1+vu_x'/c^2}\left[\left(1-\frac{v^2}{c^2}\right)E_y' - u_x'\left(1-\frac{v^2}{c^2}\right)B_z' + u_z'\left(1-\frac{v^2}{c^2}\right)B_x'\right]$$

$$= \frac{q}{\gamma(1+vu_x'/c^2)}(E_y'+u_z'B_x'-u_x'B_z'), \tag{29}$$

or, with Eq. (10),

$$F_y = \frac{1}{\gamma(1+vu_x'/c^2)}F_y', \tag{30}$$

which is the transformation equation for obtaining the y component of the force measured in the laboratory system from the y component of the force measured in the moving system.

C. Transformation equation for obtaining the z component of F

Substituting Eqs. (14), (19), (16), (20), and (15) into Eq. (8) and proceeding as we did for deriving Eq. (30), we get

$$F_z = \frac{1}{\gamma(1+vu_x'/c^2)}F_z', \tag{31}$$

which is the transformation equation for obtaining the z component of the force measured in the laboratory system from the z component of the force measured in the moving system.

III. CONCLUSIONS

The transformation equations that we have obtained are for transforming forces from the moving (primed) reference frame to the laboratory (stationary) reference frame. The inverse transformations can be derived in the same manner. However, as usual, the inverse transformations can be obtained without additional derivations by simply switching primes from the primed to the unprimed quantities and reversing the sign in front of v.

The method of deriving force transformation equations presented in this paper is simpler and more direct than the conventional method based on the use of mechanical energy and momentum. It also has the advantage of closer relation to electrodynamics, which is the logical and historical basis of the relativity theory.[9] As has been kindly noted by an anonymous reviewer, the calculations could make a good detailed exercise for the student.

[1]See, for example, A. R. French, *Special Relativity* (Norton, New York, 1968), pp. 205–225.

[2]See, for example, W. G. V. Rosser, *Classical Electromagnetism via Relativity* (Plenum, New York, 1968), pp. 9–15.

[3]Alan M. Portis, *Electromagnetic Fields: Sources and Media* (Wiley, New York, 1978), pp. 696–707.

[4]See, for example, Robert Resnick, *Introduction to Special Relativity* (Wiley, New York, 1968), pp. 119–120.

[5]See, for example, (a) A. Sommerfeld, *Electrodynamics* (Academic, New York, 1952), pp. 241–243 or (b) Emil J. Konopinski, *Electromagnetic fields and Relativistic Particles* (McGraw-Hill, New York, 1981), pp. 393–396.

[6]See, for example, Roald K. Wangsness, *Electromagnetic Fields* (Wiley, New York, 1979), pp. 557–559 and Ref. 5(b), p. 396 footnote; see also A. Einstein, *The Meaning of Relativity* (Princeton U.P., Princeton, New Jersey, 1950), 3rd ed., p. 47.

[7]See, for example, Ref. 2, p. 157.

[8]See, for example, Ref. 2, pp. 9–10.

[9]The derivation is so simple and natural that it is difficult to believe that it has been overlooked until now. Nevertheless I could not find this derivation in any of the 60, or so, textbooks on special relativity and electromagnetic theory that I consulted before writing this article: the earliest of the books was E. Cunningham, *The Principle of Relativity* (Cambridge U.P., London, 1914) and the latest was Mark A. Heald and Jerry B. Marion, *Classical Electromagnetic Radiation* (Saunders, Fort Worth, 1995). It is interesting to note, however, that for the special case of a particle which at the given moment is at rest in the moving reference frame a similar derivation is indicated in W. Pauli, *The Theory of Relativity* (Pergamon, New York, 1958), p. 82 and in R. Becker and F. Sauter, *Electromagnetic Fields and Interactions* (Blaisdell, New York, 1964), Vol. 1, p. 351.

THE ROLE OF EMOTION IN TEACHING

It is particularly important to keep out emotion, or rather to control it carefully. Fathers and mothers, husbands and wives, and people in authority very often forget this. When they explain, they shout. Their faces become distorted with anger or urgency. They make violent gestures. They feel that they are explaining things more forcibly. But in fact their emotion makes them difficult to understand. A wife screaming at her husband, a sergeant roaring at a platoon, a father bellowing at his son, create fear, and even hatred, but they do not manage to explain what they want done and persuade their hearers to do it. Whenever we sink to believing that the more emotion we display, the more effect we shall produce, we are reverting to our animal ancestry and forgetting that conscious reason is what makes us men.

Gilbert Highet, *The Art of Teaching* (Vintage Books, New York, 1989; originally published by Alfred A. Knopf, 1950), p. 249.