Electromagnetic interaction momentum and simultaneity

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Using a simple example of two interacting electric charges, we show that different observers see a different distribution of momentum between the particles and the electromagnetic field, and we discuss how this is related to the relativity of simultaneity (which has to be taken into account even if it seems that we are in a ''nonrelativistic'' approximation).
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I. INTRODUCTION

The concept of momentum in electromagnetic fields has been the object of innumerable discussions, particularly in connection with the classical model of the electron. One of the problems of the classical model of the electron was that if one calculates the integral over all space of the Poynting vector of the electron moving with velocity v, and calls m the electrostatic energy of the resting electron divided by c^2, its electromagnetic momentum does not come out mv, but $\frac{4}{3}mv$ (in other words the energy–momentum four vector is not ''covariant''). Related to this is the fact that the inertial mass (self-force divided by acceleration) is also $\frac{4}{3}m$. Poincaré[1] accounted for the $-\frac{1}{3}$ by taking into account the momentum associated with the nonelectromagnetic stresses keeping the electron together (what has later become known as ''hidden momentum,'' an expression first introduced by Shockley[2] and recently discussed and reviewed in this journal[3–5]). More recently, Rohrlich[6,7] criticized Poincaré's argument and avoided the $\frac{4}{3}$ factor by assuming that the electromagnetic energy–momentum density was not given by Poynting's vector, but made a new, covariant definition. This new definition was also criticized.[8,9]

A useful way to clarify concepts is to make simple examples, in which the concept is isolated without technical and computational complications. The example of two charges is one of the simplest, and along this line several papers have been written, which have clarified some concepts of classical electrodynamics. In particular we refer to a paper by Boyer,[10] who showed the origin of the mass–energy equivalence in classical electromagnetism, one by Griffths and Owen,[11] who discuss the mass, energy, and momentum of two rigidly connected interacting charges, and a recent paper showing the effect of retardation on the energy balance in coherent radiation.[12] In particular, an advantage of considering the interaction field of two charges is that we do not need hypotheses on the inner structure of the electron.

We want to discuss how different reference systems see the momentum of two interacting particles, showing the relativity of the partition of the total momentum into a ''particle'' momentum and an ''interaction'' one, and how this is related to the relativity of simultaneity.

Our calculation will be nonrelativistic, in the sense that we consider reference frames moving with velocity v such that $v^2/c^2 \ll 1$. This approximation is not necessary, as it would not be much more difficult to deal with arbitrarily large velocities. But it is interesting to see that effects related to the relativity of simultaneity (or the constancy of the speed of light) can appear even at ''nonrelativistic'' velocities.

II. THE EXAMPLE

Let us consider a system of two identical charged particles each of mass m and charge q, moving freely only under the effect of each other's field, with initial conditions such that the motion of both particles is on the same straight line connecting them (let us call this axis z).

In a suitable reference frame S, let particle 1 with charge q_1 be at $z=0$ and the other one, with equal charge $q_2=q_1=q$, be at $z=r$ at the same time, both with instantaneous velocity $v=0$. Their interaction will accelerate them in opposite directions, but we suppose their mass is large enough so that acceleration and radiation are negligible, as well as the change in velocity during the time r/c. The mechanical momentum of the particles is zero, and for symmetry reasons also the electromagnetic field momentum is zero. The mass of the two particles is $2m$, and the energy of the field is $U=q^2/r$ (in Gaussian units). The mass of the system (its energy divided by c^2) is $2m+q^2/rc^2$.

We want to evaluate the momentum of this system as seen by an observer S' moving in the direction $-z$ with velocity v. From classical electromagnetic theory, we know that the momentum in the field can be calculated as the volume integral of Poynting's vector.

What one would be tempted to say is that, apart from the mechanical momentum $2mv$ of the two masses (which includes the contribution of the electromagnetic field of each charge—here we do not enter into a classical model of the electron), there is an interaction momentum which can be obtained by integrating the part of Poynting's vector due to the interaction $(\mathbf{E}_1\times\mathbf{B}_2+\mathbf{E}_2\times\mathbf{B}_1)$ over the whole space. This turns out to be $2q^2v/rc^2$. This has been calculated directly by Griffiths and Owen,[11] and can be easily determined also[13] as $q_1A_1+q_2A_2$, where A_i is the vector potential at the ith particle due to the other one. The total momentum would then be $(2m+2q^2/rc^2)v$. But if we consider the field energy, which contributes to the mass of the system (as can be seen classically[10]) with a term U/c^2, we would expect the field momentum to be Uv/c^2 and not $2Uv/c^2$.

But this argument has neglected the fact that each observer sees simultaneity in a different way. In order to determine the momentum of the system in S', one has to add simultaneous values. Now, while S sees the two particles having $v=0$ as simultaneous events, the events when the particles have exactly the velocity v are not simultaneous in S', but have a time difference $\tau = rv/c^2$ (the Lorentz factor being 1 in our approximation). While, for our assumption of large masses, the interaction momentum varies little in a time rv/c^2, the same is not true of the contribution of the masses.

Let us take the instant t where q_1 has exactly velocity v, while q_2 will reach the same velocity after the time τ. During this time interval the force of q_1 on q_2 will give the latter an impulse $F\tau \simeq (q^2/r^2)v(r/c^2) = q^2v/rc^2$ (still, in the approximation of large masses, r and F can be considered constant during this small time interval). Then at time t the charge q_2 has a momentum $mv - q^2v/rc^2$.

Therefore, for S', at that instant the two masses have a momentum $mv + (mv - q^2v/rc^2)$, while the e.m. field has a momentum $2q^2v/rc^2$. In total, the momentum is $(2m + q^2v/rc^2)v$, or the total mass times v, as it should be.

III. ANOTHER REMARK

For completeness we mention the case of two particles connected with a rigid bar (though it has already been discussed in the cited literature). In this case their velocities and momenta are constant, so it does not matter whether we measure them at the same time or not, and there is no problem of simultaneity for the calculation of the particle momenta in S' (and in the approximation $v^2/c^2 \ll 1$, the lengths seen by S and S' are the same). In order to keep the distance constant, the bar is stressed. Even if in the system S the stress is not related to any kind of energy (i.e., the rod is rigid), nevertheless in the system S' a classical flow of energy appears: not only does the electromagnetic field perform work, but mechanical forces do too; as the rod pulls back the forward particle and pulls forward the back one, it does negative work on the forward one and positive work on the other one. This means a transfer of power $W = Fv = q^2v/r$ along the rod in the backwards direction, and in relativity a flow of energy implies a momentum. If s is the rod's cross section, and the stress is F/s, the observer S' sees a momentum density Fv/sc^2,[14] and a total momentum Fvr/c^2. This ''hidden'' momentum compensates for the one half of the Poynting term that is in excess.[15]

Essentially this is the argument used by Poincaré[1] for calculating the momentum due to the forces keeping the electron together (in this geometry of our example the factor 2 is the analog of the $\frac{4}{3}$ in the spherical case of the classical electron model).

IV. CONCLUSIONS

In conclusion, we see that the two reference systems S and S' in uniform motion relative to each other see a different distribution of the momentum between the particles and the field.

We see that stability of the system is not necessary to get the covariance of the total energy–momentum. In other words, Poincaré stresses are necessary for stability, not for covariance: the introduction of stabilizing forces makes the covariance of the energy–momentum independent of the time at which the two interacting points are considered, thus hiding the problem of simultaneity. In this sense, Rohrlich[6,7] was right to say that stability and covariance are two different problems. However it also appears that it is not necessary (or desirable) to redefine the energy–momentum density as Rohrlich does.

This example also teaches us something about ''relativistic'' and ''nonrelativistic'' approximations: Although we said we made a ''nonrelativistic'' approximation ($v^2/c^2 \ll 1$), the simultaneity as defined in relativity does have to be taken into account.

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[15]The discussion in Ref. 9 on the stabilizing forces in the classical electron model is interesting for understanding the connection between ''hidden momentum'' and the relativity of simultaneity.