The Wall of Death

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Much to everybody's surprise, it was recently demonstrated that according to Einstein's general theory of relativity, very close to a compact star or a black hole the centrifugal force may attract towards the center of a circular motion. We show here that Newtonian theory predicts exactly the same effect, and that the geometrical reason for it is identical in both theories. The centrifugal force always repels in the *local* outward direction. However, in a curved space (or on a curved surface) in which the perimeter of concentric circles decreases with the increasing diameter, the local outward direction points towards the global center of the circles, and thus the centrifugal force attracts to the center.

I. REVERSAL OF THE CENTRIFUGAL FORCE

Einstein's general theory of relativity predicts that the three-dimensional space close to a very compact star or a black hole is not Euclidean, but very strongly curved. In such strongly curved spaces one finds circular photon rays: some of the light trajectories are bent so much that they become perfect circles. Around a nonrotating (Schwarzschild black hole with the mass M the circular photon rays are great circles on the sphere $r=3GM/c^2$, located well outside the spherical surface of the black hole, which itself locates at $r=2GM/c^2$. Note, that for a black hole with the mass equal to that of the Sun, $2GM/c^2=3$ km. For a very compact spherical star (with the mass M and the radius $R<3GM/c^2$) in addition to the external circular photon ray at $r=3GM/c^2$ there exist also circular photon rays inside the star. In strongly curved spaces, several properties of rotating bodies predicted by Einstein's theory *appear* to be acutely paradoxical and puzzling, because they contradict our intuition based on Newtonian theory. Some of them are somehow connected to the existence of the circular photon rays in space.

For example, according to Newtonian theory, identical test rockets moving with different orbital speeds v along a fixed circular orbit with the radius r around a spherical body with the mass M need to use a *speed dependent* thrust, $T=(v_K^2-v^2)/r$, in order to stay on the orbit. Here, $v_K=(GM/r)^{1/2}$ is the orbital speed corresponding to a free orbital motion. However, according to Einstein's theory,[1] test rockets moving along the circular photon ray around a Schwarzschild black hole should all use, *irrespective* of their different orbital speeds, the same rocket thrust $T=mc^4/6GM$ in order to stay on this orbit.

According to Newtonian theory, a gyroscope moving around a circle should always precess in the opposite sense to its circular motion in order to point in a fixed direction in space: gyroscopes on clockwise orbits precess anticlockwise and *vice versa*. However, according to general relativity,[2] very close to a Schwarzschild black hole (for $r<3GM/c^2$) orbiting gyroscopes precess in the same sense as their orbital motion: gyroscopes on clockwise orbits precess clockwise.

The conventional and well-known Rayleigh criterion demands that the angular momentum must *increase* outwards for stability. However, according to Einstein's theory,[3] close enough to a Schwarzschild black hole (for $r<3GM/c^2$), the Rayleigh stability criterion is reversed: stable equilibria correspond to angular momentum *decreasing* outwards. Similarly, the well-known result from the classical theory of thin accretion disks states that viscous stresses always transport angular momentum outwards,[4] but according to Einstein's theory,[5] close to a Schwarzschild black hole (for $r<3GM/c^2$), viscous stresses in thin accretion disks transport angular momentum *inwards*.

Newtonian theory predicts that when a rotating body shrinks conserving angular momentum, it always becomes progressively more flattened. In contrast to this, according to Einstein's theory[6] the ellipticity of quasistationarily contracting rigidly rotating spheroids with fixed mass M and fixed total angular momentum J *decreases* with decreasing mean radius R when $R<5GM/c^2$. Similarly, and again in contrast to Newtonian theory, general relativity predicts[7] that rotation *increases* internal pressure of a sufficiently compact body (having its radius, $R<2.5GM/c^2$).

The above described situations are directly relevant for studying some most important and difficult problems of the modern astrophysics such as the gravitational collapse of rapidly rotating stars, coalescence of binary pulsars (believed to be connected to the origin of the gamma ray bursts) or the question of how fast a pulsar can spin. For this reason they have been at focus of interest for quite some time and many experts have tried to give an explanation for the puzzling paradoxes connected with them. However, these attempts were all unconvincing and mathematically incorrect. The correct explanation was found only a short time ago by one of us.[8] It did not pass unnoticed. Several scientific journals, both professional[9] and popular,[10] devoted editorial articles to it, and occasionally some news appeared also in the general press.[11] This public interest in a rather academic problem of classical general relativity arose, probably, because of its particular prediction which, if taken out of the context, sounds shockingly perverse: *the centrifugal force may attract to the center of a circular motion.*

The centrifugal force is a certain *thing* existing in nature, a particular inertial force which occurs in a rotating reference frame. If the angular velocity of the rotating frame is $\mathbf{\Omega}$, and the distance of a particle (with mass m) from the axis of rotation of the frame is **r**, then there is a force acting on the particle *in the rotating frame* which equals

$$\mathbf{Z}=m\mathbf{\Omega}\times(\mathbf{r}\times\mathbf{\Omega}). \tag{1.1}$$

It was C. Huygens[12] who named the force (1.1) *the centrifugal force.* In Latin this means escaping, avoiding, or repelling from the center. In his 17th century Huygens could not anticipate that physicists working two centuries after him will demonstrate that the "centri-fugal" force may indeed, in a curved space, attract to the center and thus become "centri-petal." Although in this case the name "centri-fugal" becomes self-contradictory, it should be kept unchanged, because it is already a part of our culture, and every educated person knows it. When properly understood, the statement "centrifugal force attracts" would never appear to be undefined or ambiguous. It faithfully brings the physics to forefront, describing reality in the most proper and adequate terms we have. The paradox with the self-contradicting name is just one more example of the often forgotten truth that by creating names we do not create reality, and that properties of things existing in nature may disagree with our linguistic creations—things have been created before we started naming them.[13]

More precisely, it was *formally and rigorously* demonstrated that according to Einstein's general theory of relativity, in the case of a Schwarzschild black hole the centrifugal force repels from the center (as it always does according to Newtonian theory) only for a circular motion which goes (with an arbitrary orbital speed) along circles with radii greater that of the circular photon ray, $r>3GM/c^2$. For a motion (again with an arbitrary orbital speed) along circles with radii $r<3GM/c^2$ the force attracts to the center. It is exactly zero for a motion along the circular photon orbit, $r=3GM/c^2$. Obviously, this explains all the puzzling examples given above, but how can one understand the formal results without going into rather technical details of general relativistic arguments and calculations?

In this paper we show that the attractive behavior of the centrifugal force is not at all strange. It may also occur, and indeed it occurs, in very familiar Newtonian situations, for example on the Wall of Death. The Wall of Death is a circus for motorcycle acrobatic shows. The public takes the lower floor and the motorcycle acrobats ride on the inside of the barrel-shaped wall of the circus. Each particular Wall of Death is a two-dimensional surface of revolution. It is symmetric with respect to the intrinsic rotation in which all points move along concentric latitude circles, around the fixed center C located at the bottom of the Wall. Meridians emerge radially from the center and are orthogonal to the latitude circles (Fig. 1). The shortest paths (geodesic) between a point on the Wall and the center goes always along a meridian joining them. From the point of view of the three-dimensional geometry, the center of the latitude circle O shown in Fig. 2 coincides with the point $C_{(3)}$, but from the point of view of the two-dimensional geometry *intrinsic* to the Wall, the center of this circle coincides with the point C.

An acrobat who moves along a particular latitude circle feels the centrifugal force **Z** in the noninertial reference frame corotating (comoving) with him. In the three-dimensional space with Euclidean geometry the centrifugal force acts, obviously, in the direction outward from the center of the circular motion, pushing the acrobat away from the point $C_{(3)}$. However, one may consider the centrifugal force $\mathbf{Z}^{\dagger}$ *intrinsic* to the two-dimensional surface of the Wall by projecting **Z** into this surface. Then, as Fig. 2 clearly shows, *the intrinsic centrifugal force attracts the acrobat in the direction towards the center of his circular motion, pulling him towards the point* C.

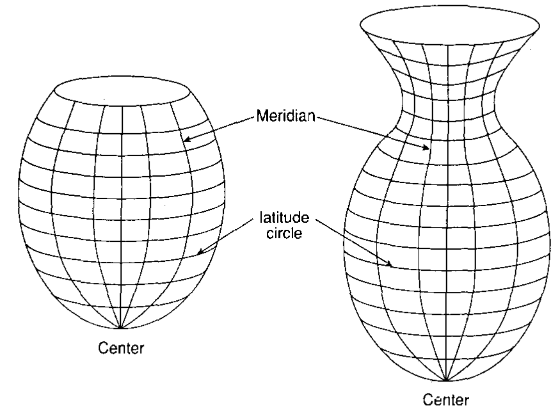

Fig. 1. Latitude circles and meridians on the Wall of Death with two different shapes.

This may look as a triviality not even worth mentioning, but the very geometrical reason for the attractive two-dimensional centrifugal force, projected on the curved surface of the Wall, is identical as in the case of the attractive three-dimensional centrifugal force in the curved space of Einstein's theory. It may therefore be useful to study the links of the curved geometry of the Wall and the dynamics of the motorcycle motion there as a model of the formidable world of a curved space close to a black hole. Our article is devoted exactly to this.

As far as we know and could check, nobody before has noticed that according to Newtonian dynamics, on the Wall of Death (and in similar situations on curved surfaces) the centrifugal force may attract to the center of a circular motion.[14] This should not be surprising: the notion of an attractive centrifugal force was probably for all of us a kind of a mental *tabu*, consciously or unconsciously imposed on our imagination, not so much by the *real nature* of things, but by the meaning of the *name* of the force. Mental tabus are often difficult to break. This one was

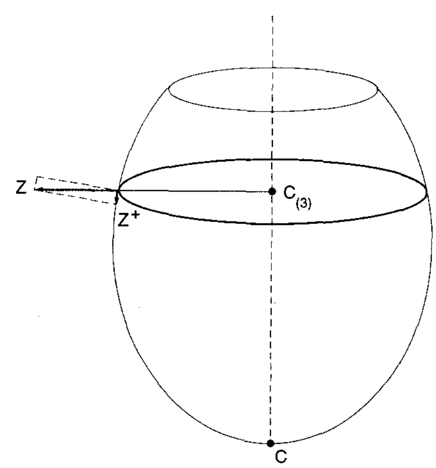

Fig. 2. The centrifugal force paradox: Centrifugal force intrinsic to a curved surface attracts to the center of a circular motion.

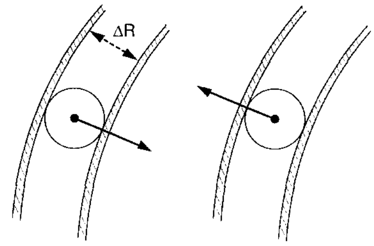

Fig. 3. Normal components of pulling and pushing forces are balanced by the Wall.

broken only when it had been *rigorously* proved, in an abstract mathematical way, that according to Einstein's theory the "centrifugal" force may attract to the center in some very *unfamiliar* situations involving black holes. Only then it was possible to realize that the same could also be true in much more familiar situations fully described by Newtonian dynamics.

II. TWO-DIMENSIONAL WORLD ON THE WALL OF DEATH

The very reason that according to Einstein's general relativity the centrifugal force may attract to the center of a circular motion is the curvature of space. The three-dimensional space of Newtonian theory is not curved but flat, and thus described by Euclidean geometry. However, one may consider Newtonian dynamics on a two-dimensional curved surface thus bringing effects of the curvature directly into Newtonian dynamics. This is a standard procedure, described in many textbooks.[15] Its importance comes from the fact that the two-dimensional Newtonian dynamics can be studied not only theoretically, but also directly experimentally on some models. This gives an opportunity to experimentally illustrate how the strong curvature of space influences dynamics, and thus to visualize some of the apparently puzzling predictions of Einstein's theory by expressing them in familiar Newtonian terms.

Following this procedure, we discuss here the two-dimensional gravitational and centrifugal forces as a part of the two-dimensional world on the surface of the Wall of Death. The geometry in this world is not Euclidean, but curved. The curvature of the Wall influences the dynamics in the same way as the curvature of space influences the dynamics in Einstein's general theory of relativity. Thus, by studying the motion of the motorcycles on the Wall of Death one can understand a lot about the motion of spacecraft and planets around black holes.

Of course, the motion of motorcycles on a real Wall of Death is not truly two-dimensional, because the solid-body reaction balances only these normal forces which push against the Wall. Forces which pull along the normal are not balanced. A fully two-dimensional model of Newtonian dynamics, with both the pushing and pulling normal forces balanced, is shown in Fig. 3. The wall is split there into two walls separated by a gap with a constant width ΔR. The acrobats riding on the motorcycles are represented by small spheres, all with radius $(\Delta R)/2$. They move without friction inside the gap. For the internal consistency of the two-dimensional model in each point on the Wall it must be

$$\Delta R \ll \mathscr{R}_0. \tag{2.1}$$

Here, $\mathscr{R}_0$ is the smallest curvature radius ${}^{(3)}\mathsf{R}$, measured in the three-dimensional space, of the geodesic lines on the two-dimensional surface crossing the point in question. (The radius of curvature R is defined later.) If (2.1) is not fulfilled, the internal structure of the particles would influence their motion. Thus, one cannot discuss within this model distances smaller than ΔR and all the distances considered in this article must obey the condition,

$$d > \Delta R. \tag{2.2}$$

The two walls could be arranged to be perfect mirrors so that the light bounces in the gap between them and this way propagates along the Wall. From the Fermat principle one deduces that the bouncing light propagates along the rays which are geodesic lines in the curved geometry on the Wall. Thus, the acrobats may use light rays in all their measurements which involve construction of geodesic lines.

By measuring things intrinsic to the Wall, and by giving some physical interpretation to the two-dimensional objects defined by these measurements, the acrobats construct a *curved two-dimensional dynamics*. In particular, they may introduce the gravitational and centrifugal forces intrinsic to the Wall. It is quite obvious that the two-dimensional gravitational and centrifugal forces should be identical with their three-dimensional originals projected into the Wall. This *uniquely* determines the only physically correct definition of these forces in the curved two-dimensional world on the Wall. However, the acrobats have no information about the three-dimensional Euclidean space around the Wall and thus they cannot "project" from three to two dimensions. They must use a definition of these forces based on terms intrinsic to the Wall, but at the same compatible with the projection from three to two dimensions. Later in this article we discuss such a *Unique Definition.*

III. GEOMETRY ON THE WALL OF DEATH

A geometrical object in a curved surface is called *local* if it is possible to construct it in such a way that the greatest distance d involved in the construction is much smaller than the smallest curvature radius, $\mathscr{R}_0$,

$$d \ll \mathscr{R}_0. \tag{3.1}$$

In the familiar Euclidean geometry on a plane it is not possible to distinguish between local and global measurements, because $\mathscr{R}_0 = \infty$ and therefore condition (3.1) is fulfilled for all geometrical constructions: all of them are therefore "local."

The (geodesic) radius of a latitude circle equals half of its diameter measured along a meridian, $R = D/2$, and the circumferential radius equals $r = P/2\pi$, with P being the perimeter of the circle. It should be obvious that neither the geodesic radius of the circle R, nor its circumferential radius r, can be measured locally.

In Euclidean geometry the ratio of perimeter to diameter of a circle equals

$$P/D = \pi = 3.141\,592\,653\,6\ldots, \tag{3.2}$$

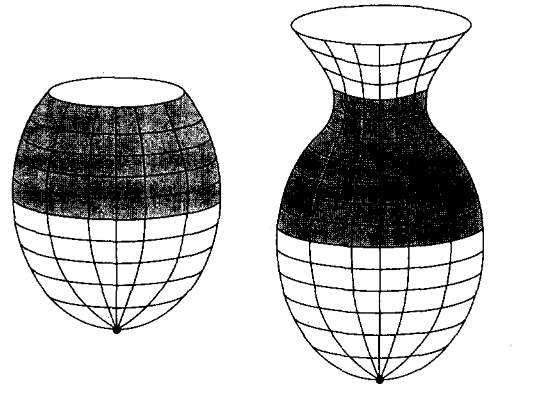

Fig. 4. The circumferential radius r as a function of the radius R.

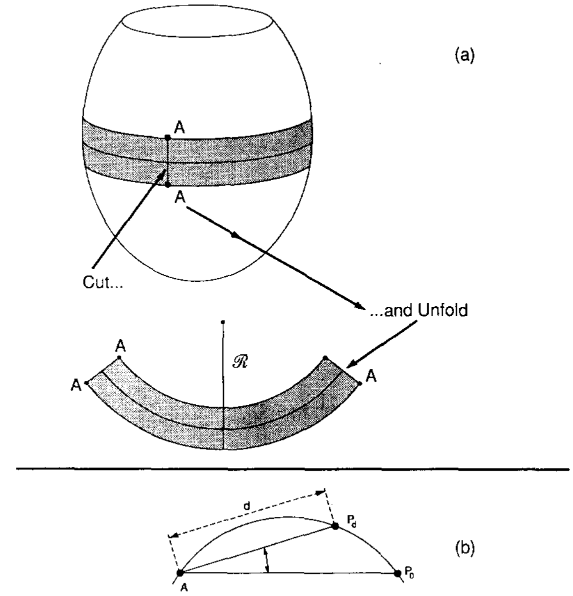

Fig. 6. The curvature radius R. (a) Geometrical construction. (b) Local measurement.

and thus $\text{sign}(dr/dR) \equiv \epsilon = +1$. However, in the upper hemisphere of the Wall of Death the circumferential radius is a decreasing function of the radius (Fig. 4), and therefore it is $\epsilon = -1$ there. This has an interesting and important consequence: the globally measured "outward direction" is exactly opposite to that measured locally.

Globally, outwards means the direction outside of the center of the circle. This direction could be found in the following way [Fig. 5(a)]. Trace, in terms of a light ray, a geodesic joining the point A on the circle with the center of the circle C. In the Euclidean plane this would be a straight line. On the curved surface of the Wall it is a meridian. The outward direction is defined by the unit vector $\mathbf{q}_0$ having the same direction as the geodesic sector $\vec{CA}$. The sector $|CA|$ could have its length $d \sim \mathscr{R}_0$ and this is why the construction is a global one.

Locally, outwards means the direction in which a small arc of the circle is bent with respect to a tangent geodesic line. The local outside direction can be found by the following construction [Fig. 5(b)]. Take a sufficiently small piece of the surface such that its geometry could be considered as being Euclidean. On this small piece geodesics lines defined by light rays do not differ from the straight lines on a plane. Using the familiar Euclidean construction, find two points A_1 and A_2 on the circle at equal distances from A, such that the arcs $\overset{\frown}{AA_1}$ and $\overset{\frown}{AA_2}$ are equal. Next, find two points P_1 and P_2 equally distant from the points A_1 and A_2, such that the lengths of the corresponding sector are equal, $|A_1P_1| = |A_2P_1|$, and $|A_1P_2| = |A_2P_2|$. Join the points P_1 and P_2 by a straight line. Finally, mark the point P_0 in which this line crosses the sector A_1A_2. The outward direction at the point A is defined by the unit vector $\mathbf{q}$ which agrees with the direction of the sector $\vec{P_0A}$. The longest distance on the small piece of the surface used in this construction obeys $d \ll \mathscr{R}_0$ and this is why the construction is a local one.

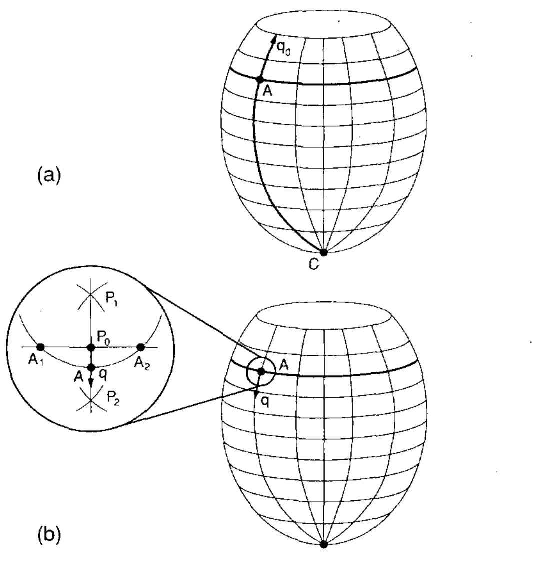

Fig. 5. The global (a) and the local (b) measurements of the outward direction for the latitude circles. The insert circle in the figure is very small and its geometry does not differ from the Euclidean geometry. The points A_1, A, and A_2 are located on the same latitude circle. Its curvature is grossly exaggerated in the figure in order to show the details of the construction.

From Fig. 4 one deduces that if the perimeter *increases* with the increasing diameter ($\epsilon = +1$), then the locally and globally defined outside directions are *the same*, but if the perimeter *decreases* with the increasing diameter ($\epsilon = -1$), then the locally and globally defined outside directions are *opposite*,

$$\epsilon \equiv \text{sign}\left(\frac{dr}{dR}\right), \quad \mathbf{q} = \epsilon \mathbf{q}_0. \tag{3.3}$$

The radius of curvature R is a quantity determined by the function $r(R)$. Figure 6(a) explains its construction based on the fact that cones are intrinsically flat—they could be cut, unfold and put flat on the plane. The latitude circle O in this figure is shown together with a strip of the surface of the Wall surrounding it. The strip is narrow enough to be considered as a part of a perfectly conical surface. After cutting and unfolding the strip, the circle O becomes a fragment of a circle $O^\dagger$ with the radius R which is the curvature radius of the original circle O.

In order to see that R is a local quantity, we consider now a different construction made on a *very small* part of the strip. This construction [shown above in Fig. 6(b)]

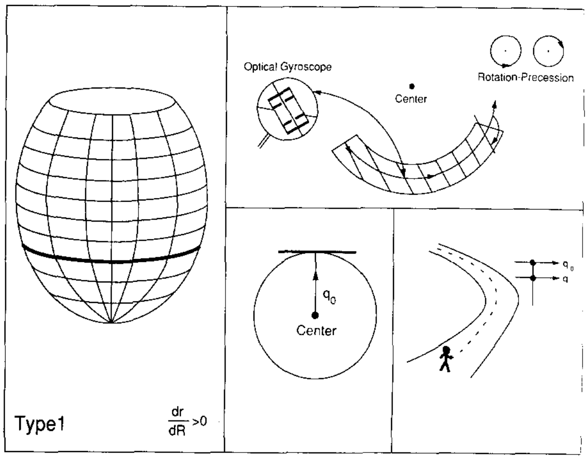

Fig. 7. Type 1 latitude circles.

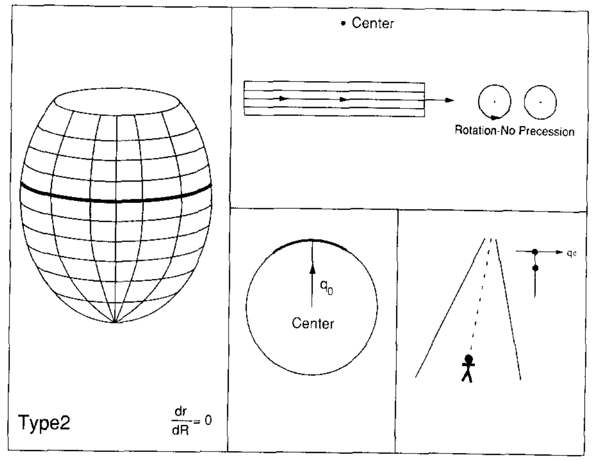

Fig. 8. Type 2 latitude circles.

determines the curvature radius in terms of a local measurement *intrinsic* to the Wall, with no reference to the three-dimensional Euclidean space in which the Wall is embedded.

The local measurement of the curvature radius [Fig. 6(b)]. From a point A on the circle O construct a geodesic which cross the circle in a point P_0, which is located very close to A. Denote the length of the sector AP_0 by d_0. From the point A construct another geodesic which crosses the circle O in a point P_d, located anywhere between P_0 and A. Denote the length of the sector AP_d by d and the angle $\angle P_0AP_d$ by $\alpha=\alpha(d)$. Find, by repeating the measurement,

$$\alpha_0=\lim_{d\to\Delta d}\alpha(d),\quad \mathsf{R}=\lim_{d\to\Delta d}\frac{d}{\sin(\alpha-\alpha_0)}. \tag{3.4}$$

One always can measure R with *a priori* assumed accuracy Δd in a finite number of steps. The accuracy of the measurement must be *finite* because of the condition (2.2); $\Delta R<\Delta d\ll\mathscr{R}_0$. Note, that this construction works not only for circles, but for *any* lines on a curved or flat surface.

In Euclidean geometry on a plane (where as we have explained one cannot distinguish between local and global geometrical objects) it is, of course, $r=R=\mathsf{R}$, but in a general two-dimensional non-Euclidean geometry on a curved surface one has,

$$\mathsf{R}=r\left|\frac{dr}{dR}\right|^{-1}=r\epsilon\left(\frac{dr}{dR}\right)^{-1}. \tag{3.5}$$

The inverse of the curvature radius, $\mathscr{C}=1/\mathsf{R}$, is called the curvature of a line. Lines with zero curvature everywhere are geodesic.

The three types of latitude circles on the surface of the Wall of Death are shown in Figs. 7–9. Type 1: The perimeter increases with increasing diameter, $\epsilon=+1$. The geodesic line tangent to the circle lies outside of the circle (in the global sense) and the local outward direction points outwards away of the global center. Circles in Euclidean geometry are always type 1. Type 2: The perimeter does not change with diameter, $dr/dR=0$. The curvature radius is infinite, $\mathsf{R}=\infty$. The circle is identical with a geodesic line. It is impossible to define locally the outward direction. No such circles are present in Euclidean geometry. Type 3: The perimeter decreases with increasing diameter, $\epsilon=-1$. The geodesic line tangent to the circle lies inside the circle. No circles of this type are possible in Euclidean geometry.

Figures 7–9 also show what an acrobat located at latitude circles of different types would actually *see*. It is clear that what he sees agrees with the local, but not global sense of outwards.

The type of the circle can be found not only by geometrical measurement of $\epsilon=\mathbf{q}\cdot\mathbf{q}_0$ but also by a dynamical measurement of the precession of an "optical gyroscope."

A conventional gyroscope, which may consist of a rapidly spinning dreidl, is a three-dimensional object. It would not fit the two-dimensional world of the Wall. More practical for the purpose is the optical gyroscope which consists of two parallel mirrors fixed in a tube, and two opaque screens between them. There is a small hole in each screen exactly on the optical axis of the device. Servomotors attached to it assure that independently of its motion the light moves exactly along the axis. The optical gyroscope is a two-dimensional device and thus it can be used by the acrobats on the Wall. A remarkable property of it is that its precession agrees with the precession of the conventional gyroscope. If the optical gyroscope moves along a light ray then obviously, it does not precess (the servomotors do no work). However, it does precess when it moves along a

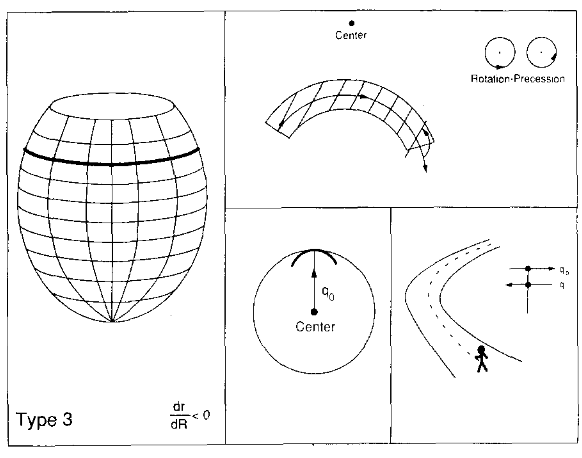

Fig. 9. Type 3 latitude circles.

curved line. Figures 7–9 show that for the type 1 circles the gyroscopes precess *backwards* with respect to the orbital motion (anticlockwise precession for clockwise rotation and *vice versa*), that they do not precess for the type 2 circles, and that they precess *forwards* for the type 3 circles.

IV. DYNAMICS ON THE WALL OF DEATH

The Unique Definition. The gravitational force depends on the location but it does not depend on the speed. The centrifugal force depends on the absolute value of the speed but not on its sign.

Consider again a motorcycle which moves with no friction and with a constant orbital speed v along a latitude circle. In general its motion cannot be free and should be supported by a real force $\mathbf{F}^\dagger$ which keeps the motorcycle moving with the assumed speed along the fixed circle. This could be achieved by means of a spring attached to the motorcycle (and then the $\mathbf{F}^\dagger$ is the spring tension), or by a supporting rocket engine pointing sideward (and then $\mathbf{F}^\dagger$ is the rocket engine thrust). Independently of the definition of the gravitational and centrifugal forces which one may prefer, in the reference frame comoving (corotating) with the motorcycle the real force $\mathbf{F}^\dagger$, the gravitational force, and the centrifugal force must be in a perfect balance,

$$\begin{pmatrix}\text{real}\\ \text{force}\end{pmatrix}+\begin{pmatrix}\text{gravitational}\\ \text{force}\end{pmatrix}+\begin{pmatrix}\text{centrifugal}\\ \text{force}\end{pmatrix}=0. \quad (4.1)$$

In a more general case other inertial forces should be considered.[16] For example, if the Wall is spinning, the Coriolis force will be present on the Wall. We shall return to this point later.

The real force $\mathbf{F}^\dagger$ is, of course, a directly measurable quantity and so is the orbital speed of the motorcycle v. We show that from the Unique Definition and Eq. (4.1) it follows how the centrifugal force can be measured. The idea of the measurement was described by Newton in *Principia*. Newton considered how the real force of the tension of a rope connecting two balls which move around a circle depends on the orbital speed. We adopt Newton's idea for the Wall of Death by considering two motorcycles with the same mass m_0 which move on the same latitude circle $R=R_0$, but with different orbital speeds. The first motorcycle moves with the speed v_0 and the second one with the speed $v=v_0+\delta v$. The real force which supports the motion of the first motorcycle equals $\mathbf{F}_0^\dagger$. For the second motorcycle it equals $\mathbf{F}_0^\dagger+\delta\mathbf{F}^\dagger$. According to the Unique Definition, the gravitational force is the same for both motorcycles. Therefore, the difference in the real force must be caused by the difference in the centrifugal force. Because the centrifugal force does not depend on the sign of the orbital speed, it is not the difference of the speeds which matters here, but the difference of the squares of the speeds, $\delta(v^2)=v^2-v_0^2$. The acrobats measure v_0, $\delta(v^2)$, and $\delta\mathbf{F}^\dagger$, and then combine results of these measurements into

$$\mathbf{Q}^\dagger=-\frac{\delta\mathbf{F}^\dagger}{\delta(v^2)}\,v_0^2. \quad (4.2)$$

This is exactly the centrifugal force on the Wall of Death.

Proof. Let us denote by $\mathbf{G}=m_0[\nabla\Phi]_{(3)}$ and $\mathbf{Z}=m_0(v^2/r)\mathbf{e}$ the gravitational and centrifugal forces in the three-dimensional space. Here $[\nabla\Phi]_{(3)}$ is the gradient of the gravitational potential in the three-dimensional space, r is the circumferential radius of the latitude circle, and $\mathbf{e}$ is the unit vector normal to the axis of rotation in three-dimensional space. We denote by $\mathbf{G}^\dagger$ and $\mathbf{Z}^\dagger$ projections of these forces onto the surface of the Wall of Death. Figure 10 shows that

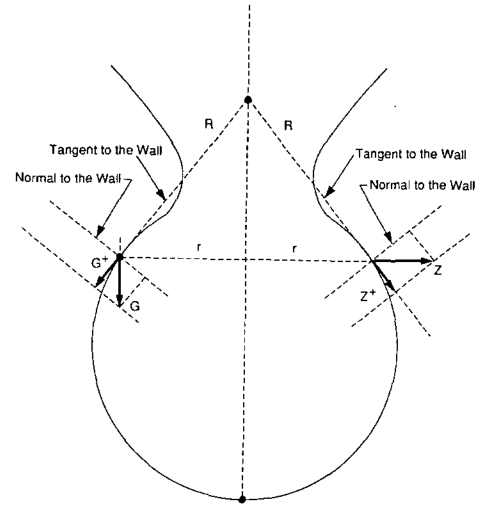

Fig. 10. Projection of the gravitation and centrifugal forces.

$$\mathbf{G}^\dagger=\frac{\mathsf{R}}{\sqrt{\mathsf{R}^2-r^2}}\,|\mathbf{G}|\,\mathbf{q}_0=m_0\nabla\Phi,\quad \mathbf{Z}^\dagger=\frac{r}{\mathsf{R}}\,|\mathbf{Z}|\,\mathbf{q}=m_0\,\frac{v^2}{\mathsf{R}}\,\mathbf{q}. \quad (4.3)$$

Here, ∇ is the gradient operator and Φ the gravitational potential on the Wall. The curvature radius R and the unit vectors $\mathbf{q}$ and $\mathbf{q}_0$ connected to the local and global outside directions have been discussed in the previous Section. We can now write Eq. (4.1) explicitly,

$$\mathbf{F}^\dagger+m_0\nabla\Phi+m_0\left(\frac{v^2}{\mathsf{R}}\right)\mathbf{q}=0. \quad (4.4)$$

From the Unique Definition it then follows that the gravitational and centrifugal forces on the Wall are given by

$$\begin{pmatrix}\text{gravitational}\\ \text{force}\end{pmatrix}=m_0\nabla\Phi,\qquad \begin{pmatrix}\text{centrifugal}\\ \text{force}\end{pmatrix}=m_0\,\frac{v^2}{\mathsf{R}}\,\mathbf{q}. \quad (4.5)$$

Thus, the intrinsic centrifugal force on the Wall of Death, defined by the Unique Definition, is identical with the projection of the three-dimensional centrifugal force into the surface of the Wall. The centrifugal force can be measured by the experiment (4.2)

$$\begin{pmatrix}\text{centrifugal}\\ \text{force}\end{pmatrix}=-\frac{\delta\mathbf{F}^\dagger}{\delta(v^2)}\,v_0^2. \quad (4.6)$$

The force always repels in the *local* outside direction $\mathbf{q}$. However, motorcycles moving along type 3 circles are attracted by the centrifugal force towards the *global* center because for this type of circles $\mathbf{q}=-\mathbf{q}_0$. The centrifugal force is identically zero for all the motorcycles moving, with any orbital speed, along type 2 circles, because $\mathscr{C}=1/\mathsf{R}=0$ there. These statements may sound paradoxical, but one look at Fig. 11, which is a slightly changed Fig. 8,

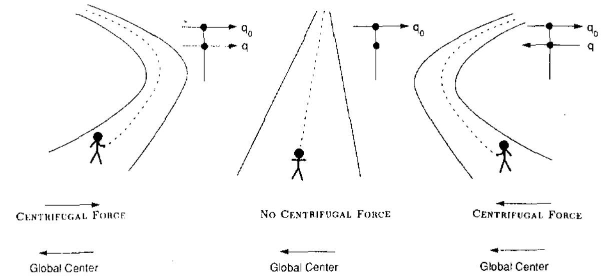

Fig. 11. Centrifugal force at latitude circles of different types.

should be enough to demonstrate that they are in accord with everybody's intuition!

The real force needed to support motorcycle motion along type 2 circles equals $\mathbf{F}^{\dagger}=m_0\nabla\Phi$, independently of the orbital speed. Note, that the free motion is possible only for circles of type 1, because only for these circles the gravitational and centrifugal forces point to opposite directions and thus they could balance each other. No free motion is possible for circles of types 2 and 3.

In general, the two-dimensional surface of the Wall $\mathscr{S}$ may be described, in cylindrical coordinates (r,z,ϕ), by the equation $z=z(r)$. In the three-dimensional space the whole surface $\mathscr{S}$ lies above the plane $z=0$ and it is tangent to this plane in the point C, which is the center of all the latitude circles on the Wall. The inner geometry of $\mathscr{S}$ has the form,

$$dl^2=\left[1+\left(\frac{dz}{dr}\right)^2\right]dr^2+r^2\,d\phi^2=dR^2+r^2(R)\,d\phi^2. \quad (4.7)$$

Here R is geodesic distance from the center,

$$R(r)=\int_0^r\left[1+\left(\frac{dz}{dr}\right)^2\right]^{1/2}dr. \quad (4.8)$$

A latitude circle has the circumferential radius r and the curvature radius [cf. Eq. (3.5)]

$$\mathsf{R}=r\left|\frac{dr}{dR}\right|^{-1}. \quad (4.9)$$

The locally defined outside pointing vector $\mathbf{q}$ has the only nonzero component

$$q_{(R)}=\left(\frac{dr}{dR}\right)\left|\frac{dr}{dR}\right|^{-1}=\epsilon. \quad (4.10)$$

The sign of $q_{(R)}$ is the same as the sign of the function dz/dr. The latitude circles with $1/\mathsf{R}=0$ are geodesic, because they coincide with geodesic lines. Geodesic circles are located in places where the surface $\mathscr{S}$ is tangent (in the three-dimensional space) to cylinders $dz/dr=\infty$ coaxial with rotation axis.

From Eq. (4.7) one deduces that the Lagrangian for a particle motion on the Wall is

$$L=\tfrac{1}{2}m[\dot{R}^2+r^2(R)\dot{\phi}^2]-\Phi(R), \quad (4.11)$$

and the associated equation of motion,

$$m\ddot{R}=mr(R)\,\frac{dr(R)}{dR}\,\dot{\phi}^2-\frac{d\Phi(R)}{dR}, \quad (4.12a)$$

$$r^2(R)\dot{\phi}=\text{const.} \quad (4.12b)$$

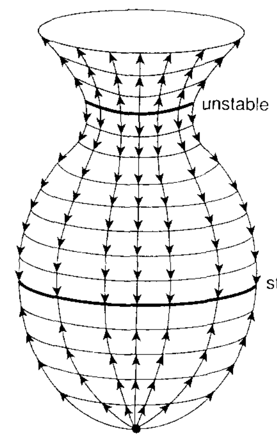

Fig. 12. Centrifugal force repels from unstable and attracts towards stable free circular photon orbits.

Both the energy $\mathscr{E}$ and the angular momentum $\mathscr{L}\equiv\mathscr{P}_\phi$ are constants of photon motion on the Wall. The momentum of a photon must be a null vector, $\mathscr{E}^2=c^2\mathscr{P}^2=0$ in the spacetime, and therefore

$$\mathscr{E}^2=c^2\mathscr{P}^2,\quad \mathscr{P}^2=\mathscr{P}_R^2+\mathscr{P}_\phi^2. \quad (4.13)$$

Here $\mathscr{P}_R$ is the momentum in the radial direction.

Equation (4.13) can also be rewritten in a form familiar from Newtonian theory,

$$\mathscr{P}_R^2=(\mathscr{E}^2/c^2)-\mathscr{U}_{\text{eff}}, \quad (4.14)$$

where $\mathscr{U}_{\text{eff}}$ is the effective potential for the photon motion,

$$\mathscr{U}_{\text{eff}}=\mathscr{L}^2/r^2. \quad (4.15)$$

The condition for a circular photon orbit, $d\mathscr{U}_{\text{eff}}/dR=0$, is equivalent to $dr/dR=0$ and therefore to $1/\mathsf{R}=0$. Stable circular orbits correspond to $d^2\mathscr{U}_{\text{eff}}/dR^2>0$, i.e., minima of the potential, while the unstable orbits correspond to $d^2\mathscr{U}_{\text{eff}}/dR^2<0$, i.e., maxima of the potential. Therefore, the circular photon orbits are located at the *extrema* of the function $r(R)$: the stable circular orbit at the *maxima* of $r(R)$, and the unstable circular orbit at the *minima* of this function:

stable circular photon orbit:

$$\frac{dr}{dR}=0,\quad \frac{d^2r}{dR^2}<0,$$

unstable circular photon orbit:

$$\frac{dr}{dR}=0,\quad \frac{d^2r}{dR^2}>0. \quad (4.16)$$

From this one concludes that $\mathbf{q}$ always points in the direction towards the *stable* circular photon orbits and outwards the *unstable* photon orbits (*Proof*: see Fig. 11). Therefore, the centrifugal force attracts towards the stable circular photon orbits and repels from the unstable ones (Fig. 12).

As an instructive example we shall now discuss a perfectly spherical Wall of Death located at the surface of Earth. In this case the two-dimensional geometry on the Wall has constant curvature $\mathscr{C}=1/R_0$, where R_0 is the radius of the Wall. Gravitational force is due to the Earth surface gravity, $g_0=9.8$ cm/s^2,

Table I. Spherical Wall of Death at the surface of the Earth.

Circumferential radius: $x=r/R_0=\sin X$
Radius: $X=R/R_0=\arcsin x$
Curvature radius: $\mathscr{R}=\frac{R}{R_0}=\frac{x}{\sqrt{1-x^2}}=\lvert\tan X\rvert$
Local outward direction: $\mathbf{q}=\epsilon\mathbf{q}_0,\quad \epsilon=\frac{\tan X}{\lvert\tan X\rvert}$
Orbital speed: $u=\frac{v}{\sqrt{g_0R_0}}$
Gravitational potential: $\tilde{\Phi}=\frac{\Phi}{g_0R_0}=\frac{1}{2}\sin^2 X$
Real force: $\tilde{\mathbf{F}}^\dagger=\mathbf{F}^\dagger/g_0m_0=\tilde{F}^\dagger\mathbf{q}_0$
Gravitational force: $\tilde{\mathbf{G}}^\dagger=\mathbf{G}^\dagger/g_0m_0=\tilde{G}^\dagger\mathbf{q}_0=-\sin X\mathbf{q}_0$
Centrifugal force: $\tilde{\mathbf{Z}}^\dagger=\frac{\mathbf{Z}^\dagger}{g_0m_0}=\tilde{Z}^\dagger\mathbf{q}=\epsilon\sqrt{1-x^2}\left(\frac{u^2}{x}\right)\mathbf{q}_0=u^2\left(\frac{1}{\tan X}\right)\mathbf{q}_0$
Precession rate of the gyroscope: $\tilde{\mathbf{\Omega}}^\dagger=\frac{\mathbf{\Omega}^\dagger\sqrt{R_0}}{\sqrt{g_0}}=\frac{u}{\mathscr{R}}\mathbf{q}$

$$R_0=\begin{pmatrix}\text{curvature radius}\\ \text{of the Wall}\end{pmatrix},\qquad g_0=\begin{pmatrix}\text{gravitational acceleration}\\ \text{on the Wall}\end{pmatrix}. \tag{4.17}$$

Table I shows how R_0 and g_0 are used to scale all the geometrical and physical quantities relevant for our discussion, and how to express them as explicit functions of the circumferential and geodesic radii.

The general equation of motion specified to the spherical Wall takes the form,

$$\tilde{F}^\dagger-x+\epsilon u^2\frac{1}{x}\sqrt{1-x^2}=0. \tag{4.18}$$

The function $\tilde{F}^\dagger(u)$ is shown in Fig. 13 for the three types of the latitude circles. The dynamical aspect of the centrifugal force reversal and vanishing of the centrifugal force

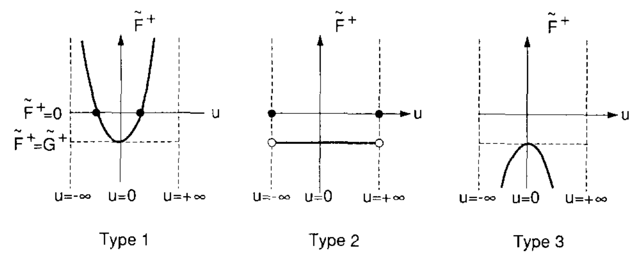

Fig. 13. Real force in function of orbital speed for the three types of latitude circles. Note that the curves $\tilde{F}^\dagger=\tilde{F}^\dagger(u)$ are symmetric with respect to the vertical axis, $\tilde{F}^\dagger(u)=\tilde{F}^\dagger(-u)$. If the Wall is spinning unbeknownst to the acrobats, then in addition to the gravitational and centrifugal forces there will be also the Coriolis force acting on the motorcycles. This will manifest as an asymmetry in the curves $\tilde{F}^\dagger(u)$. One introduces the gravitational, centrifugal and Coriolis forces by considering quantities $\tilde{G}=\tilde{F}^\dagger(0)$, $\tilde{Z}^\dagger=\frac{1}{2}[\tilde{F}^\dagger(u)+\tilde{F}^\dagger(-u)]-\tilde{F}^\dagger(0)$, $\tilde{C}^\dagger=\frac{1}{2}[\tilde{F}^\dagger(u)-\tilde{F}^\dagger(-u)]$, which obviously may be measured experimentally. A similar construction may be introduced in the most general case when the shape of the Wall has no symmetries, rotates and changes in time in any arbitrary way Ref. 16.

for type 2 circles are clearly visible in this figure. It is also clear that in the upper hemisphere, i.e., for the type 2 and 3 circles the free motion is impossible because the centrifugal and gravitational forces cannot balance, as they both point towards the center.

In the lower hemisphere, i.e., for type 1 circles, the free motion is possible for any assumed value of the radius, $0\leqslant x\leqslant 1$, and any assumed value of the orbital speed $0\leqslant u^2\leqslant\infty$. The free orbital speed at a given radius is given by

$$u_K^2=\frac{x^2}{\sqrt{1-x^2}}, \tag{4.19}$$

and the radius corresponding to a free motion for a given orbital speed by,

$$x_K^2=\frac{1}{2}u^4\left(\sqrt{1+\frac{4}{u^4}}-1\right). \tag{4.20}$$

Very close to the geodesic circle $x=1$ this formula gives

$$x_K^2=1-\left(\frac{1}{u^4}\right)+O^2\left(\frac{1}{u^4}\right), \tag{4.21}$$

which means that for all motorcycles with orbital speed $u\gg 1$, i.e., for these which move much faster than $v_0=\sqrt{R_0g_0}$, the free orbit is practically at $x=1$. The physical reason for this is obvious: gravitational force is irrelevant for ultra-fast ($v\gg v_0$) objects because the centrifugal force increases with increasing orbital speed while the gravitational force, which is speed independent, does not. Thus, for a very high orbital speed the centrifugal force always dominates.

V. PROPERTIES OF THE INERTIAL FORCES: INERTIAL FORCES IN EINSTEIN'S THEORY

The discussion in the previous sections made us prepared to list some important properties of the gravitational and centrifugal forces (in the reference frame comoving with the particle) on the Wall of Death.

(1) The condition for the occurrence of the centrifugal force is a nonzero curvature of the trajectory of the moving body. The force acts only on bodies which move along curved trajectories, and does not act on those which move along geodesics lines in space which coincide with the light rays. Gyroscopes do not precess when move along geodesics.

(2) The centrifugal force always pushes bodies in the local outward direction. The local outward direction may point to the global center, and thus the centrifugal force may attract to the center of a circular motion. This happens if and only if perimeter is a decreasing function of diameter for concentric circles. In such situation free motion is not possible.

(3) The centrifugal force attracts towards stable circular rays of light and repels outward from the unstable ones.

(4) The gravitational force per unit mass equals to the gradient of the gravitational potential and is not connected in any way to the curvature of space. The gravitational force does not depend on the speed.

It is remarkable that these properties are *identical* in the case of particles and photons motion close to compact astronomical objects such as very dense stars and black

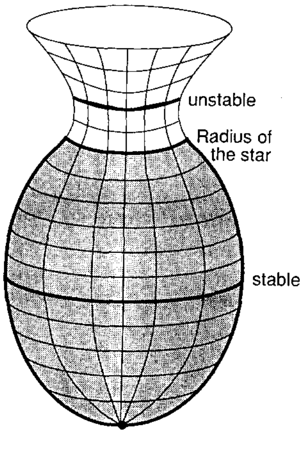

Fig. 14. The embedding diagram, showing the curvature of space inside and around a very compact star with constant density.

holes, which are described by Einstein's general theory of relativity. According to the Einstein theory the geometry of space very close to a compact object is not Euclidean, but strongly curved. The curvature of space increases with the compactness which could be measured, for a spherical object with the mass M, as the ratio of the gravitational radius R_G of the object, to its radius R,

$$C=\frac{R_G}{R}=\frac{2GM}{Rc^2}=\left(\frac{2.9\ \text{km}}{R}\right)\left(\frac{M}{M_\odot}\right). \tag{5.1}$$

Here, G denotes the gravitational constant, c the speed of light and $M_\odot$ the mass of the Sun. Neutron stars have radii not much different than $R\approx 10$ km and masses not much different than that of the Sun, and therefore $C\approx 0.3$ for them. For nonrotating black holes it is always $C=1$.

We shall now consider a spherical star about twice times more compact than a typical neutron star, $8/9<C<2/3$. Geometry of the equatorial plane both inside and outside of such a star is illustrated in Fig. 14 by so-called embedding diagram. It consists of a curved two-dimensional surface of revolution embedded in the familiar three-dimensional Euclidean geometry. The model surface has the same inner geometry as the two-dimensional equatorial plane of the star, embedded in the curved three-dimensional non-Euclidean geometry. The geometry on the embedded diagram is based on measurement of distances with the help of the light signals. In particular, the distance d_{AB} between two points A and B equals half of the round trip travel t_{AB} time (multiplied by the speed of light), in which light moves from A to B and after reflection at B goes back to A,

$$d_{AB}=\tfrac{1}{2}ct_{AB}. \tag{5.2}$$

The time t is measured by static observers who use *synchronized* clocks. This way of measuring distances defines the *optical reference geometry*.[17]

It differs from more often used alternative geometry in which the measurements of distances are based, in principle, on the use of rigid rulers. Both geometries are just some particular ways to make maps of a curved space, and none should be *a priori* consider better than the other one. The difficulty with mapping here is quite similar to that in the case of conventional cartography where it is impossible to represent accurately the spherical surface of the Earth on a flat plane without some kind of distortion. Several types of projections are used in cartography to minimize the distortion of the features of interest, while some other features may be at the same time distorted beyond recognition. The choice of a particular projection depends on the purpose of the special map. For example, the well-known Mercator projection exaggerates polar regions to an enormous extent, but it is invaluable to navigators as it shows all lines of constant directions as straight lines. The optical geometry turns out to be extremely useful for studying light propagation and dynamics, because the geodesic lines in optical geometry are a dynamical models for "straight lines." (In particular, for static spacetimes, light moves in the optical geometry along geodesic lines.) This helps to isolate particular and complicated technicalities from the basic geometrical and physical issues.

In Fig. 14 the two circular photon orbits are indicated by heavy lines. Exactly as in the case of the two-dimensional Newtonian dynamics, stability [cf. Eq. (4.10)] of the orbit is connected to the sign of the second derivative of the function $r=r(R)$. The centrifugal force attracts to the stable, and repulses from the unstable circular photon orbit.

This explains, in exactly the same way as in the case of the Wall of Death, the paradoxical and astrophysically important effects mentioned at the beginning of this article.

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[b]Also: Nordita, Blegdamsvej 17, DK-2100 Kobenhavn Ø, Denmark.

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[12]Huygens gave a formula for the centrifugal force in his treatise *De vi centrifuga* written in 1659 but only published in 1703 after his death. In 1669 Huygens established his claim to authorship of the formula by

sending it, in anagrammatic form, to Oldenburg, who was then the secretary of the Royal Society. The solution of the anagrams were given in Huygens' book *Horologium Oscillatorium* which has been published in Paris in 1673, 14 yr before publication of Newton's *Principia*.

[13]The Bible, Gen. 1. 1 and Gen. 2. 20.

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Kepler's third law and the oscillator's isochronism

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Two classes of differential equations which have Kepler-like and oscillatorlike elliptical orbits are shown to have generalizations of the conserved angular momentum, energy, and Laplace–Runge–Lenz vector (Jauch–Hill–Fradkin tensor for the oscillator case). Both possess a generator of self-similar transformations and the related period–semimajor axis relation is a generalization of Kepler's third law in which the constant of proportionality depends explicitly on the eccentricity of the orbit.

I. INTRODUCTION

It is well known that the Kepler problem, described in reduced coordinates by the equation of motion

$$\ddot{\mathbf{r}}+\frac{\mu\mathbf{r}}{r^3}=0, \tag{1.1}$$

where μ is a positive constant, possesses the constants of the motion, energy

$$E=\tfrac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}-\frac{\mu}{r}, \tag{1.2}$$

angular momentum

$$\mathbf{L}=\mathbf{r}\times\dot{\mathbf{r}}, \tag{1.3}$$

Hamilton's vector[1]

$$\mathbf{K}=\dot{\mathbf{r}}-\frac{\mu}{L}\hat{\boldsymbol{\theta}} \tag{1.4}$$

and Laplace–Runge–Lenz vector[2–4]

$$\mathbf{J}=\mathbf{K}\times\mathbf{L}=\dot{\mathbf{r}}\times\mathbf{L}-\mu\hat{\mathbf{r}}, \tag{1.5}$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are the unit vectors in plane polar coordinates in the plane of the motion. If θ is measured from $\mathbf{J}$, the scalar product of $\mathbf{r}$ with Eq (1.5) leads to the equation of the orbit

$$r=\frac{L^2}{\mu+J\cos\theta} \tag{1.6}$$

which is an ellipse, parabola, or hyperbola according to whether $\mu>J$, $\mu=J$ or $\mu<J$.

The equation of motion (1.1) is invariant under the actions of the second extensions of the three elements of the rotation group SO(3), time translation $G=\partial/\partial t$, and self-similarity

$$G=t\frac{\partial}{\partial t}+\frac{2}{3}r\frac{\partial}{\partial r}. \tag{1.7}$$

(In the case of the generator of self-similar transformations, for example, the second extension is

$$G^{[2]}=t\frac{\partial}{\partial t}+\frac{2}{3}r\frac{\partial}{\partial r}-\frac{1}{3}\dot{r}\frac{\partial}{\partial \dot{r}}-\frac{4}{3}\ddot{r}\frac{\partial}{\partial \ddot{r}}.\Big)$$

This last symmetry is generally associated with the Laplace–Runge–Lenz vector and Kepler's third law of planetary motion

$$TR^{-3/2}=\frac{2\pi}{\mu^{1/2}},$$

