How efficiently do 3-manifolds bound 4-manifolds?

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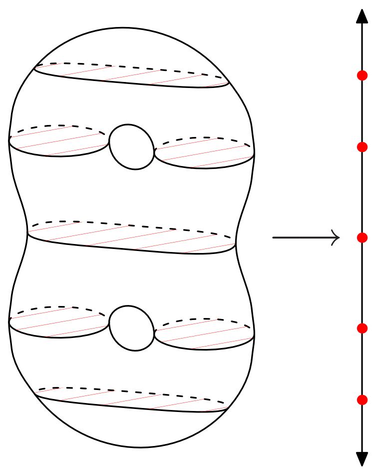
Trieste, Italy
June 23, 2005



http://www.math.harvard.edu/~dpt/speaking/

Main result

Theorem (Rohlin (1951), Thom (1954), Lickorish (1962), Rourke (1985), Matveev-Polyak (1994))**.** *Every oriented, closed 3-manifold M is the boundary f an oriented 4-manifold.*

We'll give yet another proof!

Try to control the *complexity* of the 4-manifold.

Definition. The *complexity* of a PL n-manifold M is the minimum, over all triangulations of M, of the number of n-simplices in the triangulation.

Question. What is $G_3(k)$, the least function such that every orientable 3-manifold of made of k tetrahedra is the boundary of a 4-manifold with at most $G_3(k)$ 4-simplices? What is its asymptotic growth rate?

Theorem (CT)**.** $G_3(k) = O(k^2)$.

There is an obvious linear lower bound on G_3; there is still a gap between linear and quadratic.

Elaborations

Theorem. $G_3(k) = O(k^2)$.

As a result, you don't lose too much by using a surgery diagram rather than a triangulation to represent a 3-manifold. (A surgery diagram gives a 3-manifold and a 4-manifold bounded by it.)

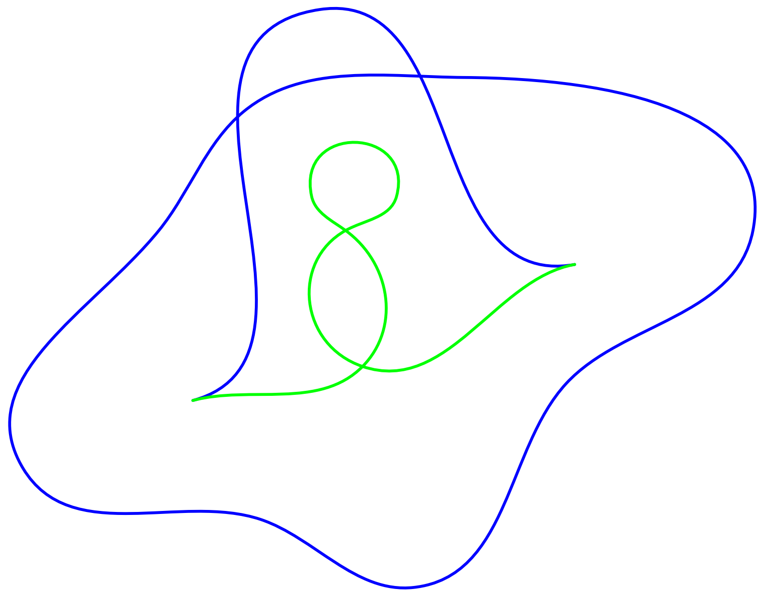
Theorem. *A hyperbolic 3-manifold with volume V is rational surgery on a link with $O(V^2)$ crossings.*

For a clean statement for more general manifolds, we need to generalize surgery diagrams to *shadow diagrams*.

Theorem. *A shadow diagram for a geometric manifold M has at least $\Omega(M)$ vertices and requires at most $O(M^2)$ vertices.*

Proof idea

Throw your 3-manifold at the screen.



That is, take a generic smooth map from M^3 to $\mathbb{R}^2$.

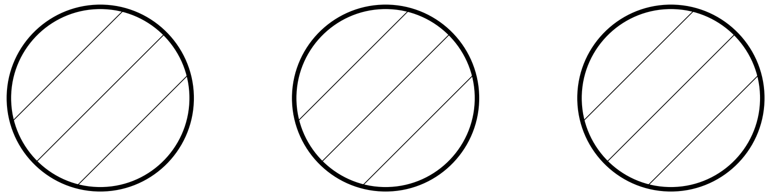
At regular value, inverse image is a 1-manifold, a disjoint union of circles.
Idea: Glue in disks to these circles.

Extend past singularities.

Warm-up: $n = 1$

Overall question: when does an oriented n-manifold (without boundary) appear as the boundary of an oriented $(n+1)$-manifold?

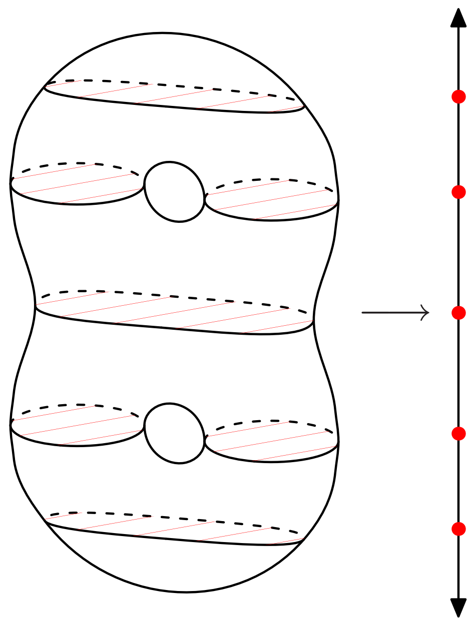
$n = 1$: An closed 1-manifold is a disjoint union of circles. It is the boundary of a disjoint union of disks.



Warm-up: $n = 2$

Does every surface bound a 3-manifold? No, but every orientable one does.

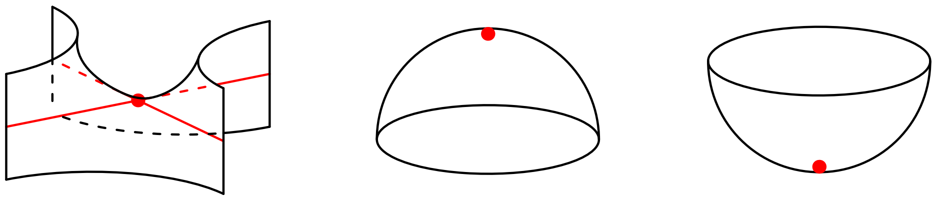
Pick a generic, smooth function from surface to $\mathbb{R}$ (a Morse function). Inverse image of a regular point is an oriented 1-manifold, a disjoint union of circles. **Idea:** glue in disks to each of these circles.



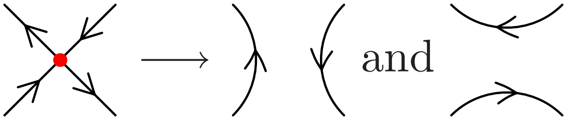
But how to continue past the critical points?

Warm-up: $n = 2$, analysing singularities

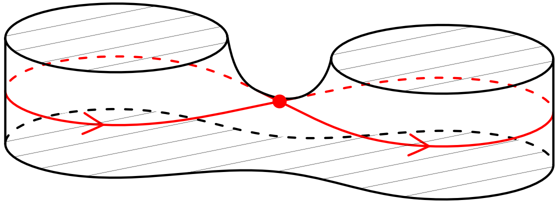
A critical point (locally in domain) is either a saddle, a maximum, or a minimum.



The orientations of the level set around a saddle point must alternate:



Therefore the inverse image of a saddle value is a figure 8, and the inverse image of its neighborhood is a pair of pants.



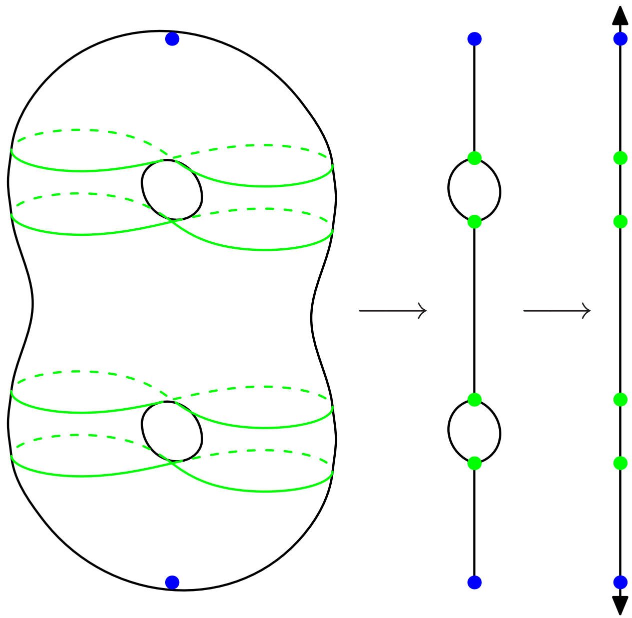
After attaching disks at the regular values, we are left with a 2-sphere boundary component. Fill it in with a 3-ball to construct the 3-manifold.

The Stein factorization: a more global point of view

For a map f from a compact Σ^2 to $\mathbb{R}$, let

$$X = \{\text{connected component of } f^{-1}(x) \mid x \in \mathbb{R}^2\}$$

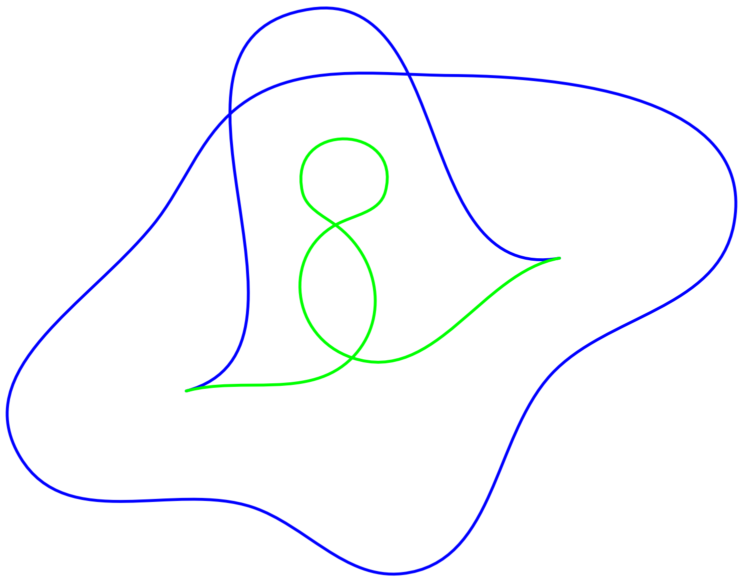
Then we can write $f = g \circ h$, where the fibers of h are connected and g is finite-to-one. This is the *Stein factorization* of f.



At generic points, Σ is a circle bundle over the *Stein graph* and the 3-manifold is a disk bundle. Alternatively, the 3-manifold collapses onto the Stein graph.

The main case: $n = 3$

Throw your 3-manifold at the screen.



That is, take a generic smooth map from M^3 to $\mathbb{R}^2$.

At regular value, inverse image is a 1-manifold, a disjoint union of circles.
Idea: Glue in disks to these circles.

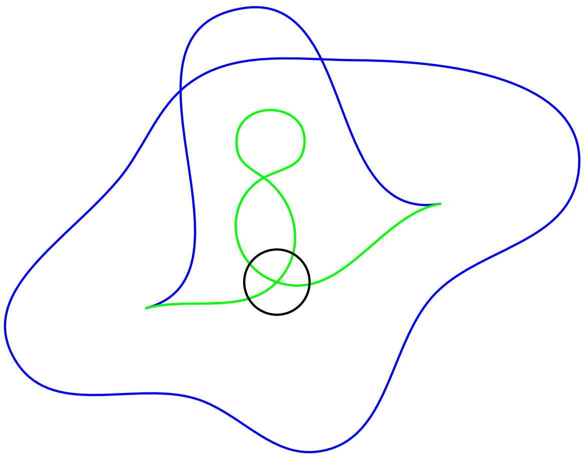
Extend past singularities, which were first analyzed by H. Levine (1988).

The main case: $n = 3$

In the case of 3-manifolds mapping to $\mathbb{R}^2$, the Stein factorization will give a surface with some type of singularities. In codimension 1 the singularities look like the previous singularities (surface to $\mathbb{R}$), crossed with an interval.

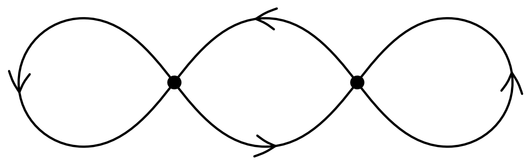
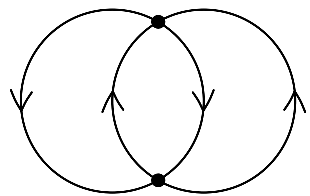
In codimension 2 locally in the domain, the only new singularity is a kind of cusp where two critical points of index 0 and 1 meet and cancel.

However, this singularity turns out to play little role. More interesting is the crossing of two codimension 1 saddle-type singularities.



Crossing singularities

In the inverse image of a crossing of two saddle type singularities, there are two singular points. There are two ways to connect up these two singularities to get a connected, oriented fiber.

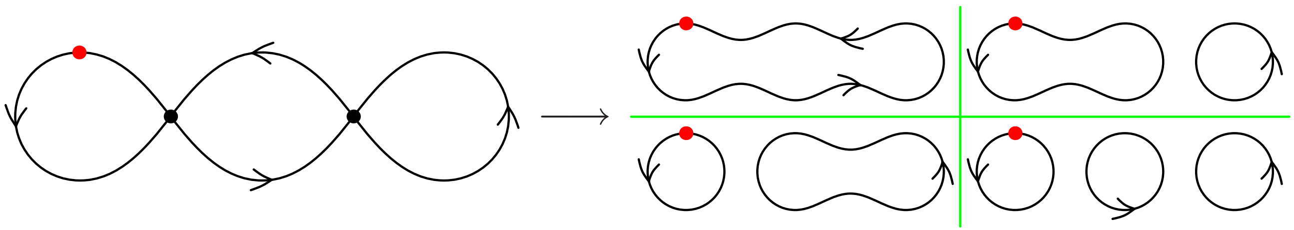
(a)  (b) 

We will assume that our generic map has no singularities of type (b), since:

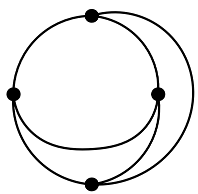
- Type (b) is similar to type (a), only slightly more complicated;
- A singularity of type (b) can be C^0 perturbed to get two singularities of type (a).

Analyzing the singularity

In a neighborhood of a singularity of type (a), the inverse image of a generic point is one of the 4 ways of resolving the singular graph into a 1-manifold.



We want to find the inverse image N of a neighborhood of this point, which is a 3-manifold with boundary. The boundary has a map to the circle. The Stein graph is



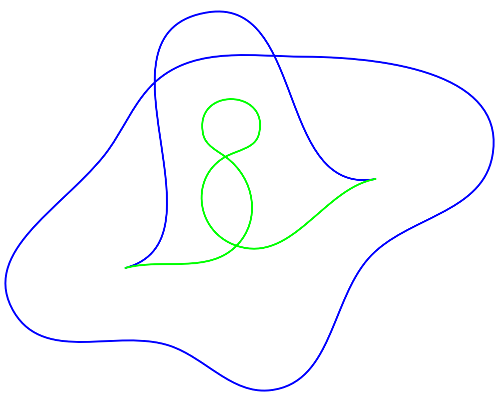
Therefore the boundary is a genus 3 surface.

Each regular point in the singular fiber gives us a disk in the boundary that bounds in N. Considering enough of these, we see that N is S^3 minus an unknotted tetrahedron graph.

Constructing the 4-manifold

Now assemble the pieces to construct a 4-manifold W^4 bounded by our 3-manifold M^3.

- Start with $M \times [0, 1]$. We want to kill one boundary component.
- Attach a 2-handle along each circle in the inverse image of a regular point.
- Attach a 3-handle transverse to each codimension 1 singularity.
- The remaining boundary components are all 3-spheres, by the analysis of the singularities. (The graph was filled in by the other components, just as the pair of pants was filled.) Attach 4-balls to them.



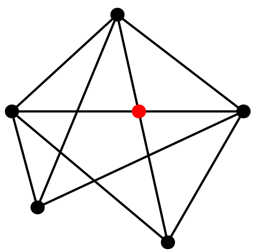
Why quadratic?

We have been working with generic smooth maps. To bound the complexity, we want to start from a triangulation.

Start with a triangulation and pick a generic piecewise linear map to the plane: Pick an arbitrary map from the vertices of the triangulation and extend linearly over the simplices.

Codimension 1 singularities come from the images of edges of triangulation.

Codimension 2 singularities may come from the crossing of the image of two edges. The number of such crossings is at most quadratic in the number of tetrahedra.



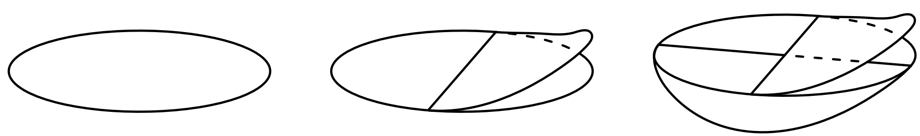
Shadow surfaces: first glimpse

As in the case of surfaces mapping to $\mathbb{R}$, we can consider the Stein factorization of our map from M^3 to $\mathbb{R}^2$.

The resulting Stein surface is a 2-complex. It has a number of local models, including

- the plane $\mathbb{R}^2$ at regular points;
- a 3-page book from codimension 1 singularities; and
- the cone over the graph we found around the crossing of singularities.

among a few others (which can be removed).



The 3-manifold is generically a circle bundle over the Stein surface and the 4-manifold is generically a disk bundle.

The shadow world

These Stein surfaces are examples of *shadow surfaces*, as introduced by Turaev.

As representations of 3- or 4-manifolds, they have many advantages: they

- Generalize many other representations;
- Let you compute quantum invariants;
- Provide efficient representations for small manifolds;
- Relate to hyperbolic geometry (and Gromov norm); and
- Arise naturally in proofs.

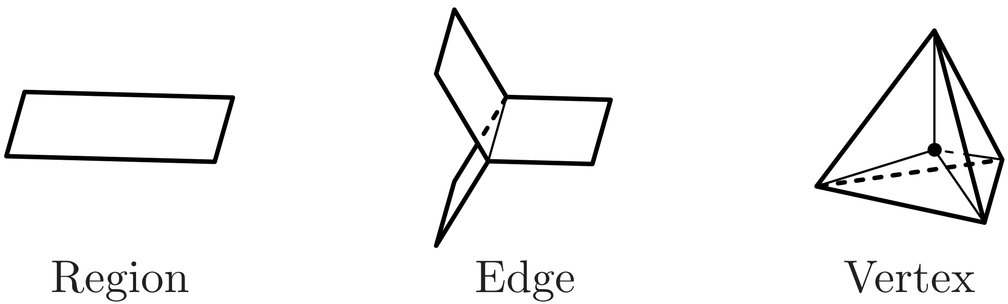
But basic questions are unknown:

- Simple moves relate any two shadows represent the same 3-manifold (CT), but
- Finding such moves for 4-manifolds is related to the Andrews-Curtis conjecture.

Definition of shadows

A soap bubble locally looks like:

- A plane;
- A 3-page book; or
- A cone over the tetrahedron graph.



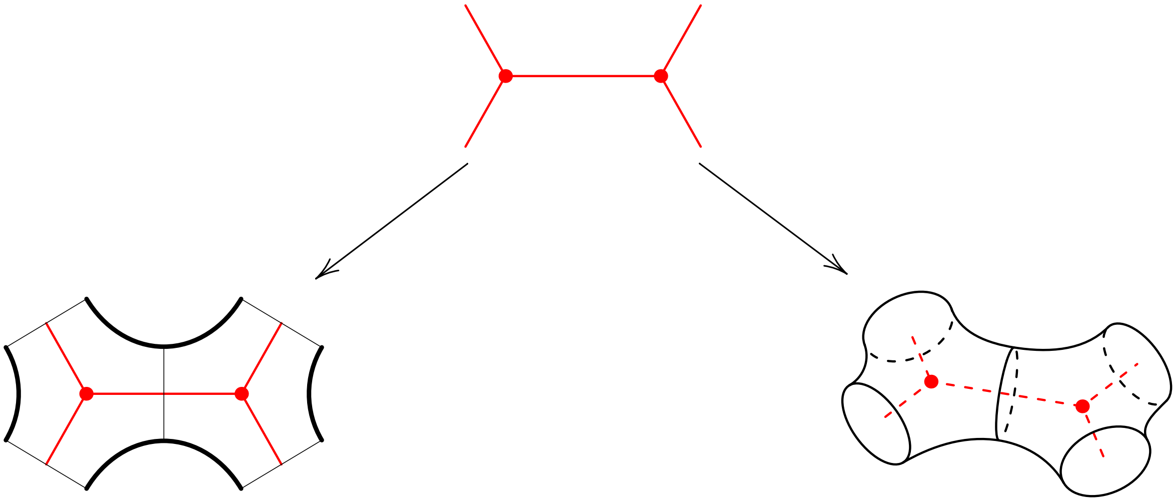
A 2-complex Σ whose only singularities are in the list above is called a *simple polyhedron*.

If such a Σ is embedded in a 4-manifold W in a locally flat way, and W collapses onto Σ, then we call Σ a *shadow presentation* of W and its boundary ∂W.

Shadows in 2 dimensions

Contrast shadows with standard spines for 3-manifolds, where the 3-manifold is generically an interval bundle over Σ, rather than a circle/disk bundle.

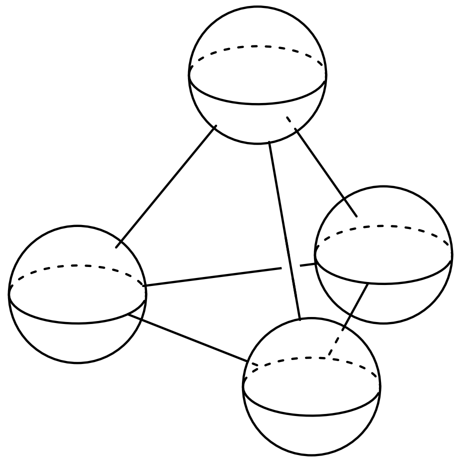
One dimension down, trivalent graphs Γ represent surfaces in two ways:



- $N(\Gamma)$, $\Gamma \subset \Sigma^2$, is a ribbon graph dual to a triangulation.
- $N(\Gamma)$, $\Gamma \subset M^3$, is a handlebody; the surface on the boundary has a pair of pants decomposition.

Shadows in 3 dimensions

In 3D, the basic building block of shadows is the complement of the tetrahedron graph.



The pairs of pants around vertices are glued according to the singular graph.

The result has torus boundary components; these are filled in with a solid torus for each region of the shadow (if regions are disks).

To determine the 4-manifold W from the 2-complex Σ, we also need to specify some integers or half-integers (the *gleams*) on the regions. These can be thought of as framings or relative Euler classes.

Shadow number and hyperbolic volume

Definition. The *shadow number* of a 3-manifold is the minimum number of vertices in a shadow representation of the manifold.

Note that there are an infinite number of 3-manifolds of fixed shadow number. Contrast with complexity.

Theorem. *A geometric manifold with Gromov norm $\|M\|$ has shadow number S with*

$$\frac{v_{\mathrm{tet}}}{2v_{\mathrm{oct}}}\|M\| \leq S \leq C\|M\|^2$$

for a suitable constant C.

For lower bound, the manifold before gluing in the regions has an explicit hyperbolic structure: each block is the double of a regular ideal octahedron along half its faces. The Gromov norm only decreases under surgery.

For upper bound, use a version of main theorem for ideal triangulations.

Corollary. *A map from a 3-manifold M to $\mathbb{R}^2$ has at least $\|M\|/10$ crossing singularities.*

Related Questions and Future Directions

Questions on shadows

- Is the shadow number additive under connect sum? This implies an explicit linear bound on G_3: a standard spine for M is a shadow for $M\#\bar{M}$.
- Is the shadow number additive under torus connect sum? This implies an upper bound on the shadow number in terms of hyperbolic volume.
- For hyperbolic manifolds, can you find shadow representatives so that hyperbolicity is automatic? (Compare: alternating knots.)

Related question: bounding knots

An unknotted circle in space is the boundary of an embedded disk.

Question. What is the least function G_{disk} so that every unknot with k straight line segments bounds a disk made of $G_{\text{disk}}(k)$ flat triangles?

Theorem (Hass-Lagarias-Snoeyink-W. Thurston). $c_1 A^k \leq G_{\text{disk}}(k) \leq c_2 B^{k^2}$ *for suitable constants* c_1, c_2, A, B.

A polygonal non-intersecting curve in space is the boundary of a surface.

Question. What is the least function G_{surf} so that every polygonal curve with k straight line segments bounds a surface made of $G_{\text{surf}}(k)$ flat triangles?

Theorem (Hass-Lagarias). $\frac{1}{2}k^2 - ck \leq G_{\text{surf}}(k) \leq 7k^2$ *for a constant* c.

Question. What is the bound G_{Pachner} of the number of Pachner moves required to turn a triangulation of S^3 with k tetrahedra into a standard one?

Theorem (King, Mijatović). $G_{\text{Pachner}}(n) \leq c_1 A^{n^2}$ *for constants* c_1, A.

Embedded Andrews-Curtis

- What moves suffice to relate any two shadow surfaces representing the same 4-manifold?
- Given two 2-dimensional spines of a 4-manifold. Are they equivalent by embedded 3-deformations?
- (Andrews-Curtis, group theory version) Are any two balanced presentations of the trivial group related by Nielsen transformations?

We have a set of moves relating any two shadow surfaces representing the same 3-manifold. In general, finding set of moves is not hard once you allow any sort of stabilization that increases the second homology of the 4-manifold.

4-manifolds bounding 5-manifolds?

Why doesn't this proof work to show that 4-manifolds bound 5-manifolds? Start the same way: pick a generic map from your 4-manifold to $\mathbb{R}^3$...

There are some new codimension 3 singularities. For some of them, the inverse image of a neighborhood, filled in around the boundary, is not S^4, but rather $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$.

This does give a proof that every smooth 4-manifold is cobordant to a union of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$'s.

If you want to get bounds on the complexity, you also need to worry about generic PL maps which are not generic smooth maps in codimension 2.

Similarly, not ever unoriented surface is the boundary of a 3-manifold. For unoriented surfaces, there is another possible structure of the singular level, which yields an $\mathbb{RP}^2$ on the boundary.

Questions on bounding manifolds

- Lower bounds or better upper bounds for G_3, 3-manifolds bounding 4-manifolds.
- Lower bounds for G_{Pachner}, Pachner moves to make a triangulation of S^3 standard.
- Bounds for 3-manifolds to bound special 4-manifolds, like simply-connected or spin.
- Lower bounds with a coarser notion of the complexity of the 4-manifold (e.g., the order of the second homology).

One approach to a lower bound for G_3:

- Pick an invariant I of 3-manifolds that is defined from a 4-manifold bounded by the 3-manifold;
- Show that I is linearly bounded by the complexity of the 4-manifold;
- Find a family of 3-manifolds for which I grows quadratically in the complexity of the 3-manifold.

Previous work

Each constructive proof that 3-manifolds bound 4-manifolds gives, in principle, an estimate for $G_3(n)$.

- Rohlin (1951): Based on a generic map $f : M^3 \to \mathbb{R}^5$. Probably gives $G_3(n) \leq kn^4$.
- Thom (1954): Homotopy-theoretic. Hard to get explicit bounds on $G_3(n)$.
- Lickorish (1962), Rourke (1985), Matveev-Polyak (1994): Inductive proofs, based on mapping class group. Use the inductive hypothesis twice, so get exponential bounds for $G_3(n)$ at best.
- Costantino-T.: Based on a generic map $f : M^3 \to \mathbb{R}^2$. Gives $G_3(n) \leq kn^2$.