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Will We Ever Classify Simply-Connected Smooth 4-manifolds?

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Abstract. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

1. Introduction

Despite spectacular advances in defining invariants for simply-connected smooth and symplectic 4-dimensional manifolds and the discovery of important qualitative features about these manifolds, we seem to be retreating from any hope to classify simply-connected smooth or symplectic 4-dimensional manifolds. The subject is rich in examples that demonstrate a wide variety of disparate phenomena. Yet it is precisely this richness which, at the time of these lectures, gives us little hope to even conjecture a classification scheme. In these notes, adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute, we will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

Algebraic Topology. The critical algebraic topological information for a closed, simply-connected, smooth 4-manifold X is encoded in its Euler characteristic $e(X)$, its signature $\sigma(X)$, and its type $t(X)$ (either 0 if the intersection form of X is even and 1 if it is odd). These invariants completely classify the homeomorphism

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type of X ([3, 12]). We recast these algebraic topological invariants by defining $\chi_h(X) = (e(X) + \sigma(X))/4$, which is the holomorphic Euler characteristic in the case that X is a complex surface, and $c(X) = 3\sigma(X) + 2e(X)$, which is the self-intersection of the first Chern class of X in the case that X is complex.

Analysis. To date, the critical analytical information for a smooth, closed, simply-connected 4-manifold X is encoded in its Seiberg-Witten invariants [30]. When $\chi_h(X) > 1$ this integer-valued function SW_X is defined on the set of *spin*^{c} structures over X. Corresponding to each *spin*^{c} structure $\mathfrak{s}$ over X is the bundle of positive spinors $W_{\mathfrak{s}}^+$ over X. Set $c(\mathfrak{s}) \in H_2(X)$ to be the Poincaré dual of $c_1(W_{\mathfrak{s}}^+)$. Each $c(\mathfrak{s})$ is a characteristic element of $H_2(X; \mathbf{Z})$ (i.e. its Poincaré dual $\hat{c}(\mathfrak{s}) = c_1(W_{\mathfrak{s}}^+)$ reduces mod 2 to $w_2(X)$). The sign of SW_X depends on a homology orientation of X, that is, an orientation of $H^0(X; \mathbf{R}) \otimes \det H_+^2(X; \mathbf{R}) \otimes \det H^1(X; \mathbf{R})$. If $\mathrm{SW}_X(\beta) \neq 0$, then β is called a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with $\mathrm{SW}_X(-\beta) = (-1)^{\chi_h(X)}\,\mathrm{SW}_X(\beta)$. The Seiberg-Witten invariant is an orientation-preserving diffeomorphism invariant of X (together with the choice of a homology orientation). We recast the Seiberg-Witten invariant as an element of the integral group ring $\mathbf{Z}H_2(X)$, where for each $\alpha \in H_2(X)$ we let t_α denote the corresponding element in $\mathbf{Z}H_2(X)$. Suppose that $\{\pm\beta_1, \dots, \pm\beta_n\}$ is the set of nonzero basic classes for X. Then the Seiberg-Witten invariant of X is the Laurent polynomial

$$\mathcal{SW}_X = \mathrm{SW}_X(0) + \sum_{j=1}^{n} \mathrm{SW}_X(\beta_j) \cdot (t_{\beta_j} + (-1)^{\chi_h(X)}\, t_{\beta_j}^{-1}) \in \mathbf{Z}H_2(X).$$

When $\chi_h = 1$ the Seiberg-Witten invariant depends on a given orientation of $H_+^2(X; \mathbf{R})$, a given metric g, and a self-dual 2-form as follows. There is a unique g-self-dual harmonic 2-form $\omega_g \in H_+^2(X; \mathbf{R})$ with $\omega_g^2 = 1$ and corresponding to the positive orientation. Fix a characteristic homology class $k \in H_2(X; \mathbf{Z})$. Given a pair (A, ψ), where A is a connection in the complex line bundle whose first Chern class is the Poincaré dual $\widehat{k} = \frac{i}{2\pi}[F_A]$ of k and ψ a section of the bundle W^+ of self-dual spinors for the associated *spin*^{c} structure, the perturbed Seiberg-Witten equations are:

$$D_A\psi = 0$$
$$F_A^+ = q(\psi) + i\eta$$

where F_A^+ is the self-dual part of the curvature F_A, D_A is the twisted Dirac operator, η is a self-dual 2-form on X, and q is a quadratic function. Write $\mathrm{SW}_{X,g,\eta}(k)$ for the corresponding invariant. As the pair (g, η) varies, $\mathrm{SW}_{X,g,\eta}(k)$ can change only at those pairs (g, η) for which there are solutions with $\psi = 0$. These solutions occur for pairs (g, η) satisfying $(2\pi\widehat{k} + \eta) \cdot \omega_g = 0$. This last equation defines a wall in $H^2(X; \mathbf{R})$.

The point ω_g determines a component of the double cone consisting of elements of $H^2(X; \mathbf{R})$ of positive square. We prefer to work with $H_2(X; \mathbf{R})$. The dual component is determined by the Poincaré dual H of ω_g. (An element $H' \in H_2(X; \mathbf{R})$ of positive square lies in the same component as H if $H' \cdot H > 0$.) If $(2\pi\widehat{k} + \eta) \cdot \omega_g \neq 0$ for a generic η, $\mathrm{SW}_{X,g,\eta}(k)$ is well-defined, and its value depends only on the sign of $(2\pi\widehat{k} + \eta) \cdot \omega_g$. Write $\mathrm{SW}_{X,H}^+(k)$ for $\mathrm{SW}_{X,g,\eta}(k)$ if $(2\pi\widehat{k} + \eta) \cdot \omega_g > 0$ and $\mathrm{SW}_{X,H}^-(k)$ in the other case.

The invariant $\mathrm{SW}_{X,H}(k)$ is defined by $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^+_{X,H}(k)$ if $(2\pi\widehat{k})\cdot\omega_g > 0$, or dually, if $k \cdot H > 0$, and $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^-_{X,H}(k)$ if $H \cdot k < 0$. The wall-crossing formula [15, 16] states that if H', H'' are elements of positive square in $H_2(X; \mathbf{R})$ with $H' \cdot H > 0$ and $H'' \cdot H > 0$, then if $k \cdot H' < 0$ and $k \cdot H'' > 0$,

$$\mathrm{SW}_{X,H''}(k) - \mathrm{SW}_{X,H'}(k) = (-1)^{1+\frac{1}{2}d(k)}$$

where $d(k) = \frac{1}{4}(k^2 - (3\,\mathrm{sign} + 2\,e)(X))$ is the formal dimension of the Seiberg-Witten moduli spaces.

Furthermore, in case $b^- \leq 9$, the wall-crossing formula, together with the fact that $\mathrm{SW}_{X,H}(k) = 0$ if $d(k) < 0$, implies that $\mathrm{SW}_{X,H}(k) = \mathrm{SW}_{X,H'}(k)$ for any H' of positive square in $H_2(X; \mathbf{R})$ with $H \cdot H' > 0$. So in case $b^+_X = 1$ and $b^-_X \leq 9$, there is a well-defined Seiberg-Witten invariant, $\mathrm{SW}_X(k)$.

Possible Classification Schemes. From this point forward and unless otherwise stated all manifolds will be closed and simply-connected. In order to avoid trivial constructions we consider *irreducible* manifolds, i.e. those that cannot be represented as the connected sum of two manifolds except if one factor is a homotopy 4-sphere. (We still do not know if there exist smooth homotopy 4-spheres not diffeomorphic to the standard 4-sphere S^4).

So the existence part of a classification scheme for irreducible smooth (symplectic) 4-manifolds could take the form of determining which $(\chi_h, c, t) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_2$ can occur as $(\chi_h(X), c(X), t(X))$ for some smooth (symplectic) 4-manifold X. This is referred to as *the geography problem.* The game plan would be to create techniques to realize all possible lattice points. The uniqueness part of the classification scheme would then be to determine all smooth (symplectic) 4-manifolds with a fixed $(\chi_h(X), c(X), t(X))$ and determine invariants that would distinguish them. Again, the game plan would be to create techniques that preserve the homeomorphism type yet change these invariants.

In the next two sections we will outline what is and is not known about the existence (geography) and uniqueness problems without detailing the techniques. Then we will list the techniques used, determine their interplay, and explore questions that may yield new insight. A companion approach, which we will also discuss towards the end of these lectures, is to start with a particular well-understood class of 4-manifolds and determine how all other smooth (symplectic) 4-manifolds can be constructed from these.

2. Existence

Our current understanding of the geography problem is given by Figure 1 where all known simply-connected smooth irreducible 4-manifolds are plotted as lattice points in the (χ_h, c)-plane. In particular, all known simply-connected irreducible smooth or symplectic 4-manifolds have $0 \leq c < 9\chi_h$ and every lattice point in that region can be realized by a symplectic (hence smooth) 4-manifold.