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Exotic smooth structures on 4-manifolds, II ☆

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Abstract

In this article we continue to investigate exotic smooth structures of 4-manifolds studied in [Forum Math. 14 (2002) 915–929]. As a conclusion, we claim that most known simply connected, closed, irreducible, nonspin, smooth 4-manifolds with b_2^+ odd and large enough admit infinitely many, both symplectic and non-symplectic, exotic smooth structures.
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1. Introduction

One of the fundamental problems in the topology of 4-manifolds is to determine whether a given topological 4-manifold admits a smooth structure and, if it does, whether such a smooth structure is unique or not. Though the complete answer is far from reach, the inception of gauge theory makes us to answer these questions [1,2,5–8]. Regarding this, R. Fintushel and R. Stern recently conjectured the following

Conjecture 1. *All but finitely many simply connected, closed, smooth* 4*-manifolds with* $SW \neq 0$*, admit an infinite family of both symplectic and non-symplectic distinct structures.*

In this article we continue to investigate the existence problem of exotic smooth structures on smooth 4-manifolds studied in [8]. According to a convention, we say that a smooth 4-manifold admits an *exotic smooth structure* if it has more than one distinct

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smooth structure. As we see in Section 2, the simple way to get exotic smooth structures on a smooth 4-manifold X is to construct smooth 4-manifold X' which is homeomorphic, but not diffeomorphic, to X. Using this technique, the author obtained two results—one is for spin case [7] and the other is for nonspin and nonpositive signature case [8]. But one case is still left, that is, it is natural to ask the same question for nonspin smooth 4-manifolds with positive signature. In the next section we first review known techniques and results. And then we apply similar techniques to get the following theorem.

Theorem 1.1. *There is an increasing sequence $\{m_i\}$ with $m_i \to 9$ such that every simply connected, closed, nonspin, irreducible, smooth 4-manifold satisfying $0 \leqslant \mathbf{c} \leqslant m_i\chi$ and $b_2^+ \geqslant C_i$ odd (C_i is a constant depending on m_i) admits infinitely many, both symplectic and non-symplectic, exotic smooth structures.*

Remarks.

(1) Here we define $\mathbf{c}$ and χ by $3\sigma + 2e$ and $(\sigma + e)/4$, respectively, where σ and e denote the signature and the Euler characteristic of a given 4-manifold. Note that every smooth 4-manifold with positive signature satisfies $\mathbf{c} := 3\sigma + 2e > 8\chi$. Furthermore, all known simply connected, closed, irreducible, smooth 4-manifolds with $SW \neq 0$ satisfy both $0 \leqslant \mathbf{c} = 3\sigma + 2e < 9\chi$ and b_2^+ odd.

(2) The main new ingredient in the proof of Theorem 1.1 above is to use complex surfaces lying on the BMY-line studied by A. Stipsicz [10].

2. A construction

As we mentioned in Introduction, the simple way to get exotic smooth structures on a smooth 4-manifold X is to construct a family of homeomorphic types of X which have mutually different Seiberg–Witten invariants (or Donaldson invariants). For this, we first briefly review main techniques—called a fiber sum, a logarithmic transform, and R. Fintushel and R. Stern's knot surgery—and state related theorems (see [7,8] for details).

Definition. For $i = 1, 2$, let X_i be a closed smooth 4-manifold containing a smoothly embedded surface Σ of the same genus g and of the opposite square. Suppose $X_i^0 = X_i - \nu_i(\Sigma)$ is a complement of a tubular neighborhood $\nu_i(\Sigma)$ of Σ in X_i and N_i is a boundary of the tubular neighborhood $\nu_i(\Sigma)$. Then, by choosing an orientation-reversing, fiber-preserving diffeomorphism $\varphi : \nu_1(\Sigma) \to \nu_2(\Sigma)$ and by gluing X_1^0 to X_2^0 along their boundaries via the diffeomorphism $\varphi| : N_1 \to N_2$, we define a new closed smooth 4-manifold $X_1 \sharp_\Sigma X_2$, called a *fiber sum* of X_1 and X_2 along Σ. Furthermore R. Gompf, J. McCarthy and J. Wolfson independently extended *a fiber sum* technique to symplectic category. That is, they proved

Theorem 2.1 [2,4]. *For $i = 1, 2$, let X_i be a closed symplectic 4-manifold containing a symplectic (or Lagrangian) genus g surface Σ of the opposite square. In case Σ is Lagrangian, assume that Σ represents a non-zero homology class in $H_2(X_i : \mathbb{Z})$. Then* *there exists an orientation-reversing, bundle diffeomorphism* $\varphi : \nu_1(\Sigma) \to \nu_2(\Sigma)$, *so that the resulting* 4*-manifold* $X_1 \sharp_\Sigma X_2$ *admits a symplectic structure.*

Remarks.

(1) A spin condition is preserved under a fiber sum operation. That is, if both X_1 and X_2 are spin 4-manifolds, so is $X_1 \sharp_\Sigma X_2$ (Proposition 1.2 in [2]).

(2) Minimality is also preserved under a symplectic fiber sum operation. That is, if both X_1 and X_2 are irreducible symplectic 4-manifolds, so is $X_1 \sharp_\Sigma X_2$ due to W. Lorek. Here is a sketch of Lorek's argument: If $X_1 \sharp_\Sigma X_2$ is not minimal, since any exceptional curve in a symplectic 4-manifold with $b^+ \geqslant 2$ can be symplectically embedded, there exists a family of symplectic 2-sphere C_t, parameterized by $t \in D^2$, of square -1 in $X_1 \sharp_\Sigma X_2$. By considering C_t as a pseudo-holomorphic curve and applying Gromov's compactness theorem for the family C_t $(t \to (0,0))$, one can get a limiting curve $C = C_1 \cup \cdots \cup C_k$ in $X_1 \cup X_2$ such that each C_i is an embedded 2-sphere lying in either X_1 or X_2 and $C^2 = C_1^2 + \cdots + C_k^2 = -1$. But it is impossible because any embedded 2-sphere in a symplectic 4-manifold with $b^+ \geqslant 2$ has a negative square (Theorem 2.5 in [6]).

Definition. Let X be a smooth 4-manifold containing a smoothly embedded torus T of square 0. Then, removing a tubular neighborhood $\nu(T) \cong T \times D^2$ of T in X and regluing it along the boundary via a diffeomorphism $\varphi : T \times \partial D^2 \to T \times \partial D^2$ such that the absolute value of the degree of the map $\mathrm{proj}_{\partial D^2} \circ \varphi : \{pt\} \times \partial D^2 \to \partial D^2$ is p, we define a new smooth 4-manifold $X(p)$, called a *p-logarithmic transform* on X, which depends on the choice of φ as well as T. Note that, if T is a regular torus lying in a cusp neighborhood, $X(p)$ is uniquely determined up to diffeomorphism. In the case T is a regular torus lying in a fishtail neighborhood which is not contained in a cusp neighborhood, $X(p)$ is also uniquely determined by a choice of an auxiliary data regarding the vanishing cycle of T (Theorem 8.5.9 in [3] for details). Furthermore, M. Symington proved that $X(p)$ admits a symplectic structure in some cases [11].

Theorem 2.2 (Corollary 10.2.7 in [3]). *Suppose X is a symplectic* 4*-manifold containing a symplectic fishtail neighborhood. Then the manifold $X(p)$ obtained by p-logarithmic transform along a symplectic torus T in a fishtail neighborhood admits a symplectic structure.*

Definition. Suppose K is a fibered knot in S^3 with a punctured surface Σ_g° of genus g as fiber. Let M_K be a 3-manifold obtained by performing 0-framed surgery on K, and let m be a meridional circle to K. Then the 3-manifold M_K can be considered as a fiber bundle over circle with a closed Riemann surface Σ_g as a fiber, and there is a smoothly embedded torus $T_m := m \times S^1$ of square 0 in $M_K \times S^1$. Thus $M_K \times S^1$ fibers over $S^1 \times S^1$ with Σ_g as fiber and with $T_m = m \times S^1$ as section. It is a theorem of Thurston that such a 4-manifold $M_K \times S^1$ has a symplectic structure with symplectic section T_m. Thus, if X is a symplectic 4-manifold with a symplectically embedded torus T of square 0, then the fiber sum 4-manifold $X_K := X \sharp_{T=T_m} (M_K \times S^1)$, obtained by taking a fiber sum along $T = T_m$, is symplectic—We call this a *knot surgery* introduced by R. Fintushel and R. Stern. They also proved that X_K is homotopy equivalent to X under a mild condition on X, and computed the Seiberg–Witten invariant of X_K.

Theorem 2.3 [1]. *Suppose X is a simply connected symplectic 4-manifold which contains a smoothly embedded torus T of square 0 in a cusp neighborhood with $\pi_1(X \setminus T) = 1$ and representing a non-trivial homology class $[T]$. If K is a knot in S^3, then X_K is a smooth 4-manifold which is homeomorphic to X and whose Seiberg–Witten invariant is*

$$SW_{X_K} = SW_X \cdot \Delta_K(t)$$

where $\Delta_K(t)$ is the Alexander polynomial of K and $t = \exp(2[T])$. Furthermore, if K is a fibered knot and T is symplectically embedded, then X_K is a symplectic 4-manifold.

Next, we introduce some basic symplectic 4-manifolds which will be served as building blocks of our construction.

Building Block 1. Let $Q := Z_1 \sharp_\psi Z_2$ be a symplectic 4-manifold constructed as follows: First, consider a Thurston's manifold $Z := \mathbb{R}^4/G$, where G is a discrete subgroup of symplectomorphisms generated by unit translations parallel to the x^1-, x^2-, and x^3-axes, together with the map $(x^1, \dots, x^4) \mapsto (x^1 + x^2, x^2, x^3, x^4 + 1)$. Note that projection onto the last two coordinates induces a bundle structure $\pi : Z \to \mathbb{T}^2$ with symplectic torus fibers. Next, using two copies, $\pi_i : Z_i \to \mathbb{T}^2$ $(i = 1, 2)$, of Thurston's manifold and using an orientation-reversing bundle map ψ induced from 90° rotation $\psi_0 : \pi_1^{-1}(0) \to \pi_2^{-1}(0)$ defined by $\psi_0(x^1, x^2) = (-x^2, x^1)$, we obtain a symplectic fiber sum $Q := Z_1 \sharp_\psi Z_2$ which is a torus bundle over a genus 2 surface $\mathbb{T}^2 \sharp \mathbb{T}^2$ and has a symplectic section Σ of square 0, glued a torus section in Z_1 and Z_2. Similarly, there is also a Lagrangian torus $T \subset Q$ of square 0, disjoint from Σ, in Q. For example, one obtains such a torus T by setting $x^1 = x^4 = 1/2$ in Z_1. Note that both the Euler characteristic $e(Q)$ and the signature $\sigma(Q)$ of Q are zero because $e(Z_i) = \sigma(Z_i) = 0$ [2].

Building Block 2. Let $E(n)$ be a simply connected elliptic surface with no multiple fibers and holomorphic Euler characteristic n. Then $E(n)$ can be obtained as an algebraic surface $B(2, 3, 6n - 1) \cup_{\Sigma(2,3,6n-1)} C(n)$, where $B(2, 3, 6n - 1)$ is a Brieskorn manifold and $C(n)$, usually called *a Gompf nucleus*, is the neighborhood of a cusp fiber and a section which is an embedded 2-sphere of square $-n$. Note that, for $n \geqslant 2$, a Brieskorn manifold $B(2, 3, 6n - 1)$ also contains a Lagrangian torus of square 0, which intersects 2-sphere transversely at a single point, in another cusp neighborhood [9].

Building Block 3. Let $H(n^2)$ be a complex surface lying on the Bogomolov–Miyaoka–Yau line which is constructed as follows: Given a Riemann surface Σ_2 of genus 2, take a $\mathbb{Z}_5$-action on Σ_2 generated by $\gamma : \Sigma_2 \to \Sigma_2$ which has exactly 3 fixed points, say p_1, p_2, p_3. Then the quotient $\Sigma_2/\mathbb{Z}_5$ is $\mathbf{C}P^1$. Let us denote the quotient map by $\varphi : \Sigma_2 \to \mathbf{C}P^1$ and denote the inverse image of the diagonal via $\varphi \times \varphi : \Sigma_2 \times \Sigma_2 \to \mathbf{C}P^1 \times \mathbf{C}P^1$ by $F \subset \Sigma_2 \times \Sigma_2$. If we blow up $\Sigma_2 \times \Sigma_2$ in the 3 points (p_i, p_i) $(i = 1, 2, 3)$, we can take the 5-cyclic branched cover of $(\Sigma_2 \times \Sigma_2) \sharp 3\overline{\mathbf{C}P}^2$ along $\widetilde{F}_1 \cup \cdots \cup \widetilde{F}_5$, where $\widetilde{F}_1, \dots, \widetilde{F}_5$ are disjoint curves obtained by the proper transform of F. We denote the resulting complex surface by $H(1)$. Note that it admits a Lefschetz fibration $H(1) \to \Sigma_2 \times \Sigma_2 \xrightarrow{\mathrm{pr}_1} \Sigma_2$ with fibers of genus 16 and it contains a genus 2 surface Σ_2, as a section, of square -1. And then, taking an n-fold unbranched cover $\phi_n : \Sigma_{n+1} \to \Sigma_2$ and pulling $H(1) \to \Sigma_2 \times \Sigma_2$ back via $\phi_n \times \phi_n$, we obtain a complex surface, denoted by $H(n^2)$, which also admits a Lefschetz fibration $H(n) \to \Sigma_{n+1} \times \Sigma_{n+1} \xrightarrow{\mathrm{pr}_1} \Sigma_{n+1}$ with fibers of genus $15n+1$ and it contains a genus $(n+1)$ surface Σ_{n+1}, as a section, of square $-n$. Furthermore, it has the Euler characteristic $e(H(n^2)) = 75n^2$ and the signature $\sigma(H(n^2)) = 25n^2$ [10].

Before going on, let us summarize results obtained in our previous papers.

Theorem 2.4 [7]. *There is a line $\mathbf{c} = f(\chi)$ with a slope > 8.76 in the $(\chi, \mathbf{c})$-plane such that any allowed lattice point satisfying $\mathbf{c} \leqslant f(\chi)$ in the first quadrant can be realized as (χ, c_1^2) of a simply connected spin non-complex symplectic 4-manifold which admits infinitely many distinct exotic smooth structures. In particular, all allowed lattice points $(\chi, \mathbf{c})$ except finitely many lying in the region $0 \leqslant \mathbf{c} \leqslant 8.76\chi$ satisfy $\mathbf{c} \leqslant f(\chi)$.*

Lemma 2.1 [8]. *For each integer k, $10 \leqslant k \leqslant 18$, there exists a simply connected, nonspin, irreducible symplectic 4-manifold $X_{3,k}$ with $b_2^+ = 3$ and $b_2^- = k$ which contains a symplectic genus 2 surface Σ_2 of square 0 and a symplectic torus T of square 0, disjoint from Σ_2, in a fishtail neighborhood and $\pi_1(X_{3,k} - \Sigma_2) = \pi_1(X_{3,k} - T) = 1$.*

Theorem 2.5 [8]. *Every simply connected, closed, nonspin, smooth 4-manifold with $b_2^+ > 1$ odd satisfying $b_2^- \geqslant b_2^+ + 7$ or $b_2^- \geqslant b_2^+ \geqslant 123$ admits infinitely many distinct exotic smooth structures. Furthermore, if it also satisfies $\mathbf{c} \geqslant 0$, then they are all irreducible.*

Remarks.

(1) It is easily proved that every smooth 4-manifold satisfying the hypothesis in Theorem 2.4 or Theorem 2.5 above admits actually an infinite family of, both symplectic and non-symplectic, exotic smooth structures by applying Theorem 2.3 with an infinite family of both fibered and non-fibered knots in S^3.
(2) In fact, the symplectic 4-manifolds $X_{3,k}$ described in Lemma 2.1 above also satisfy $\pi_1(X_{3,k} - (\Sigma_2 \cup T)) = 1$ because a circle $\{pt\} \times \partial D^2 \subset \Sigma_2 \times D^2$, a tubular neighborhood of Σ_2, bounds a disk in $X_{3,k} - T \subset X_{3,k}$.
(3) B. Park proved independently that the symplectic 4-manifolds $X_{3,k}$ $(10 \leqslant k \leqslant 13)$ in Lemma 2.1 above admit infinitely many, both symplectic and non-symplectic, distinct exotic smooth structures [5].

Now we investigate exotic smooth structures for nonspin smooth 4-manifolds. First we need the following two lemmas.

Lemma 2.2. *For each integer $n \geqslant 1$, there exists an irreducible Lefschetz fibration $Y_n \to \Sigma_{n+3}$ with a fiber of genus $15n+1$ which contains a genus $(n+3)$ surface*

Σ_{n+3} of square $-n$ such that the image of $\pi_1(\partial\nu(\Sigma_{n+3}))$ in $\pi_1(Y_n - \nu(\Sigma_{n+3}))$ normally generates, where $\nu(\Sigma_{n+3})$ is a tubular neighborhood of Σ_{n+3} in Y_n. Furthermore it satisfies $(e(Y_n), \sigma(Y_n)) = (75n^2 + 120n, 25n^2)$.

Proof. First, taking a fiber sum $15n$ times along a genus 2 surface Σ_2 of square 0 in a building block $Q = Z \sharp_T Z$, we get a fiber bundle $(15n+1)Q := Q \sharp_{\Sigma_2} \cdots \sharp_{\Sigma_2} Q \to \Sigma_2$ whose regular fiber is a genus $(15n+1)$ surface obtained by gluing $(15n+1)$ copies of a torus fiber in Q. Next, taking a fiber sum again along a genus $(15n+1)$ surface Σ_{15n+1} in $H(n^2)$ and in $(15n+1)Q$, we construct a Lefschetz fibration

$$Y_n := H\bigl(n^2\bigr) \sharp_{\Sigma_{15n+1}} (15n+1)Q \to \Sigma_{n+3}$$

over a genus $(n+3)$ surface Σ_{n+3} of square $-n$, which is obtained by a connected sum $\Sigma_{n+1} \sharp \Sigma_2$ of two sections Σ_{n+1} and Σ_2.

Next, since two generators of the fundamental group of a torus fiber of $Q \to \Sigma_2$ are trivial in $\pi_1(Q - \nu(\Sigma_2))/\langle \pi_1(\partial\nu(\Sigma_2))\rangle \cong 1$, all $(30n+2)$ generators of $\pi_1(\Sigma_{15n+1})$ are also trivial in $\Lambda := \pi_1((15n+1)Q - \nu(\Sigma_2))/\langle \pi_1(\partial\nu(\Sigma_2))\rangle$. Hence Λ is trivial because $\pi_1((15n+1)Q - \nu(\Sigma_2))$ is generated by $\pi_1(\Sigma^\circ_{15n+1})$ and $\pi_1(\Sigma_2)$, where $\Sigma_{15n+1}(\Sigma^\circ_{15n+1})$ is a (punctured) genus $(15n+1)$ fiber of $(15n+1)Q \to \Sigma_2$. Furthermore, since each fiber of $H(n^2)$ can be pushed into a fiber of $(15n+1)Q$, the map $\pi_1(\partial\nu(\Sigma_{n+3})) \to \pi_1(Y_n - \nu(\Sigma_{n+3}))$ is surjective. Note that $e(Y_n) = e(H(n^2)) + e((15n+1)Q) + 60n = 75n^2 + 120n$ and $\sigma(Y_n) = \sigma(H(n^2)) + (15n+1)\sigma(Q) = 25n^2$. $\square$

Lemma 2.3. *For each odd integer $n \geqslant 1$, there exists a simply connected irreducible symplectic 4-manifold $E(1)_K$ which contains a genus $(n+3)$ surface Σ_{n+3} of square n and $\pi_1(E(1)_K - \Sigma_{n+3}) = 1$.*

Proof. Let K be a torus fibered knot $T(2, n+6)$ in S^3. Then, performing 0-framed surgery on K and taking a product with S^1, we get a fibration $M_K \times S^1 \to T$ with a genus $(n+5)/2$ surface $\Sigma'_{(n+5)/2}$ as a fiber and with a torus T as a section. Then we get a desired symplectic manifold $E(1)_K$ by taking a fiber sum along a torus T in $E(1)$ and in $M_K \times S^1$. Note that $E(1)_K$ contains a symplectic genus $(n+5)/2$ surface $\Sigma_{(n+5)/2}$ of square -1, obtained by gluing $\Sigma'_{(n+5)/2}$ and a 2-sphere S which is a section of $E(1)$, and also contains a symplectic torus T of square 0 which intersects $\Sigma_{(n+5)/2}$ transversely at a point. Hence, by symplectically resolving $(n+1)/2$ intersection points repeatedly, we get a symplectic genus $(n+3)$ surface Σ_{n+3} whose square is $[\Sigma_{(n+5)/2} + \frac{(n+1)}{2}T]^2 = n$. Note that $\pi_1(E(1)_K - \Sigma_{n+3}) = 1$ follows from the Van-Kampen theorem using the fact that the meridian of a tubular neighborhood $\nu(\Sigma_{n+3})$ of Σ_{n+3} collapses in $E(1) - \nu(T \cup S) \subset E(1)_K - \nu(\Sigma_{n+3})$. $\square$

Proposition 2.1. *For each odd integer $n \geqslant 1$ and $10 \leqslant k \leqslant 18$, there exists a simply connected, nonspin, irreducible symplectic 4-manifold $Z_{n,k}$ containing a symplectic genus 2 surface Σ_2 of square 0 and a torus T of square 0, disjoint from Σ_2, in a fishtail neighborhood which satisfies $\pi_1(Z_{n,k} - \Sigma_2) = \pi_1(Z_{n,k} - T) = 1$. Furthermore, it has $\chi(Z_{n,k}) = 25n^2 + 31n + 5$ and $\mathbf{c}(Z_{n,k}) = 225n^2 + 248n + 35 - k$.*

Proof. First, in order to construct desired manifolds, take a symplectic fiber sum along a genus $(n+3)$ surface Σ_{n+3} of the opposite square $\pm n$ in Y_n and in $E(1)_K$ (Lemmas 2.2 and 2.3). After then, taking a symplectic fiber sum again along a torus T of square 0 in $Q \subset Y_n$ and in $X_{3,k}$, we get a desired irreducible symplectic 4-manifold

$$Z_{n,k} := E(1)_K \sharp_{\Sigma_{n+3}} Y_n \sharp_T X_{3,k}.$$

Note that the simple connectivity of $Z_{n,k}$ follows from Lemmas 2.2 and 2.3:

$$\begin{aligned}
&\pi_1\big(E(1)_K \sharp_{\Sigma_{n+3}} Y_n\big)\\
&\quad = \pi_1\big(\big(E(1)_K\big) - \nu(\Sigma_{n+3})\big) * \pi_1\big(Y_n - \nu(\Sigma_{n+3})\big) \big/ \big\langle \pi_1\big(\partial\nu(\Sigma_{n+3})\big)\big\rangle\\
&\quad \cong \pi_1\big(Y_n - \nu(\Sigma_{n+3})\big) \big/ \big\langle \pi_1\big(\partial\nu(\Sigma_{n+3})\big)\big\rangle\\
&\quad \cong 1.
\end{aligned}$$

An easy computation shows that $e(Z_{n,k}) = e(E(1)_K) + e(Y_n) + 4(n+2) + e(X_{3,k}) = 75n^2 + 124n + k + 25$ and $\sigma(Z_{n,k}) = \sigma(E(1)_K) + \sigma(Y_n) + \sigma(X_{3,k}) = 25n^2 - k - 5$. Hence it has $\chi(Z_{n,k}) = 25n^2 + 31n + 5$ and $\mathbf{c}(Z_{n,k}) = 225n^2 + 248n + 35 - k$. Furthermore since Σ_{n+3} is not characteristic in $H_2(E(1)_K : \mathbb{Z})$, $E(1)_K - \Sigma_{n+3}$ cannot be spin, so that $Z_{n,k}$ is nonspin. The other properties of $Z_{n,k}$ in Proposition 2.1 above follow from the fact that $X_{3,k}$ has the same properties described in Lemma 2.1. □

Theorem 2.6. *There is an increasing sequence $\{m_i\}$ with $m_i \to 9$ such that every simply connected, closed, nonspin, irreducible, smooth 4-manifold satisfying $0 \leqslant \mathbf{c} \leqslant m_i\chi$ and $b_2^+ \geqslant C_i$ odd (C_i is a constant depending on m_i) admits infinitely many, both symplectic and non-symplectic, exotic smooth structures.*

Proof. For each odd integer $i \geqslant 1$, define two numbers m_i and C_i by

$$\begin{cases} m_i := \dfrac{\mathbf{c}(Z_{i,10} \sharp_{\Sigma_2} (112.5i + 173.5)Q)}{\chi(Z_{i,10} \sharp_{\Sigma_2} (112.5i + 173.5)Q)} = \dfrac{225i^2 + 1148i + 1413}{25i^2 + 143.5i + 178.5},\\ C_i := 50i^2 + 287i + 356. \end{cases}$$

Then $\{m_i\}$ is an increasing sequence converging to 9 and any lattice point $(\chi, \mathbf{c})$ satisfying $8\chi < \mathbf{c} \leqslant m_i\chi$ and $\chi \geqslant (C_i + 1)/2$ (equivalently, $b_2^+ \geqslant C_i$) is realized as (χ, c_1^2) of a simply connected, closed, nonspin, irreducible, symplectic 4-manifold

$$Z_{n,k} \sharp_{\Sigma_2} lQ \sharp_T E(l')$$

for some integers n, k, l and l'. The fact that $Z_{n,k} \sharp_{\Sigma_2} lQ \sharp_T E(l')$ admits infinitely many, both symplectic and non-symplectic, exotic smooth structures follows from Theorem 2.3 by using a regular torus lying in a cusp neighborhood in $E(l')$ (or $X_{3,k} \subset Z_{n,k}$) and using an infinite family of fibered and non-fibered knots in S^3. Hence we are done by combining this result with Theorem 2.5. □

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