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Stein 4-manifolds with boundary and contact structures

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Abstract

We discuss several applications of Seiberg–Witten theory in conjunction with an embedding theorem (proved elsewhere) for complex 2-dimensional Stein manifolds with boundary. We show that a closed, real 2-dimensional surface smoothly embedded in the interior of such a manifold satisfies an adjunction inequality, regardless of the sign of its self-intersection. This inequality gives constraints on the minimum genus of a smooth surface representing a given 2-homology class. We also discuss consequences for the contact structures existing on the boundaries of these Stein manifolds. We prove a slice version of the Bennequin–Eliashberg inequality for holomorphically fillable contact structures, and we show that there exist families of homology 3-spheres with arbitrarily large numbers of homotopic, nonisomorphic tight contact structures. Another result we mention is that the canonical class of a complex 2-dimensional Stein manifold with boundary is invariant under self-diffeomorphisms fixing the boundary. © 1998 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Contact structures are the odd-dimensional analogue of symplectic structures. A contact structure on a smooth 3-dimensional manifold M is a distribution ξ of tangent 2-planes locally defined as the kernel of a 1-form α such that $\alpha \wedge \mathrm{d}\alpha$ is nowhere vanishing. For such a *contact form* α, $\alpha \wedge \mathrm{d}\alpha$ defines an orientation on M and this orientation is independent of the choice of α. If M is already oriented and $\alpha \wedge \mathrm{d}\alpha$ is a positive multiple of the volume form, then ξ is called *positive*, otherwise it is called *negative*.

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ξ is orientable as a vector bundle if and only if α can be chosen to be a global 1-form. In this article we will only consider orientable contact structures. To find out more about contact structures, the interested reader should consult [1,9,12].

Any smooth hypersurface $M \subset X$ in a smooth 4-manifold with an almost complex structure $J: TX \to TX$ has a canonically induced orientable distribution of tangent 2-planes $\xi = TM \cap J(TM)$. When M is the level set of a strictly J-convex function (i.e., a function which is strictly subharmonic when restricted to J-holomorphic discs) this distribution is a contact structure.

A *Stein 4-manifold with boundary* is a triple (W, J, ϕ), where

(i) W is a smooth 4-manifold with boundary,

(ii) J is a complex structure on W such that $(\mathrm{Int}(W), J)$ is Stein, i.e., a complex manifold biholomorphic to a complex submanifold of $\mathbb{C}^n$, and

(iii) $\phi: W \to \mathbb{R}$ is a Morse function with $\phi|_{\partial W}$ constant, and such that ϕ is strictly J-convex.

If we denote by J the multiplication by $\sqrt{-1}$ in TW and by J^* its dual, the 2-form $\omega_\phi = \mathrm{d}J^*\,\mathrm{d}\phi$ defined by the J-convex function ϕ is nondegenerate and closed, hence defines a symplectic structure on W. The metric g_ϕ given by $g_\phi(v, v') = \omega_\phi(v, Jv')$ defines a Kähler structure on W.

The simplest example of a Stein 4-manifold with boundary is the unit ball $B^4 \subset \mathbb{C}^2$ with the restriction of the standard complex structure on $\mathbb{C}^2$ and the standard J-convex function $\phi = \log(1 + \sum_i |z_i|^2)$. Any sublevel set $\phi \leqslant c$ of a strictly J-convex function ϕ on a Stein manifold is a Stein manifold with boundary. In Section 3 we will describe a procedure due to Eliashberg for producing Stein structures on 4-manifolds with boundary having a handlebody decomposition with only 1- and 2-handles.

In this paper we will illustrate several applications of the following embedding theorem when combined with results from Seiberg–Witten theory:

Theorem 1.1 [21, Corollary 3.3]. *Let (W, J, ϕ) be a Stein 4-manifold with boundary. Then, there exists a minimal complex surface X of general type with $b_2^+ > 1$ and:*

(1) *a holomorphic embedding of W as a domain inside X,*

(2) *a Kähler form ω_X on X such that $\omega_X|_W = \omega_\phi$.*

We will divide the applications of this theorem into two groups. In Sections 2 and 3 we will talk about applications to the topology of Stein 4-manifolds with boundary and their boundary contact structures coming from the first part of the statement. In Section 4 we will briefly describe results following from the second part of the theorem.

2. First applications

Let us recall a few results from Seiberg–Witten theory.

For a smooth closed 4-manifold X with $b_2^+(X) > 1$ the Seiberg–Witten monopole equations [24,25,28] give rise to invariants of differentiable structures. The *Seiberg–Witten invariant* of X is a map SW from the set of Spin^c-structures on X to the integers (see [28,16,27,11,22]) which takes nonzero values at only finitely many Spin^c-structures. A determinant line bundle is associated to each Spin^c-structure. The first Chern classes of the determinant line bundles associated to the Spin^c-structures for which SW is nonzero are called *Seiberg–Witten basic classes* (by analogy to the cohomology classes introduced by Kronheimer and Mrowka in [15]). The set $\mathrm{SWB}(X)$ of basic classes of X is a finite subset of $H^2(X)$ and it is a differentiable invariant of X. It is a simple fact that when B is basic than so is $-B$. Witten [28] showed that this set contains the canonical class in the case of a Kähler surface. Taubes proved [26] the same to be the case for symplectic 4-manifolds. For minimal Kähler surfaces of general type a more precise result holds. In this case there are only two Spin^c-structures with nontrivial Seiberg–Witten invariants, i.e., those canonically associated to the complex structure and its conjugate, hence the set of basic classes consists of plus or minus the canonical class.

Seiberg–Witten basic classes satisfy the following *adjunction inequality* [16,23]: for $K \in \mathrm{SWB}(X)$ and a smoothly embedded surface of positive genus $\Sigma \hookrightarrow X$ with $\Sigma \cdot \Sigma \geqslant 0$,

$$|K \cdot \Sigma| + \Sigma \cdot \Sigma \leqslant 2g(\Sigma) - 2. \tag{1}$$

The first half of Theorem 1.1, i.e., the existence of a holomorphic embedding of a Stein 4-manifold with boundary into a minimal Kähler surface of general type, has the following two immediate applications when coupled with results from Seiberg–Witten theory.

Proposition 2.1. *Let W be a Stein 4-manifold with boundary. Given a smoothly embedded oriented surface of positive genus $\Sigma \subset W$ with self-intersection $\Sigma \cdot \Sigma \geqslant 0$, then*

$$\left|\langle c_1(W), \Sigma\rangle\right| + \Sigma \cdot \Sigma \leqslant 2g(\Sigma) - 2.$$

Proof. Obvious from the first part of Theorem 1.1, using (1). □

Proposition 2.2. *Let W be a Stein 4-manifold with boundary. Then, W does not contain nontrivial spheres with self-intersection greater than -2 smoothly embedded in its interior, and if $S \hookrightarrow W$ is a smoothly embedded sphere with self-intersection -2, then $c_1(W) \cdot S = 0$.*

Proof. We will sketch below that, given a homologically nontrivial 2-sphere S embedded in the interior of W, W embeds inside a minimal Kähler surface of general type X in such a way that the homology class supported by the sphere S is of infinite order in X. The conclusion will then be an immediate consequence of a well-known argument for X.

Any homologically nontrivial 2-sphere S smoothly embedded in the interior of W has to intersect algebraically nontrivially at least one of the co-cores of the 2-handles. We can enlarge W by attaching a 2-handle to a Legendrianization of the boundary of such a co-core and then apply the embedding theorem to this enlargement to construct X. Hence, W can be embedded in an X containing a smooth 2-sphere S' satisfying $S \cdot S' \neq 0$, which shows that S is of infinite order in X.

It is a well-known fact following from either Donaldson or Seiberg–Witten theory (see, for example, [18]) that an algebraic surface with $b_2^+ > 1$ has no embedded spheres with nonnegative self-intersection (representing a class of infinite order in $H_2(X, \mathbb{Z})$).

We will concentrate on the cases of (-1)- and (-2)-spheres. Since the set of basic classes is a diffeomorphism invariant and for a surface of general type the only Seiberg–Witten basic classes are $\pm K_X$, any self-diffeomorphism of X must preserve K_X up to sign.

If S is either a (-1)- or a (-2)-sphere in X, the reflection r_S in $H^2(X;\mathbb{Z})$ with respect to the Poincaré dual of S can be realized by a diffeomorphism (see, e.g., [10, Proposition 2.4]). Therefore

$$r_S(K_X) = K_X + 2\left(\frac{K_X \cdot S}{S \cdot S}\right)S$$

has to be equal to either K_X or $-K_X$. In the first case, $K_X \cdot S = 0$, which is possible only if $S \cdot S = -2$, because K_X is characteristic. In the second case, K_X would be a rational multiple of S, which is impossible because for X minimal of general type $K_X \cdot K_X > 0$, and S has negative square. □

Recall that a knot $K \subset M$ inside a contact 3-manifold M is called *Legendrian* if K is everywhere tangent to the contact 2-plane distribution. Given a Legendrian knot K inside the boundary of a Stein 4-manifold with boundary, and given any nonvanishing tangent vector field t_K on K, Jt_K determines a framing of K which we will call the *canonical framing*.

Theorem 2.3 (Eliashberg [6]; [8, Theorem 6.1]). *Let W be a Stein 4-manifold with boundary. Suppose $K \subset \partial W$ is a Legendrian knot. Then, the smooth 4-manifold W' obtained by attaching a 2-handle to W along K with framing -1 with respect to the canonical contact framing of K has a Stein structure which extends the Stein structure on W.*

Applying this theorem when $W = B^4 \subset \mathbb{C}^2$ one easily sees, for example, that the 4-manifolds with boundary described by the framed links of Fig. 1 have Stein structures. Notice that this fact is not at all clear a priori: in order to check that Theorem 2.3 applies one has to show that the links of Fig. 1 can be isotoped to links which are Legendrian

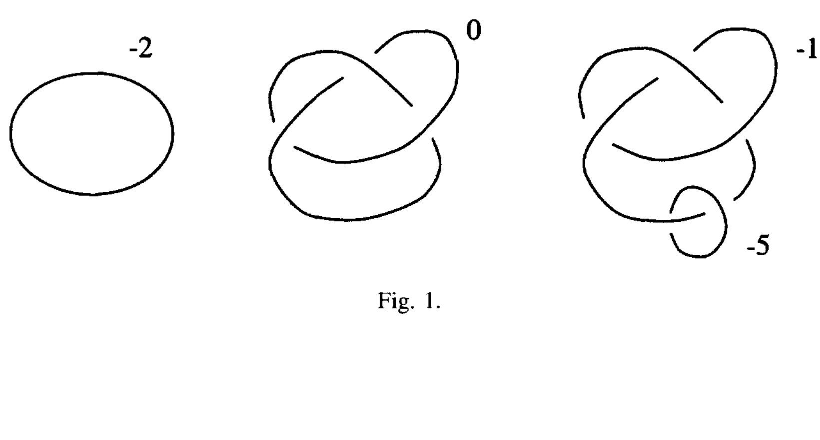

Fig. 1.

with respect to the standard contact structure in S^3 and whose canonical framings are one unit bigger than their prescribed framings (see, e.g., [13,20,21]).

The restriction of the standard contact structure on $S^3 = \partial B^4 \subset \mathbb{C}^2$ to the complement of a point is isomorphic to the contact structure on $\mathbb{R}^3$ defined by the 1-form $\alpha = x\,\mathrm{d}y + \mathrm{d}z$. A Legendrian link in $\mathbb{R}^3 \cong S^3 \setminus \{\text{point}\}$ can be represented by its *front*, namely its projection to the yz-plane in $\mathbb{R}^3$. For a generic Legendrian knot this projection has no vertical tangent lines and it has finitely many singularities which are either ordinary double points or horizontal cusps. At the double point the over-arc is always the one with the smaller slope. It is always possible to reconstruct from a front the unique Legendrian link projecting to it.

Given a front projection of an oriented Legendrian knot in S^3, the canonical framing determined by the standard contact structure is easily calculated. Since all knots K in S^3 are nullhomologous, there is a one-to-one correspondence between their framings and the integers obtained by associating to each framing of K the linking number of K with the push-off determined by the framing. Under this identification the canonical framing corresponds to the *Thurston–Bennequin invariant* $\mathrm{tb}(K)$, an invariant of the Legendrian isotopy class of K. $\mathrm{tb}(K)$ can be calculated from a generic projection $\mathcal{P}$ in the following way. Let w denote the writhe of K, namely the algebraic number of self-crossings of $\mathcal{P}$. Let c denote the number of cusps. Then, $\mathrm{tb}(K) = w - c/2$. There is another invariant of Legendrian isotopy that one can easily compute from a front projection, called the *rotation number*. If we denote by a the number of ascending cusps and d the number of descending cusps it is expressed as $r(K) = (d - a)/2$. Fig. 2 illustrates the front of a Legendrian right-handed trefoil, and tb and r are calculated, showing that the manifold obtained by adding a 2-handle to a 4-ball along a trefoil with zero framing carries a structure of a Stein 4-manifold with boundary. It is an easy exercise to check that the same is true for the other examples in Fig. 1.

While Theorem 2.3 guarantees, under certain circumstances, the existence of Stein structures on 4-manifolds with boundary, Propositions 2.1 and 2.2 may sometimes be used to exclude the existence of such structures.

For example, the 4-manifolds of Fig. 3 do not admit Stein structures with boundary, the manifold on the left because it contains a (-1)-sphere, the manifold on the right because it contains a $(+1)$-torus.

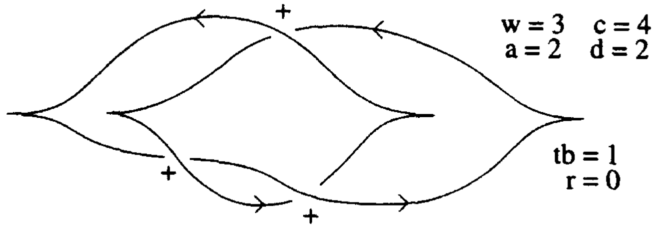

Fig. 2.

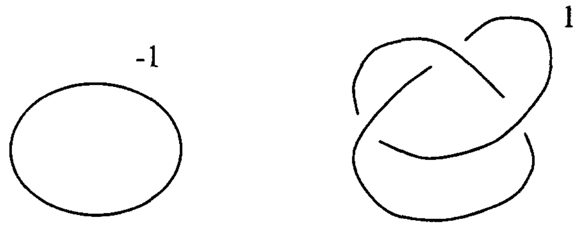

Fig. 3.

Adjunction-like inequalities as (1) give lower bounds for the minimum genus of a smooth representative of a given 2-homology class in a 4-manifold. However, (1) only applies to classes of nonnegative self-intersection.

The following theorem establishes an adjunction inequality for surfaces with possibly negative self-intersection (including the case of genus zero) embedded in the interior of Stein 4-manifolds with boundary.

Theorem 2.4. *Let W be a Stein 4-manifold with boundary, and $\Sigma \hookrightarrow \mathrm{Int}(W)$ a smoothly embedded surface. Then,*

$$\left|\left\langle c_1(W), \Sigma\right\rangle\right| + \Sigma \cdot \Sigma \leqslant 2g(\Sigma).$$

Proof. It is not difficult to see that, inside any 3-ball contained in the standard contact 3-sphere $S^3 \subset \mathbb{C}^2$ one can find a Legendrian $(p, 2)$-torus knot $K(p\,\text{odd})$ with $\mathrm{tb}(K) = p - 2$, $r(K) = 0$ (see, e.g., [13,20,21]). Moreover, by the Darboux theorem for contact structures any point $p \in \partial W$ is contained inside a contact 3-ball B contactomorphic to a 3-ball in S^3. Thus, inside B there is a copy of K. Attaching a 2-handle along K with framing $p - 3$ gives a Stein manifold W'. Moreover, it is not difficult to see, by looking at the construction used in the proof of Theorem 2.3, that $\langle c_1(W'), F_K\rangle = r(K) = 0$, where F_K is the surface formed as the union of a Seifert surface for K and the core of the 2-handle. It is easy to check that $g(F_K) = (p - 1)/2$ and $F_K \cdot F_K = p - 3$, thus Proposition 2.1 applied (for p large enough) to the piping Σ' of Σ and F_K gives

$$\left|\left\langle c_1(W'), \Sigma'\right\rangle\right| + \Sigma \cdot \Sigma + p - 3 \leqslant 2\big(g(\Sigma) + (p - 1)/2\big) - 2,$$

thus proving the statement, since $\langle c_1(W'), \Sigma'\rangle = \langle c_1(W'), (\Sigma + F_K)\rangle = \langle c_1(W), \Sigma\rangle$. $\square$

As an application of this theorem consider the following example: one can check that Theorem 2.3 applies to the manifold W described in Fig. 4, showing that W has a Stein structure with boundary J. The boundary of W is a homology-sphere, and the Poincaré dual of $c_1(W, J) \in H^2(W; \mathbb{Z}) \cong H^2(W, \partial W; \mathbb{Z})$ is $4T$, where T is the 2-homology class generated by the 2-handle attached along the trefoil. Let S be the homology class corresponding to the unknot, and consider $x = (3a - 1)T + aS$, with a a nonnegative integer. Then, $x^2 = -2a$, and applying Theorem 2.4 one sees that the smallest genus of a smoothly embedded, oriented surface representing x is at least a.

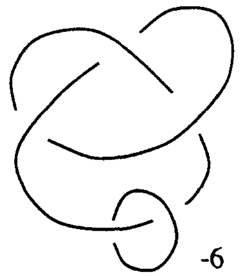

Fig. 4.

3. A slice Bennequin–Eliashberg inequality

Let (M, ξ) be a contact 3-manifold. On a generically embedded surface $S \subset M$, ξ induces a foliation S_ξ with isolated singularities at the points where S is tangent to ξ. S_ξ is called the *characteristic foliation*. A contact structure is called *overtwisted* if there exists an embedded disc $D \subset M$ such that its characteristic foliation contains a closed orbit with exactly one critical point inside. Otherwise, the contact structure is called *tight*. Bennequin proved that the contact structure induced on S^3 considered as $\partial B^4 \subset \mathbb{C}^2$ is tight [4]. More generally, it was shown by Gromov and Eliashberg [7,14] that the contact structure induced on the boundary of any Stein manifold (actually on any level set of a strictly J-convex function) is tight.

Let W be a Stein 4-manifold with boundary, and $F \hookrightarrow W$ a smoothly embedded oriented surface transversal to ∂W with $K = \partial F \subset \partial W$ connected and Legendrian. Let t_K be the vector field tangent to the oriented knot K. Let ξ be the tight contact structure induced on ∂W and $\nu_K \in \Gamma(T\partial W|K)$ the positively oriented normal to the contact structure restricted to K. We will now generalize to Legendrian knots in W the definitions of the Thurston–Bennequin invariant $\mathrm{tb}(K)$ and the rotation number $r(K)$ which we had introduced earlier for Legendrian knots in S^3.

Definition 3.1. Define $\mathrm{tb}_F(K) \in \mathbb{Z}$ as the obstruction to extending ν_K to a nonvanishing section of the normal bundle of F in W, and $r_F(K) \in \mathbb{Z}$ as the obstruction to extending $\{t_K, \nu_K\}$ to a complex framing of $TW|F$.

Remark 3.2. The definitions of tb_F and r_F depend only on the relative homology class $[(F, \partial F)] \in H_2(W, \partial W; \mathbb{Z})$

Lemma 3.3. *Let W_K be the Stein 4-manifold with boundary obtained by attaching to W a 2-handle along K with framing one less than the one prescribed by the contact structure on the boundary of W. Let Σ be the closed surface obtained by joining F to the core of the 2-handle. Then,*

$$|\langle c_1(W_K), \Sigma\rangle| + \Sigma^2 \leqslant 2g(\Sigma) - 2.$$

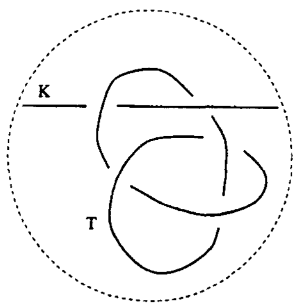

Fig. 5.

Proof. Consider a standard contact 3-ball B^3 centered at a point of K. There is a 0-framed Legendrian right-handed trefoil knot $T \subset B^3$, geometrically linked once with K, having $\mathrm{tb}(T) = 1$ and $r(T) = 0$ (Fig. 5).

Endow K with the contact framing. Consider the Legendrian framed link L which is the union of T and K. Let W_L be the Stein manifold with boundary obtained by gluing 2-handle to W_K along the Legendrian knot T according to Theorem 2.3. Let $T \in H_2(W_L)$ be the class supported by a genus-one Seifert surface for T (slightly pushed into the interior of the 4-ball) union the core of the attaching 2-handle. Then, as in the proof of Theorem 2.4, we have $\langle c_1(W_L), T\rangle = r(T) = 0$. Observe that $nT + \Sigma$ has square $\Sigma^2 + 2n$ and can be represented by a smoothly embedded surface of genus $g(\Sigma) + n$. Hence, for n sufficiently large Proposition 2.1 applies, giving $\langle c_1(W_L), nT + \Sigma\rangle + \Sigma^2 + 2n \leqslant 2g(\Sigma) - 2 + 2n$. Since the restriction of $c_1(W_L)$ to W_K is $c_1(W_K)$, the result follows. □

Theorem 3.4. *Let W be a Stein 4-manifold with boundary and $F \hookrightarrow W$ a smoothly embedded oriented surface transversal to ∂W with $K = \partial F \subset \partial W$ Legendrian. Then,*

$$|r_F(K)| + \mathrm{tb}_F(K) \leqslant 2g(F) - 1. \tag{2}$$

Proof. Let W_L be the Stein manifold with boundary constructed in the proof of the theorem, and let Σ be the union of F and the core of the 2-handle attached along K. It suffices to observe that by construction and the definitions of $\mathrm{tb}_F(K)$ and $r_F(K)$, we have $\mathrm{tb}_F(K) = \Sigma^2 + 1$ and $r_F(K) = \langle c_1(W_L), \Sigma\rangle$, and then to apply the theorem. □

Remark 3.5. One can define invariants $\mathrm{tb}_F(K)$ and $r_F(K)$ (the classical *Thurston* and *rotation number* invariants) in a similar way, for a surface F with Legendrian boundary embedded in a contact 3-manifold M. In this context, the inequality (2) has been proved by Bennequin for $M = S^3$ with its standard contact structure [4] and then by Eliashberg for a general M with a tight contact structure [8]. Theorem 3.4 is a "slice" extension of this result to the contact structures occurring on the boundaries of Stein 4-manifolds. A proof of this statement for the potentially more general case of symplectically semi-fillable contact structures has been recently announced by Kronheimer and Mrowka.

An important result due to Eliashberg [5] is that the classification up to isotopy of overtwisted contact structures on closed 3-manifolds coincides with their homotopy classification as 2-plane fields. On the other hand, the classification of tight contact structures is far from understood. In this respect, the following is a nice application of (2).

Example 3.6 (Akbulut–Matveev [3]). Using [13] one can show that the Mazur's contractible 4-manifold W studied in [2] has a Stein structure with boundary, and that there is a Legendrian knot $K \subset \partial W$ with $\mathrm{tb}(K) = 0$ which is sent to a slice knot by a self-diffeomorphism $f : \partial W \to \partial W$. It follows that if ξ is the tight contact structure on ∂W, $f^*(\xi)$ is not isotopic to ξ (although it is homotopic to it, as well as obviously isomorphic). In fact, if $f^*(\xi)$ was isotopic to ξ, we would have

$$0 = \mathrm{tb}_\xi(K) = \mathrm{tb}_{f^*(\xi)}(\widetilde{K}),$$

where $\widetilde{K}$ is the knot obtained by pulling back K via the isotopy. Hence

$$\mathrm{tb}_{f^*(\xi)}(\widetilde{K}) = \mathrm{tb}_\xi\left(f(\widetilde{K})\right) \leqslant -1,$$

where the equality is clear, while the inequality follows from (2).

4. Automorphisms of Stein 4-manifolds with boundary

As we mentioned in the previous section, the classification of overtwisted contact structures up to isotopy can be reduced to a homotopy problem. It is natural to wonder whether one can say something about the set of isotopy classes of tight contact structures induced on the boundary of a given smooth 4-manifold W by the various Stein structures on W. The following theorem, which deals with this question, is a consequence of the full strength of Theorem 1.1 (i.e., both parts of the statement) together with basic results from Seiberg–Witten theory.

Theorem 4.1 [21, Theorem 1.2].[1] *Let W be a smooth 4-manifold with boundary. Suppose (W, J_1, ϕ_1), (W, J_2, ϕ_2) are two Stein structures with boundary on W, with associated* Spin^c*-structures Θ_1 and Θ_2. If the induced contact structures ξ_1 and ξ_2 on ∂W are isotopic, then Θ_1 and Θ_2 are isomorphic (and in particular have the same c_1).*

Proof. We briefly sketch the proof of this result for the benefit of the reader. By Theorem 1.1, there is a minimal Kähler surface of general type X with $b_2^+(X) > 1$, and a holomorphic embedding of (W, J_1) as a domain inside X, such that the Kähler form ω_X restricts to ω_{ϕ_1} on W. By hypothesis, there is a diffeomorphism $f : \partial W \to \partial W$ isotopic to the identity such that $f_*(\xi_1) = \xi_2$. This fact and ω-convexity enable us to build a new symplectic manifold $X' = W \cup_f (X \setminus W)$ by gluing the two symplectic manifolds (W, ω_{ϕ_2}) and $(X \setminus W, \omega_X)$ via a symplectomorphism between two collars around their boundaries which extends f (in a proper sense, see [21, Lemma 4.1]). Since f is isotopic

[1] Another proof of this theorem has recently appeared in [17].

to the identity, there is a diffeomorphism $\psi : X \to X'$ which is the identity on the complement of a collar around ∂W. Recall that a symplectic structure determines a unique homotopy class of compatible almost complex structures. Moreover, to any almost complex structure J one can associate a canonical Spin^c-structure, whose isomorphism class only depends on the homotopy class of J. Using the minimality of X, and following the argument in [21], one can conclude that ψ pulls back the Spin^c-structure associated to the symplectic structure on X' to the Spin^c-structure associated to the complex structure on X. Hence, restricting to W, and denoting by Θ_1 and Θ_2, respectively the Spin^c-structures associated to J_1 and J_2 on W, this argument shows that $\Theta_2 = \Theta_1$. $\square$

An immediate corollary of Theorem 4.1 is the following.

Corollary 4.2. *Let W be a smooth 4-manifold with boundary. Let (J_1, ϕ_1) and (J_2, ϕ_2) be two Stein structures with boundary on W and let ξ_1 and ξ_2 be the corresponding contact structures induced on ∂W. Let $F : W \to W$ be a self-diffeomorphism whose restriction to ∂W sends ξ_1 onto a contact structure isotopic to ξ_2. Then, $F^*(J_2)$ is homotopic to J_1 as an almost complex structure. In particular, if ξ_1 and ξ_2 are isotopic, then J_1 and J_2 are homotopic as almost complex structures.*

Proof. Let Θ_1 and Θ_2 be, respectively the Spin^c-structures associated to J_1 and J_2. Applying Theorem 4.1 to $F^*(J_2)$ and J_1 we see that $F^*(\Theta_2) = \Theta_1$. Since $H^3(W; \mathbb{Z}) = H^4(W; \mathbb{Z}) = 0$, a simple argument using obstruction theory shows that on W there is a one-to-one correspondence between almost complex structures up to homotopy and Spin^c-structures up to isomorphism [19]. This correspondence is given by sending an almost complex structure J to the isomorphism class of the induced Spin^c-structure Θ_J. Since $F^*(\Theta_2)$ is the Spin^c-structure associated to $F^*(J_2)$, the statement follows. $\square$

Another immediate corollary of Theorem 4.1 is the following analogue, in the Stein world, of the C^∞-invariance of the canonical class for algebraic surfaces.

Corollary 4.3. *Let W be a Stein 4-manifold with boundary. Then, $c_1(W)$ is invariant under self-diffeomorphisms of W fixing the boundary.*

To illustrate the possible applications of Theorem 4.1, let us consider the following example. Let W be the 4-manifold with boundary described by the framed link presentation of Fig. 6 (p and q coprime).

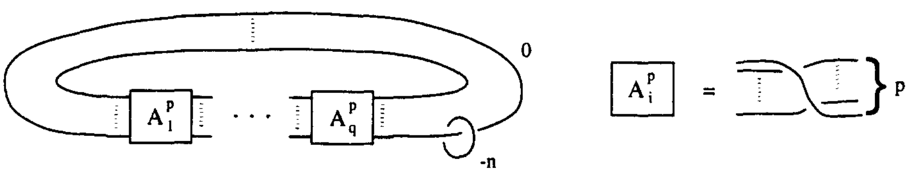

Fig. 6.

The boundary of W is orientation-reversing diffeomorphic to the Seifert fibered Brieskorn homology sphere $\Sigma(p,q,pqn-1)$ (which is naturally oriented as the link of the corresponding singularity). Let F and S denote the Poincaré duals of the homology classes determined by the (p,q)-torus knot and the unknot, respectively. On W there are Stein structures with boundary whose first Chern classes are of the form $c_1(W) = (s+rn)F+rS$, where $|s| \leqslant n-2$, $s \equiv n \bmod 2$, $|r| \leqslant (p-1)(q-1)-2$, $r \equiv 0 \bmod 2$. As shown in [21], the contact structures induced on the boundary of these Stein 4-manifolds are homotopic exactly when the numbers $c_1(W)^2 = 2rs + nr^2$ are equal. Thus, fixing the quantity $2rs + nr^2$ letting r and s vary and applying Theorem 4.1 we obtain (finite) families of homotopic, nonisotopic tight contact structures on ∂W. Moreover, by letting p, q and n vary, such families become arbitrarily large.

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