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Conformally flat Lorentz hypersurfaces

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Abstract

We shall investigate conformally flat Lorentz hypersurfaces in indefinite space forms. Some particular classes of such hypersurfaces are explicitly described and classified.

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1. Introduction

Recall that a pseudo-Riemannian manifold (M, g) is said to be conformally flat if each $x \in M$ belongs to a neighborhood $U \subset M$ such that, for certain $\sigma \in C^{\infty}(U)$, the submanifold $(U, \mathrm{e}^{\sigma} g)$ is flat. Nonflat conformally flat Riemannian hypersurfaces in Euclidean spaces E^{n+1}, $n \geq 4$, had been firstly investigated by Cartan [3], who showed that the second fundamental form of those hypersurfaces admits at each point an eigenvalue of multiplicity $\geq n-1$. Conformally flat Riemannian hypersurfaces in positive definite space forms had been extensively studied by Chen (cf. [4]), and classified by Do Carmo et al. in the compact case (cf. [6]).

In this paper, we deal with conformally flat Lorentz hypersurfaces of dimension $n \geq 4$ in indefinite space forms $\tilde{M}^{n+1}(\tilde{c})$, i.e., complete simply connected and connected $(n+1)$-dimensional Lorentz manifolds of constant curvature $\tilde{c}$. In case $\tilde{M}^{n+1}(\tilde{c}) = \mathbb{R}_1^{n+1}$, $n \geq 4$,

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a local classification of these hypersurfaces was obtained by Van de Woestijne and Verstraelen (see [19, Theorem 2]). They claimed that, if the induced metric on a Lorentz hypersurface $M^n \subset \mathbb{R}_1^{n+1}$ is conformally flat, then it can be described as follows. Locally, M^n is either congruent to a part of a hypercylinder, a Lorentz hypersphere, a generalized cylinder, or a generalized umbilical hypersurface; or else M^n is foliated by $(n-1)$-dimensional Euclidean or Lorentzian hyperspheres, paraboloids, or hyperbolic spaces. Those hypersurfaces which are foliated by paraboloids or hyperbolic spaces would consist only of what we will call "bad points". However, it should be remarked that while Theorem 2 in [18] may be correct, its proof do not seem to be clear (one should actually provide evidence of the argument used in that paper and which consists in the fact that if the shape operator A takes a certain form at a point $x \in M^n$, then the same form still holds in a neighborhood of x and the eigenvalues of A have constant multiplicities in that neighborhood).

The paper is organized as follows. Section 2 contains the basic facts about general hypersurfaces in space forms. It also contains notation and formulas we will be using. Section 3 contains a classification result for shape operators of conformally flat Lorentz hypersurfaces in space forms. Section 4 presents the standard examples of Lorentz hypersurfaces which will serve as models in our classification. Section 5 is the main section, it contains various results on conformally flat Lorentz hypersurfaces in indefinite space forms in general. For instance, those hypersurfaces in Minkowski space which we call "*good hypersurfaces*" are explicitly described and classified.

2. Preliminaries

2.1. Lorentz symmetric endomorphisms

Let V be a vector space over $\mathbb{R}$ endowed with a nondegenerate inner product $\langle , \rangle$. An endomorphism $A \in \mathrm{End}(V)$ is said to be symmetric with respect to $\langle , \rangle$ (or briefly, symmetric) if it satisfies $\langle AX, Y\rangle = \langle X, AY\rangle$ for all $X, Y \in V$.

Unlike the positive definite case, it is well known that a symmetric endomorphism A of a an indefinite vector space $(V, \langle , \rangle)$ fails in general to be diagonalizable.

In case $\langle , \rangle$ is Lorentzian, symmetric endomorphisms are classified by the following result which may be found in [16, pp. 261–262], or [16].

Proposition 2.1. *Let V be an n-dimensional vector space endowed with a Lorentz inner product $\langle , \rangle$, and let A be a symmetric endomorphism of $(V, \langle , \rangle)$. If D_k denotes the diagonal matrix* $\mathrm{diag}\{\lambda_1, \dots, \lambda_k\}$, *then relative to a chosen basis, A has one of the following forms:*

(i) $$A = D_n,$$

(ii) $$A = \begin{pmatrix} a & b & 0 \\ -b & a & \\ 0 & & D_{n-2} \end{pmatrix}, \quad \text{with } b \neq 0,$$

(iii) $$A = \begin{pmatrix} \mu & 0 & 0 \\ 1 & \mu & \\ 0 & & D_{n-2} \end{pmatrix},$$

(iv) $$A = \begin{pmatrix} \mu & 0 & 1 & \\ 0 & \mu & 0 & 0 \\ 0 & -1 & \mu & \\ & 0 & & D_{n-3} \end{pmatrix},$$

where, in cases (*i*) *and* (*ii*), *A is represented relative to an orthonormal basis* $\{e_1, \dots, e_n\}$, *with nonzero products* $\langle e_1, e_1\rangle = -1$, *and* $\langle e_i, e_i\rangle = 1\ 2 \le i \le n$. *In cases* (*iii*) and (*iv*), *A is represented relative to a pseudo-orthonormal basis* $\{u, v, e_1, \dots, e_{n-2}\}$, *with nonzero products* $-\langle u, v\rangle = \langle e_i, e_i\rangle = 1$ *for* $1 \le i \le n-2$.

2.2. Basic formulas for hypersurfaces

Let $\tilde{M}^{n+1}(\tilde{c})$ be an $(n+1)$-dimensional Lorentz space form, i.e., a complete simply connected and connected $(n+1)$-dimensional Lorentz manifold of constant curvature $\tilde{c}$. A hypersurface M^n in $\tilde{M}^{n+1}(\tilde{c})$ is said to be Lorentzian if the tangent space T_xM^n at each $x \in M^n$ inherits a Lorentz metric from $\tilde{M}^{n+1}(\tilde{c})$.

Throughout this paper we shall assume $n \ge 4$, and if M^n is an n-dimensional connected Lorentz manifold which is isometrically immersed in $\tilde{M}^{n+1}(\tilde{c})$, we shall denote by f the isometric immersion $f : M^n \to \tilde{M}^{n+1}(\tilde{c})$ representing M^n in $\tilde{M}^{n+1}(\tilde{c})$. In that case, $f(M^n)$ is a Lorentz hypersurface which we will simply denote by M^n. If M^n and N^n are Lorentz hypersurfaces in $\tilde{M}^{n+1}(\tilde{c})$, we say that M^n and N^n are *congruent* if there is an isometry ϕ of $\tilde{M}^{n+1}(\tilde{c})$ such that $\phi(M^n) = N^n$. We shall also denote by $\langle, \rangle$ both the Lorentz metrics with the same signature $(-, +, \cdots, +)$ on the hypersurface M^n and $\tilde{M}^{n+1}(\tilde{c})$. A tangent vector to M^n or to $\tilde{M}^{n+1}(\tilde{c})$ is said to be timelike, spacelike or null (lightlike) if $\langle X, X\rangle < 0$, $\langle X, X\rangle > 0$ or $\langle X, X\rangle = 0$.

Let ξ be a local spacelike unit normal field on M^n. For any vector fields X and Y tangent to M^n, we have the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = f_*(\nabla_X Y) + h(X, Y)\xi, \qquad \tilde{\nabla}_X \xi = -f_*(AX),$$

where $\tilde{\nabla}$ and ∇ denote the Levi-civita connexions on $\tilde{M}^{n+1}(\tilde{c})$ and M^n, respectively, and A is the shape operator of the isometric immersion, i.e., a field of symmetric endomorphisms which is related to the second fundamental form h by $h(X, Y) = \langle AX, Y\rangle$. If R is the curvature tensor of M^n, then the Gauss equation is given by

$$R(X, Y) = AX \wedge AY + \tilde{c}(X \wedge Y),$$

where $X \wedge Y$ denotes the skew-symmetric endomorphism defined by $(X \wedge Y)Z = \langle Z, Y\rangle X - \langle Z, X\rangle Y$. The Coddazi's equation for hypersurfaces of spaces of constant curvature reduces to

$$(\nabla_X A)Y = (\nabla_Y A)X,$$

or equivalently

$$A([X, Y]) = X \cdot (AY) - Y \cdot (AX).$$

The Ricci tensor field of M^n can be written (cf. [11] or [17]) as

$$\mathrm{Ric}(X, Y) = \tilde{c}(n-1)\langle X, Y\rangle + \mathrm{tr}\, A\langle AX, Y\rangle - \langle A^2X, Y\rangle.$$

The Weyl curvature tensor $\mathcal{W}$ is defined by

$$\begin{aligned}\mathcal{W}(X, Y, Z, W) &= \langle R(X, Y)Z, W\rangle - \psi(X, W)\langle Y, Z\rangle - \psi(Y, Z)\langle X, W\rangle \\ &\quad + \psi(X, Z)\langle Y, W\rangle + \psi(Y, W)\langle X, Z\rangle,\end{aligned}$$

where $\psi(X, Y) = (1/(n-2))\{\mathrm{Ric}(X, Y) - r\langle X, Y\rangle/2(n-1)\}$ (here r is the scalar curvature of M^n). Now, if we assume, in addition, that M^n is conformally flat and $n \geq 4$, then $\mathcal{W} \equiv 0$, and so the Gauss equation states that

$$\begin{aligned}&\langle Y, Z\rangle SX - \langle X, Z\rangle SY + \psi(Y, Z)X - \psi(X, Z)Y \\ &\quad = \langle AY, Z\rangle AX - \langle AX, Z\rangle AY + \tilde{c}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\},\end{aligned}$$

where S denotes the endomorphism defined by

$$SX = \frac{1}{n-2}\left\{(\mathrm{tr}\, A)AX - A^2X + \left(\tilde{c}(n-1) - \frac{r}{2(n-1)}\right)X\right\}.$$

Note that ψ and S satisfy $\psi(X, Y) = \langle SX, Y\rangle$.

For each $x \in M^n$, the subspace $T_0(x) = \{X \in T_xM^n / A_xX = 0\}$ is called the relative nullity space at x. The index of relative nullity $\nu(x)$ at x is defined as the dimension of the subspace $T_0(x)$, while the rank of the shape operator A_x is called the type number $k(x)$ of M^n at x. It follows that, for any $x \in M^n$, we have $\nu(x) + k(x) = n$ (see [7,8] and particularly [1] for more details about relative nullity foliations).

3. Shape operators of conformally flat hypersurfaces

As we have mentioned in Section 1, the shape operator of a conformally flat Riemannian hypersurface in an Euclidean space E^{n+1} has at each point an eigenvalue of multiplicity $\geq n-1$. In [13], Moore classified the shape operators for conformally flat Riemannian submanifolds in Euclidean spaces. A similar work has been done by Magid in [11] for shape operators of a different class of submanifolds, namely, Einstein hypersurfaces of indefinite space forms. The main purpose of the present section is to deal with the case of conformally flat Lorentz hypersurfaces in indefinite space forms. Of course, there is no direct adaptation of the proof given in [13] since, as we have previously mentioned, a Lorentz symmetric endomorphism fails to be everywhere diagonalizable. More precisely, we obtain the following proposition.

Proposition 3.1. *Let $n \geq 4$ and let $f : M^n \to \tilde{M}^{n+1}(\tilde{c})$ be an isometric immersion of a conformally flat Lorentz manifold M^n into $\tilde{M}^{n+1}(\tilde{c})$. Then, at each point $x \in M^n$, relative* *to a chosen basis which will be precise through the proof, the shape operator A_x can be put into one of the following two forms:*

$$A_x = \begin{pmatrix} \mu & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix}, \quad or \quad A_x = \begin{pmatrix} \lambda & 0 & & & 0 \\ \pm 1 & \lambda & & & \\ & & \lambda & & \\ & & & \ddots & \\ 0 & & & & \lambda \end{pmatrix}.$$

In either case, the index of relative nullity is either $0, 1, n-1$ or n.

This result has been proved in [18] for Lorentz hypersurfaces in Minkowski spaces, and the proofs are essentially the same. Actually, only recently we have been informed of the existence of that reference. Since some formulas in the proof of Proposition 3.1 will be used later for the classification result, we shall give here the part of the proof that we will be using.

3.1. *Proof of Proposition 3.1*

According to Proposition 2.1, we distinguish four cases, but since the proof is similar to that when the ambient space is flat, we will only prove Proposition 3.1 in the case where the shape operator takes form (iii) of Proposition 2.1. For a proof in the diagonalizable case (i.e., case (i) of Proposition 2.1), see for example the proof of Proposition E.1 in [10], or the proof of Theorem 1 in [18]. The cases corresponding to forms (ii) and (iv) cannot occur (cf. [19, Theorem 1]).

Assume that the shape operator A_x has form (iii) of Proposition 2.1, i.e., the characteristic polynomial of A_x is of the form $(t-\mu)^2\prod_{i=1}^{n-2}(t-\lambda_i)$. Relative to a real basis $\{u, v, e_1, \dots, e_{n-2}\}$ with all scalar products zero except $-\langle u, v\rangle = \langle e_i, e_i\rangle = 1$, $1 \le i \le n-2$, the shape operator has the form (iii) of Proposition 2.1. As before, let $s = \operatorname{tr} A_x = 2\mu + \sum_{i=1}^{n-2}\lambda_i$, and let

$$\alpha = \frac{1}{n-2}\left\{s\mu - \mu^2 + \tilde{c}(n-1) - \frac{r}{2(n-1)}\right\}, \qquad \beta = \frac{1}{n-2}(s-2\mu),$$
$$\nu_i = \frac{1}{n-2}\left\{s\lambda_i - \lambda_i^2 + \tilde{c}(n-1) - \frac{r}{2(n-1)}\right\}.$$

One computes that

$$Su = \alpha u + \varepsilon\beta v, \qquad Sv = \alpha v, \qquad Se_i = \nu_i e_i, \quad 1 \le i \le n-2.$$

Now, Gauss' equation applied to $X = u$, $Y = Z = v$ and to $X = u$, $Y = Z = e_i$ gives

$$\tilde{c} + \mu^2 = 2\alpha,$$

and

$$\alpha + \nu_i = \mu\lambda_i + \tilde{c}, \quad \lambda_i = \beta, \quad 1 \le i \le n-2,$$

respectively. In particular, we obtain $\lambda_1 = \cdots = \lambda_{n-2}$. Let λ be the common value of $(\lambda_i)_{1\leq i\leq n-2}$ (note that λ is just β). On the other hand, the first two equations imply

$$2(n-2)\mu\lambda - (n-4)\mu^2 + n\tilde{c} = \frac{r}{n-1}, \tag{3.1}$$

and

$$\mu^2 + (n-3)\lambda^2 + 2\mu\lambda + n\tilde{c} = \frac{r}{n-1}, \tag{3.2}$$

respectively.

From (3.1) and (3.2) it follows that $(n-3)(\mu-\lambda)^2 = 0$, and since $n \neq 3$ we get that $\mu = \lambda$. Hence, the shape operator has the second form indicated in Proposition 3.1. □

Remark 1. If A_x is diagonalizable, and if μ and λ denote the principal curvatures, then the following basic formula relating the scalar curvature with the length of the mean curvature vector and the square of the norm of the second fundamental form follows immediately from the Gauss equation:

$$(n-2)\lambda^2(x) + 2\mu(x)\lambda(x) + n\tilde{c} = \frac{r(x)}{n-1}. \tag{3.3}$$

Similarly, by taking $\mu = \lambda$ in (3.1) or (3.2), we obtain for a nondiagonalizable A_x that

$$\lambda^2(x) + \tilde{c} = \frac{r(x)}{n(n-1)}. \tag{3.4}$$

4. Standard examples of hypersurfaces

In the Riemannian case, the n-dimensional sphere S^n is the model of conformally flat Riemannian manifolds, and it may be isometrically immersed in the Euclidean space E^{n+1}, with scalar shape operator, i.e., S^n is totally umbilic.

In the indefinite case, the hyperquadrics $\tilde{M}^n(\tilde{c})$ are examples of conformally flat Lorentz manifolds. If $\mathbb{R}_s^{n+1}$ denotes the standard flat Lorentz space form $(\mathbb{R}^{n+1}, \langle,\rangle)$, where $\langle,\rangle$ is given by

$$\langle x, y\rangle = -\sum_{i=1}^{s} x_i y_i + \sum_{i=s+1}^{n+1} x_i y_i, \quad \text{for all } x = (x_1, \ldots, x_{n+1}),\ y = (y_1, \ldots, y_{n+1}),$$

then each $\tilde{M}^n(\tilde{c})$ might be realized as a totally umbilic hypersurface in $\mathbb{R}_s^{n+1}$ for certain s, so that the shape operator is scalar. In particular, the n-dimensional Minkowski space $\tilde{M}^n(0) = \mathbb{R}_1^n$ may in turn be realized as a totally geodesic hypersurface in $\mathbb{R}_1^{n+1}$. We have for $\tilde{c} > 0$, the so-called *de Sitter space* $S_1^n(\tilde{c})$ which is defined as

$$S_1^n(\tilde{c}) = \{x \in \mathbb{R}_1^{n+1} : \langle x, x\rangle = \tilde{c}^{-1}\},$$

and for $\tilde{c} < 0$ we have the universal covering of the so-called *anti-de Sitter space* $H_1^n(\tilde{c})$ which is defined as

$$H_1^n(\tilde{c}) = \{x \in \mathbb{R}_2^{n+1} : \langle x, x\rangle = \tilde{c}^{-1}\}.$$

For ease of notation, we write S_1^n (respectively, H_1^n) rather than $S_1^n(\tilde{c})$ (respectively, $H_1^n(\tilde{c})$) when $\tilde{c}$ is not given.

Notice that, $\tilde{M}^n(\tilde{c})$ can also be regarded as a totally geodesic conformally flat hypersurface in $\tilde{M}^{n+1}(\tilde{c})$, and so the shape operator vanishes. We refer to [2, pp. 181–185], [16, pp. 108–114], and [21, pp. 67–68], for more details about hyperquadrics.

Notice also that the diagonalizable form for the shape operator in Proposition 3.1 has the two possible eigenvalues μ and λ, but for the examples above only one eigenvalue shows up. So, we shall also provide examples with two distinct eigenvalues. In this setting, the direct products $\mathbb{R}_1^1 \times S^{n-1}$ and $E^1 \times S_1^{n-1}$ are good examples of conformally flat Lorentz hypersurfaces in $\mathbb{R}_1^{n+1}$ having diagonalizable shape operators with $\mu = 0$ and $\lambda \neq 0$. Note however that such products are not of Einstein type. Note also that if $1 < k < n-1$, the product $\mathbb{R}_1^k \times S^{n-k}$ as well as $E^k \times S_1^{n-k}$ fails to be conformally flat.

Examples of those hypersurfaces with $0 \neq \mu \neq \lambda \neq 0$ may be given as the direct products $S_1^1 \times S^{n-1}$ and $S^1 \times S_1^{n-1}$, which will later be treated as bad hypersurfaces.

4.1. Hypercylinders over plane curves

A hypercylinder in $\mathbb{R}_1^{n+1}$ is defined by one of the following isometric immersions:

$$c \times \mathrm{id} : E^1 \times \mathbb{R}_1^{n-1} \to E^2 \times \mathbb{R}_1^{n-1}, \qquad c \times \mathrm{id} : \mathbb{R}_1^1 \times E^{n-1} \to \mathbb{R}_1^2 \times E^{n-1},$$

where c is a unit speed plane curve in the sense that $\langle c', c' \rangle = \pm 1$.

Note that such immersions have nondegenerate relative nullities. More generally, Theorem 8.7 of [9] states that, up to a Lorentz motion, an isometric immersion $\mathbb{R}_1^n \to \mathbb{R}_1^{n+1}$ with nondegenerate relative nullities is split, namely, it has one of the above orthogonal products. This leads to the following proposition.

Proposition 4.1. *Let $f : \mathbb{R}_1^n \to \mathbb{R}_1^{n+1}$ be an isometric immersion with nondegenerate relative nullities. Then for each $x \in \mathbb{R}_1^n$, up to a Lorentz motion, the shape operator has the following form:*

$$A_x = \begin{pmatrix} -k & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}.$$

4.2. B-scroll immersions and generalized cylinders

In this and the next sections, we shall prove that conformally flat Lorentz hypersurfaces with nondiagonalizable shape operators really exist.

Let $x(s)$ be a null curve in the Minkowski space $\mathbb{R}_1^3$ and $T(s) = x'(s)$ its tangent vector field, and assume that $T'(s)$ is never colinear to $T(s)$. Then, differentiating $\langle T, T \rangle = 0$ it follows that $T'(s)$ is everywhere spacelike, and so in analogy with Euclidean curves $k(s) = |T'(s)|$ might be called the curvature of the null curve x. Furthermore, the principal normal vector field to x may be defined as the unit spacelike vector field $N(s)$ along x satisfying $T'(s) = k(s)N(s)$. (Note that the plane span$\{T, N\}$ is degenerate and that $\langle T, T\rangle = \langle N, T\rangle = 0$, $\langle N, N\rangle = 1$).

Now, since N is spacelike, the plane $N^\perp$ orthogonal to N is Lorentzian. Thus, it contains a second null direction other than $\mathbb{R}T$. So, in $N^\perp$ we may choose the (unique) null vector field $B(s)$ along x called the binormal such that $\langle B, T\rangle = -1$. As in the Euclidean case, if we define the torsion of x to be the real-valued function $\tau(s) = \langle B'(s), N(s)\rangle$, we easily obtain

$$N'(s) = \tau(s)T(s) + k(s)B(s), \qquad B'(s) = \tau(s)N(s).$$

These formulas together with $T'(s) = k(s)N(s)$ play the role of Frenet–Serret equations for non-null curves. Similarly, the frame $\{T(s), N(s), B(s)\}$, called Cartan frame of x, must be regarded as the Frenet–Serret apparatus of x. However, it should be noticed that even if the above construction is correct, the Cartan frame that we define here is not at all a Frenet like frame, since the parameter s is not invariant.

If, in addition, $\tau(s) = 0$ for all s, namely, if B is parallel (x is not necessarily a plane curve), the null curve $x(s)$ with the Cartan frame (T, N, B) is called a generalized cubic (cf. [9]). In this case, $x(s)$ can be explicitly determined via the initial conditions.

We are now in position to discuss the concept of B-scroll immersions introduced in [9]. Let $(x(s), (T, N, B))$ be a generalized cubic in $\mathbb{R}^3_1$. The B-scroll associated to that cubic is the Lorentz surface defined by the parameterization $h(s, u) = x(s) + uB(s)$. Since $\tau(s) = 0$ for all s, it follows that $h_*(\partial/\partial s) = T(s)$ and $h_*(\partial/\partial u) = B(s)$. Thus, we easily verify that h is an isometric immersion called B-scroll immersion of $\mathbb{R}^2_1$ into $\mathbb{R}^3_1$, where here $\mathbb{R}^2_1$ has the flat metric $-\mathrm{d}s\,\mathrm{d}u$.

Since $N(s)$ is a spacelike unit normal vector field, the Weingarten equation with $\xi = N(s)$ tells us that the shape operator has the form

$$\begin{pmatrix} 0 & 0 \\ -k & 0 \end{pmatrix}, \quad k > 0.$$

Now, to get higher dimension examples, we simply consider the isometric immersion

$$\mathrm{id} \times h : E^{n-2} \times \mathbb{R}^2_1 \to E^{n-2} \times \mathbb{R}^3_1,$$

where $h : \mathbb{R}^2_1 \to \mathbb{R}^3_1$ is, as above, a B-scroll immersion, and the factors in each product are orthogonal. Such an isometric immersion is explicitly described as $f : \mathbb{R}^n_1 \to \mathbb{R}^{n+1}_1$ defined by

$$f(x_1, \dots, x_n) = (x_1, \dots, x_{n-2}, x_{n-1} + x_n, x_{n-1} - x_n, x_n^2),$$

where the signature is $(+, \cdots, +, -, +)$.

The Lorentz manifolds $h(\mathbb{R}^2_1) \times E^{n-2}$ are called *generalized cylinders* [12]. We further note that the B-scrolls described above have degenerate relative nullity distribution. More generally, the isometric immersions $\mathbb{R}^2_1 \to \mathbb{R}^3_1$ with degenerate relative nullities are precisely the B-scroll immersions (this is Theorem 9.7 of [9]). In high dimensions, Theorem 9.8 of [9] states that, up to Lorentz motions, the isometric immersions $\mathbb{R}^n_1 \to \mathbb{R}^{n+1}_1$ with degenerate relative nullities are split, i.e., they are of the form

$$h \times \mathrm{id} : \mathbb{R}^2_1 \times E^{n-2} \to \mathbb{R}^3_1 \times E^{n-2}$$

where $h : \mathbb{R}^2_1 \to \mathbb{R}^3_1$ is a B-scroll immersion, and the factors in each product are orthogonal. In analogy with the nondegenerate case, we have the following proposition.

Proposition 4.2. *Let $f : \mathbb{R}^n_1 \to \mathbb{R}^{n+1}_1$ be an isometric immersion with degenerate relative nullities. Then for each $x \in \mathbb{R}^n_1$, up to a Lorentz motion, the shape operator has the following form:*

$$A_x = \begin{pmatrix} 0 & 0 & & 0 \\ -k & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}, \quad k > 0.$$

Proof. By Theorem 9.8 of [9] we may assume that, up to a Lorentz motion, the immersion f has the form $h \times \mathrm{id}$ where h is a B-scroll immersion. Thus, the shape operator of f takes the desired form, with k as the curvature of the generalized cubic determining the B-scroll $h(\mathbb{R}^2_1)$. It remains to show that $k \neq 0$. Indeed, Gauss' equation implies that $AX \wedge AY = 0$ for all $X, Y \in T_x(\mathbb{R}^n_1)$, from which we conclude that the index of relative nullity $\nu(x)$ is $n-1$ or n. Since the relative nullity distribution is assumed to be everywhere degenerate, it follows that $\nu(x) = n-1$, namely, $A_x \neq 0$ at each $x \in \mathbb{R}^n_1$. This in turn implies that $k \neq 0$ at every point. (Note that we can conclude that $k > 0$ simply by the facts that $k(s) = |T'(s)|$ and $T'(s)$ is everywhere spacelike.) □

4.3. Generalized umbilical hypersurfaces

This notion was introduced by Magid in [12], and it can be described as follows. Consider a Minkowski space $\mathbb{R}^{n+1}_1$ as the direct product $\mathbb{R}^3_1 \times E^{n-2}$, and let $x(s)$ be a null curve in $\mathbb{R}^{n+1}_1$ which lies entirely in the factor $\mathbb{R}^3_1$. With the same notation of the subsection above, let $\{T(s), N(s), B(s)\}$ be the Cartan frame of x regarded as a frame in $\mathbb{R}^{n+1}_1$, and $\{Z_1, \ldots, Z_{n-2}\}$ a family of orthonormal spacelike vectors in $\mathbb{R}^{n+1}_1$ which are orthogonal to $\mathrm{span}\{T(0), N(0), B(0)\}$ at $x_0 = x(0)$. For each $i \in \{1, \ldots, n-2\}$, let $Z_i(s)$ be the vector field along x obtained by parallel translation in $\mathbb{R}^{n+1}_1$ of Z_i along the curve x. Since the Cartan frame of x spans a subspace which is parallel along x, we see that each $Z_i(s)$ is orthogonal to $\mathrm{span}\{T(s), N(s), B(s)\}$, for all s. Assume now that the torsion $\tau(s)$ is a constant, say $\tau(s) \equiv \tau$, different from zero, and consider the parametrization $f : \mathbb{R}^n \to \mathbb{R}^{n+1}_1$ defined by

$$f(s, u, z_1, \ldots, z_{n-2}) = x(s) + uB(s) - GN(s) + \sum_{i=1}^{n-2} z_i Z_i(s),$$

where $G = 1/\tau + \sqrt{1/\tau^2 - \sum_{i=1}^{n-2} z_i^2}$.

Now, straightforward computation yields

$$f_*\left(\frac{\partial}{\partial s}\right) = (1-\tau G)T(s) - k(s)GB(s) + u\tau N(s), \qquad f_*\left(\frac{\partial}{\partial u}\right) = B(s),$$
$$f_*\left(\frac{\partial}{\partial z_i}\right) = Z_i(s) + \frac{z_i}{\sqrt{1/\tau^2 - \sum_{i=1}^{n-2} z_i^2}} N(s).$$

This clearly defines a Lorentz hypersurface in $\mathbb{R}_1^{n+1}$, called a generalized umbilical hypersurface. In this case, we easily check that

$$\xi = -u\tau B(s) + \tau\sqrt{\frac{1}{\tau^2} - \sum_{i=1}^{n-2} z_i^2}\, N(s) - \tau \sum_{i=1}^{n-2} z_i Z_i(s),$$

and the shape operator has the nondiagonalizable form of Proposition 3.1 with nonzero $\lambda = \tau$.

Particular case: $n = 2$. In this case, $f(s, u) = x(s) + uB(s)$ is an isometric immersion of $(\mathbb{R}^2, -2\,\mathrm{d}s\,\mathrm{d}u + u^2\tau^2\,\mathrm{d}s^2)$ into $\mathbb{R}_1^3$. By analogy to the notion of B-scrolls, we call $f(\mathbb{R}^2)$ a generalized B-scroll. Clearly, a generalized B-scroll is not geodesically complete, and relative to the coordinate system $\{s, u\}$ on $\mathbb{R}^2$, its shape operator is

$$\begin{pmatrix} -\tau & 0 \\ -k(s) & -\tau \end{pmatrix}, \qquad \tau \neq 0.$$

5. Main results

Let M^n be a Lorentz hypersurface in $\tilde{M}^{n+1}(\tilde{c})$. In [17], a point $x \in M^n$ is called *bad* if A_x is nonsingular and has a simple eigenvalue. All other points are called *good*. If all points are good, we will say that M^n is "*good*".

Let M^n be a conformally flat Lorentz hypersurface in $\tilde{M}^{n+1}(\tilde{c})$, with $n \geq 4$. In light of Proposition 3.1, we have for each $x \in M^n$, either

1. $A_x = \lambda\mathrm{Id} \neq 0$,
2. $A_x = 0$,
3. A_x is diagonalizable and has two unequal nonzero eigenvalues of multiplicity 1 and $n-1$,
4. A_x is diagonalizable and has two unequal eigenvalues, $\mu = 0$ of multiplicity 1, and $\lambda \neq 0$ of multiplicity $n-1$,
5. A_x is diagonalizable and has two unequal eigenvalues, $\mu \neq 0$ of multiplicity 1, and $\lambda = 0$ of multiplicity $n-1$, or
6. A_x is nondiagonalizable and has only one eigenvalue λ (possibly equal to zero), and the minimal polynomial is $(t-\lambda)^2$.

Remark 2. Clearly, Case 3 corresponds to bad points, so that we may interpret the definition of good points as that Case 3 could never occur.

Proposition 5.1. *Let M^n be a conformally flat Lorentz hypersurface in $\tilde{M}^{n+1}(\tilde{c})$. Then, the set of bad points is open.*

Proof. Let x_0 be a bad point, and choose an open neighborhood U of x_0 in which A remains nonsingular. Since the diagonalizable case is the only case where two eigenvalues of A can be distinct, we may choose U so that $\mu(x) \neq \lambda(x)$ for any $x \in U$. Thus, all points of U are bad. □

Proposition 5.2. *Let M^n be a conformally flat Lorentz hypersurface in $\tilde{M}^{n+1}(\tilde{c})$, with $\tilde{c} > r/n(n-1)$ everywhere, where as above r denotes the scalar curvature of M^n. Then, M^n consists purely of bad points.*

Proof. This is immediate, since at a good point $x \in M^n$ we must have $\tilde{c} \leq r(x)/n(n-1)$ by (3.3) and (3.4) in Remark 1. □

We shall now give our main results. For this purpose, we first need the following simple but basic result. Before stating it, notice that in general if $f : M^n(\tilde{c}) \to \tilde{M}^{n+1}(\tilde{c})$ is an isometric immersion between Lorentz manifolds with the same constant curvature $\tilde{c}$, it follows from the Gauss equation that $k(x) \leq 1$ everywhere. In [9], Graves classified such immersions in the situation where $\tilde{c} = 0$ and M^n is complete. Conversely, we have the following result which is still true for general pseudo-Riemannian hypersurfaces of indefinite space forms.

Proposition 5.3. *Let $f : M^n \to \tilde{M}^{n+1}(\tilde{c})$ be an isometric immersion from a Lorentz manifold M^n into $\tilde{M}^{n+1}(\tilde{c})$. If $k(x) \leq 1$ for all x, then M^n is also of constant curvature $\tilde{c}$.*

Proof. Let $x \in M^n$ such that $k(x) \leq 1$. Clearly, the case $k(x) = 0$ is trivial. So, assume that $k(x) = 1$. If $T_0(x)$ is nondegenerate, let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for $T_0(x)$, and let $e_n \notin T_0(x)$ be a unit vector such that $Ae_n = \lambda e_n$, with $\lambda \neq 0$ of course. In this case, for any $X = \sum_{i=1}^{n} x_i e_i \in T_x M^n$ we have $AX = \lambda x_n e_n$. Thus, for all $X, Y \in T_x M^n$ we get

$$R(X, Y) = AX \wedge AY + \tilde{c}(X \wedge Y) = \tilde{c}(X \wedge Y).$$

If $T_0(x)$ is degenerate. In this case A_x has the nondiagonalizable form in Proposition 3.1, with $\lambda = 0$. This means that there is a pseudo-orthonormal basis $\{e_1, \dots, e_n\}$ for $T_x M^n$ such that $A_x e_1 = e_2$ and $A_x e_i = 0$ for $2 \leq i \leq n$, where e_1 and e_2 are null vectors. Thus, for all $X, Y \in T_x M^n$ we easily see that

$$R(X, Y) = \tilde{c}(X \wedge Y).$$

Hence, M^n is a space of constant curvature $\tilde{c}$. □

We are now ready to state our main results. The first one deals with good hypersurfaces.

Theorem 5.4. *Let $n \geq 4$, and let M^n be a connected, good, conformally flat Lorentz hypersurface in $\tilde{M}^{n+1}(\tilde{c})$.Then $\tilde{c} \leq r(x)/n(n-1)$ for all $x \in M^n$, where r is the scalar curvature of M^n. Furthermore, we have*

(i) *If* $\tilde{c} = r/n(n-1)$ *identically, then* $k(x) \leq 1$ *for all* $x \in M^n$*, and so* M^n *is a space of constant curvature* $\tilde{c}$.

(ii) *If* $\tilde{c} < r/n(n-1)$ *everywhere, then either* $k(x) = n-1$ *for all* $x \in M^n$*, or* $k(x) = n$ *for all* $x \in M^n$.

Proof. The inequality $\tilde{c} \leq r/n(n-1)$ follows easily from (3.3) and (3.4) in Remark 1, and the fact that μ and λ in (3.3) cannot be simultaneously nonzero and distinct, as all points are good. If $\tilde{c} = r/n(n-1)$ identically, then each of (3.3) and (3.4) gives $\lambda \equiv 0$ because all the points are good, and so $k(x) \leq 1$ everywhere, i.e., M^n is a space of constant curvature by Proposition 5.3.

If now $\tilde{c} < r/n(n-1)$ everywhere, and since all points are good it follows from (3.3) and (3.4) that $\lambda \neq 0$ everywhere, and so $k(x) \geq n-1$ at any $x \in M^n$. Let $W = \{x \in M^n : k(x) = n\}$ and $W' = \{x \in M^n : k(x) = n-1\}$. Since M^n is connected and $M^n = W \cup W'$, it follows that if we show that W is open and closed then we are done. Clearly W is open, as it is the set of points for which A_x is invertible. So, we need only to show that W is closed. Indeed, let $\{x_i\}$ be a sequence in W converging to some $x \in M^n$. Since all points are good, the eigenvalues of A_{x_i} must verify $\lambda^2(x_i) + \tilde{c} = r(x_i)/n(n-1)$ independently of the fact that A_{x_i} is diagonalizable or not. By continuity, we get

$$\lambda^2(x) + \tilde{c} = \frac{r(x)}{n(n-1)}. \tag{5.1}$$

Since by hypothesis $\tilde{c} \neq r(x)/n(n-1)$, we conclude from (5.1) that $\lambda(x) \neq 0$. If A_x is nondiagonalizable, then according to the only possible nondiagonalizable form that A_x could take it follows that $k(x) = n$, and thus $x \in W$. If now A_x is diagonalizable, then comparing (3.3) with (5.1) we see that $\mu(x) = \lambda(x) \neq 0$. Thus $k(x) = n$, and so $x \in W$. This shows that W is closed, and the proof of Theorem 5.4 is complete. □

Our second main result follows as a corollary of Theorem 5.4.

Theorem 5.5. *Let* $n \geq 4$*, and let* M^n *be a connected, good, and complete conformally flat Lorentz hypersurface in* $\mathbb{R}_1^{n+1}$*. Then the scalar curvature* r *is* ≥ 0 *everywhere, and we have*

(i) *If* $r \equiv 0$*, then* M^n *is locally flat and congruent to either* $\mathbb{R}_1^n$*,* $\mathbb{R}_1^{n-2} \times g(E^2)$*, or* $E^{n-2} \times g(\mathbb{R}_1^2)$*, where* $g(E^2)$ *is a Euclidean cylinder in a subspace* E^3 *of* $\mathbb{R}_1^{n+1}$ *orthogonal to* $\mathbb{R}_1^{n-2}$*, and* $g(\mathbb{R}_1^2)$ *is a Lorentz cylinder or a B-scroll in a subspace* $\mathbb{R}_1^3$ *of* $\mathbb{R}_1^{n+1}$ *orthogonal to* E^{n-2}.

(ii) *If* $r > 0$ *everywhere, then* M^n *is congruent to either* S_1^n*,* $E^1 \times S_1^{n-1}$*,* $\mathbb{R}_1^1 \times S^{n-1}$*, or else* M^n *is locally congruent to a generalized umbilical hypersurface in* $\mathbb{R}_1^{n+1}$*, where here* S^{n-1} *is a Euclidean hypersphere in a subspace* E^n *of* $\mathbb{R}_1^{n+1}$ *orthogonal to* $\mathbb{R}_1^1$*, and* S_1^{n-1} *is a Lorentz hypersphere in a subspace* $\mathbb{R}_1^n$ *of* $\mathbb{R}_1^{n+1}$ *orthogonal to* E^1.

Proof. The fact that $r \geq 0$ follows from Theorem 5.4. If $r = 0$ identically, again Theorem 5.4, assertion (i), tells us that $k(x) \leq 1$ and M^n is locally flat. In case M^n is not simply connected, we let $\widetilde{M^n}$ be its universal covering with projection $\pi : \widetilde{M^n} \to M^n$. Now, Theorems 8.7

and 9.8 of [9] applied to $\widetilde{M^n}$ and its immersion $\tilde{f} = f \circ \pi$ imply that $\tilde{f}(\widetilde{M^n}) = f(M^n)$ is congruent to one of the hypersurfaces mentioned in (i) of the current theorem.

If now $r > 0$ everywhere, Theorem 5.4, assertion (ii), tells us that $k(x) = n-1$ everywhere or $k(x) = n$ everywhere. If $k(x) = n - 1$ everywhere, A_x is diagonalizable for all $x \in M^n$. Thus, by [10], Proposition D.4, or [17], Proposition 2.3, it follows that the distributions T_0 and T_λ are differentiable and integrable, where $T_\lambda(x) = \{X \in T_xM^n : A_x(X) = \lambda X\}$, and that λ is constant on each leaf of T_λ. In fact, we will see by Lemma 5.7 that λ is constant on the whole of M^n. Now, for each $x \in M^n$ we have

$$T_xM^n = T_0(x) \oplus T_\lambda(x).$$

Let $M_0(x)$ and $M_\lambda(x)$ denote the maximal integral submanifolds through x of $T_0(x)$ and $T_\lambda(x)$, respectively.

Lemma 5.6. *$M_0(x)$ can be affinely parameterized as a complete non-null geodesic $\gamma(s)$ of M^n, and $f(\gamma(s))$ is a geodesic in $\mathbb{R}_1^{n+1}$.*

Proof. Assume there is a neighborhood of x on which T_0 is generated by a nonsingular vector field X such that $\langle X, X\rangle \neq 0$. For any (other) tangent vector field Y, the Codazzi equation implies that

$$\nabla_X AY = A([X, Y]),$$

and differentiating $\langle X, AY\rangle = \langle AX, Y\rangle = 0$ yields

$$\begin{aligned} 0 &= \langle \nabla_X X, AY\rangle + \langle X, \nabla_X AY\rangle = \langle \nabla_X X, AY\rangle + \langle X, A([X, Y])\rangle \\ &= \langle A(\nabla_X X), Y\rangle + \langle AX, [X, Y]\rangle = \langle A(\nabla_X X), Y\rangle. \end{aligned}$$

Hence $\nabla_X X \in T_0(x)$, which means that $M_0(x)$ is a pregeodesic. Therefore, multiplication by an appropriate function makes it a geodesic. Let $\gamma(s)$ denote this affine parameterized geodesic, which is complete, as M^n is. Next, let us consider f locally. Since $\dot\gamma(s)$ lies in $T_0(x)$ and since $\gamma(s)$ is a geodesic of M^n, we have

$$\tilde\nabla_s\dot\gamma(s) = f_*(\nabla_s\dot\gamma(s)) + h(\dot\gamma(s), \dot\gamma(s))\xi = 0,$$

so that $f(\gamma(s))$ is a geodesic in $\mathbb{R}_1^{n+1}$. □

Remark 3. According to [16, p. 202], the geodesic $f(\gamma(s))$ of Lemma 5.6 is complete, so that $M_0(x)$ is isometrically mapped via f onto a one-dimensional subspace $E^1(x)$ or $\mathbb{R}_1^1(x)$ according to whether $\gamma(s)$ is a spacelike or a timelike geodesic, respectively.

The following lemma summarizes Lemmas 5 and 6 of [14] and Lemma 2 of [19].

Lemma 5.7. *If $k(x) = n - 1$ everywhere, then the function λ is constant on the whole of M^n, and both distributions T_0 and T_λ are parallel, i.e., $\nabla_X T_0 \subset T_0$ and $\nabla_X T_\lambda \subset T_\lambda$ for every $X \in T_xM^n$.*

Using this lemma, Remark 3, and the generalized de Rham's theorem [21] we can easily prove, just as in [14,19], the following lemma.

Lemma 5.8. *If $k(x) = n - 1$ everywhere, then*

1. *$M_\lambda(x)$ is a complete totally geodesic hypersurface of M^n.*
2. *If $M_0 = M_0(x)$ and $M_\lambda = M_\lambda(x)$, then M^n is isometric to the direct product $M_0 \times M_\lambda$.*
3. *The one-dimensional spaces $f(M_0(x))$ are parallel to each other.*
4. *In case $f(M_0(x)) = E^1(x)$, the restriction f_λ of f to M_λ is an isometry of M_λ onto a Lorentz hypersphere S_1^{n-1} in a subspace $\mathbb{R}_1^n$ of $\mathbb{R}_1^{n+1}$ orthogonal to E^1.*
5. *In case $f(M_0(x)) = \mathbb{R}_1^1(x)$, the restriction f_λ of f to M_λ is an isometry of M_λ onto a Euclidean hypersphere S^{n-1} in a subspace E^n of $\mathbb{R}_1^{n+1}$ orthogonal to $\mathbb{R}_1^1$.*
6. *If f_0 denotes the restriction of f to M_0, then $f = f_0 \times f_\lambda$, i.e., $f(u, v) = (f_0(u), f_\lambda(v))$ for every $(u, v) \in M_0 \times M_\lambda = M^n$.*

This lemma completes the proof of assertion (ii) of Theorem 5.5 in the case $k(x) = n-1$. It remains to prove it in case $k(x) = n$ everywhere. In this case, λ is constant according to [10], Proposition D.4. Hence, one can easily see that either $A = \lambda\mathrm{Id}$ or A takes the nondiagonalizable form in Proposition 3.1, with $\lambda \neq 0$ in both cases. In the former case M^n is totally umbilical, and the same arguments used in case $k(x) = n - 1$ show that M^n is a Lorentz hypersphere. In the latter case, M^n is isoparametric, and so we apply Theorem 4.5 of [12] to deduce that M^n is locally congruent to a generalized umbilical hypersurface in $\mathbb{R}_1^{n+1}$ as in Section 4.3. This completes the proof of assertion (ii) of Theorem 5.5 in the case $k(x) = n$. The proof of Theorem 5.5 is then complete. □

Remark 4. In the case where M^n is not complete, the situation is more complicated. For instance, let $x(s)$ be a unit speed curve in E^3, and consider the parametrization $h(s, u) = x(s) + ux'(s)$, $u > 0$. We can easily check that h defines a flat surface called the tangent surface of x, with metric $g = (1 + u^2k^2)\,\mathrm{d}s^2 + 2\,\mathrm{d}s\,\mathrm{d}u + \mathrm{d}u^2$, where $k = k(s)$ is the curvature of x. However, this surface is not geodesically complete since each point of c presents a singularity. Now, by considering the product $h(\mathbb{R}^2) \times \mathbb{R}_1^{n-2}$, we obtain a good, but noncomplete, flat Lorentz hypersurface in $\mathbb{R}_1^{n+1}$, which is not locally isometric to any one of those hypersurfaces appearing in the list of Theorem 5.5. (We can similarly start with a timelike curve in $\mathbb{R}_1^3$ to obtain the same things).

Another important corollary of Theorem 5.4 is the following theorem.

Theorem 5.9. *Let $n \geq 4$, and let M^n be a connected, good, and complete conformally flat Lorentz hypersurface in $S_1^{n+1}(\tilde{c})$ such that $\tilde{c} = r/n(n-1)$ identically. Then M^n is congruent to a great Lorentz hypersphere $S_1^n(\tilde{c})$.*

Proof. By Theorem 5.4, $k(x) \leq 1$ everywhere and M^n is a space of constant curvature $\tilde{c}$. Let f be the isometric immersion representing M^n in $S_1^{n+1}(\tilde{c})$, and let $\pi : \widetilde{M^n} \to M^n$ be the universal covering of M^n. Then $(\widetilde{M^n}, f \circ \pi)$ is a complete simply connected Lorentz space of constant curvature $\tilde{c}$, and hence is congruent to $S_1^n(\tilde{c})$. Thus $\widetilde{M^n}$ is orientable, and the local unit vector field ξ can be defined on the whole of $\widetilde{M^n}$. By [5], $(\widetilde{M^n}, f \circ \pi)$ is totally geodesic, i.e., a great hypersphere. In particular, $f \circ \pi$ is one to one and thus π is one to one. It follows that M^n is simply connected and (M^n, f) is congruent to a great hypersphere, as desired. $\square$

In the case where all points are bad, we have the following theorem.

Theorem 5.10. *Let $n \geq 4$, and let M^n be a conformally flat Lorentz hypersurface in $\tilde{M}^{n+1}(\tilde{c})$ with $\tilde{c} > r/n(n-1)$ everywhere. Then all points are bad, and M^n is foliated by $(n-1)$-dimensional spaces of constant curvature $> \tilde{c}$.*

Proof. By Proposition 5.2, all points are bad. Let μ and λ be the two distinct eigenvalues of A with multiplicities 1 and $n-1$, respectively. As previously, by Proposition D.4 of [10] (see also Proposition 2.3 of [17]), the distributions T_μ and T_λ are differentiable and integrable, and λ is constant on each leaf M_λ (however, we know nothing about μ). Furthermore, each M_λ is a nondegenerate totally geodesic hypersurface of M^n, and so the curvature tensor R_λ of M_λ coincides with the restriction of R to M_λ. This means that if $X, Y \in T_\lambda$, then $R_\lambda(X, Y) = R(X, Y) = (\lambda^2 + \tilde{c}) X \wedge Y$. Since $\dim M_\lambda = n - 1$, it follows that M_λ is a space of constant curvature $\lambda^2 + \tilde{c}$. $\square$

Corollary 5.11. *Let $n \geq 4$, and let M^n be a conformally flat Lorentz hypersurface in $\mathbb{R}^{n+1}_1$ with scalar curvature $r < 0$ everywhere. Then M^n is foliated by either $(n-1)$-dimensional Riemannian hyperspheres or $(n-1)$-dimensional Lorentz hyperspheres.*

Proof. This follows from the proof of Theorem 5.10 with $\lambda \neq 0$, and a close reading of the proof of Lemma 2.2 of [6]. $\square$

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