REVERSE ENGINEERING SMALL 4-MANIFOLDS

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Abstract. We introduce a general procedure called 'reverse engineering' that can be used to construct infinite families of smooth 4-manifolds in a given homeomorphism type. As one of the applications of this technique, we produce an infinite family of pairwise nondiffeomorphic 4-manifolds homeomorphic to $\mathbf{CP}^2\# 3\overline{\mathbf{CP}}^2$.

1. Introduction

This paper introduces a technique which we call *reverse engineering* that can be used to construct infinite families of distinct smooth structures on many smoothable 4-manifolds. As one example of the utility of this technique we will construct infinitely many distinct smooth structures on $\mathbf{CP}^2\# 3\overline{\mathbf{CP}}^2$. Exotic smooth structures on these manifolds were first constructed in [AP, BK1].

Reverse engineering is a three step process for constructing infinite families of distinct smooth structures on a given simply connected 4-manifold. One starts with a model manifold which has nontrivial Seiberg-Witten invariant and the same euler number and signature as the simply connected manifold X that one is trying to construct, but with $b_1 > 0$. The second step is to find b_1 essential tori that carry generators of H_1 and to surger each of these tori in order to kill H_1 and, in favorable circumstances, to kill π_1. The third step is to compute Seiberg-Witten invariants. After each of the first $b_1 - 1$ surgeries one needs to preserve the fact that the Seiberg-Witten invariant is nonzero. The fact that the next to last manifold in the string of surgeries has nontrivial Seiberg-Witten invariant allows the use of the Morgan, Mrowka, Szabó formula [MMS] to produce an infinite family as was done in [FS2].

In many instances this procedure can be successfully applied without any computation, or even mention, of Seiberg-Witten invariants. If the model manifold for X is symplectic and $b_1 - 1$ of the tori are Lagrangian so that a Luttinger surgery

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will reduce b_1, then the there are infinitely many distinct smooth manifolds with the same cohomology ring as X. If the resulting manifold is simply connected, there are infinitely many distinct smooth structures on X. It seems that the most difficult aspect to the reverse engineering procedure is the computation of π_1.

We will prove the main theorem that shows that this procedure provides infinitely many distinct manifolds in §2. We then provide two examples. In §3 we apply the reverse engineering procedure to a model for $\mathbf{CP}^2\# 3\overline{\mathbf{CP}}^2$, the 2-fold symmetric product of a genus 3 surface. We will identify the Lagrangian tori, and show that the Luttinger surgeries result in a simply connected manifold, thus producing infinitely many distinct smooth structures on $\mathbf{CP}^2\# 3\overline{\mathbf{CP}}^2$. In §4 we apply the reverse engineering procedure to the product of two genus 2 surfaces, which is a model for $S^2\times S^2$. We will identify Lagrangian tori that kill H_1 resulting in infinitely many distinct smooth manifolds with the cohomology ring of $S^2\times S^2$. However, to date we have been unsuccessful in showing that these manifolds are simply connected.

2. Reverse Engineering

One of the key questions in smooth 4-manifold topology is whether a fixed homeomorphism type containing a smooth 4-manifold must actually contain infinitely many diffeomorphism types. The idea of this section is to state and prove a general theorem pointing in this direction which may be useful to those who are constructing exotic 4-manifolds.

To state our theorem, we need to discuss some notation related to surgery on a torus with trivial normal bundle. Suppose that T is such a torus with tubular neighborhood N_T. Let α and β be generators of $\pi_1(T^2)$ and let S^1_α and S^1_β be loops in $T^3=\partial N_T$ homologous in N_T to α and β respectively. Let μ_T denote a meridional circle to T in X. By *p/q-surgery* on T with respect to β we mean

$$X_{T,\beta}(p/q)=(X\smallsetminus N_T)\cup_\varphi (S^1\times S^1\times D^2),$$
$$\varphi: S^1\times S^1\times \partial D^2\to \partial(X\smallsetminus N_T)$$

where the gluing map satisfies $\varphi_*([\partial D^2])=q[S^1_\beta]+p[\mu_T]$ in $H_1(\partial(X\smallsetminus N_T);\mathbf{Z})$. We denote the 'core torus' $S^1\times S^1\times\{0\}\subset X_{T,\beta}(p/q)$ by $T_{p/q}$.

Note we have framed N_T using S^1_α and S^1_β so that the pushoffs of α and β in this framing are S^1_α and S^1_β. When the curve S^1_β is nullhomologous in $X\smallsetminus N_T$, then $H_1(X_{T,\beta}(1/q);\mathbf{Z})=H_1(X;\mathbf{Z})$. In addition, when T itself is nullhomologous, then $H_1(X_{T,\beta}(p/q);\mathbf{Z})=H_1(X;\mathbf{Z})\oplus\mathbf{Z}/p\mathbf{Z}$.

If X is a symplectic manifold and T is any Lagrangian torus, then there is a canonical framing, called the Lagrangian framing, of N_T. This framing is uniquely determined by the property that pushoffs of T in this framing remain Lagrangian. If one performs $1/n$ surgeries with respect to the pushoff in this framing of any curve on T, then the result is also a symplectic manifold. We refer the reader to [ADK] for a full discussion of this phenomenon, which is referred to there as *Luttinger surgery*. However, one must be careful that it is often the case that the pushoff of a curve using the Lagrangian framing may not be nullhomologous, so that a $1/n$ surgery may in fact may change H_1.

Our theorem is:

Theorem 1. *Let X be a smooth closed oriented 4-manifold which contains a nullhomologous torus Λ, and let λ be a simple loop on Λ so that S^1_λ is nullhomologous in $X \smallsetminus N_\lambda$. If the Seiberg-Witten invariant of $X_{\Lambda,\lambda}(0)$ is nontrivial in the sense that for some basic class k_0, $\sum_i SW'_{X_{\Lambda,\lambda}(0)}(k_0 + 2i[\Lambda_0]) \neq 0$, then among the manifolds $\{X_{\Lambda,\lambda}(1/n)\}$, infinitely many are pairwise nondiffeomorphic.*

The meaning of 'SW′' is explained below. The following is a very simple but effective corollary to the proof.

Corollary 1. *Suppose that $X_0 = X_{\Lambda,\lambda}(0)$ has, up to sign, just one Seiberg-Witten basic class. Then the manifolds $X_n = X_{\Lambda,\lambda}(1/n)$, $n = 1, 2, 3, \ldots$ are pairwise nondiffeomorphic.* □

The theorem and corollary are particularly interesting when X is simply connected and if it can be shown that the $\{X_{\Lambda,\lambda}(1/n)\}$ are also simply connected; for then all the $\{X_{\Lambda,\lambda}(1/n)\}$ are homeomorphic.

As outlined in the introduction, one very useful application of the theorem is to start with a model manifold which has nontrivial Seiberg-Witten invariant and with the same euler number and signature as a (say) homologically simply connected manifold that we are trying to construct, but with $b_1 > 0$. Then, provided that we can find them, we surger essential tori which carry generators of H_1. If we can do this b_1 times, we kill H_1. At each stage we wish to to preserve the fact that the Seiberg-Witten invariant should be nonzero in order to satisfy the hypothesis that $X_{\Lambda,\lambda}(0)$ has nontrivial Seiberg-Witten invariant. For example, if we start with a symplectic 4-manifold with $b^+ > 1$ and each time perform a Luttinger surgery on an embedded Lagrangian torus, this will be true. The fact that the next to last manifold in our string of surgeries has nontrivial Seiberg-Witten invariant will allow the application of the theorem and/or its corollary. We will discuss two examples in §3 and §4.

The proof of Theorem 1 involves calculation of Seiberg-Witten invariants. We give a short discussion for the purpose of setting notation. The Seiberg-Witten invariant of a smooth closed oriented 4-manifold X with $b^+_X > 1$ is an integer-valued function SW_X which is defined on the set of *spin^c* structures over X. Corresponding to each *spin^c* structure $\mathfrak{s}$ over X is the bundle of positive spinors $W^+_\mathfrak{s}$ over X. Set $c(\mathfrak{s}) \in H_2(X;\mathbf{Z})$ to be the Poincaré dual of $c_1(W^+_\mathfrak{s})$. Each $c(\mathfrak{s})$ is a characteristic element of $H_2(X;\mathbf{Z})$ (i.e. its Poincaré dual $\hat{c}(\mathfrak{s}) = c_1(W_{\mathfrak{s}}^+)$ reduces to $w_2(X)$ mod 2). We shall work with the modified Seiberg-Witten invariant

$$\mathrm{SW}'_X : \{k \in H_2(X;\mathbf{Z}) \mid \hat{k} \equiv w_2(X) \pmod 2\} \to \mathbf{Z}$$

defined by $\mathrm{SW}'_X(k) = \sum_{c(\mathfrak{s})=k} \mathrm{SW}_X(\mathfrak{s})$. Up to sign, this is a diffeomorphism invariant of X. If $H_1(X;\mathbf{Z})$ has no 2-torsion, then $\mathrm{SW}'_X = \mathrm{SW}_X$.

In case $b_X^+ = 1$, the invariant requires the choice of a class $H \in H_2(X;\mathbf{R})$ with $H \cdot H > 0$. We now need to be a bit more explicit. Suppose we have a given orientation of $H_+^2(X;\mathbf{R})$ and a given metric for X. The Seiberg-Witten invariant depends on the metric g and a self-dual 2-form as follows. There is a unique g-self-dual harmonic 2-form $\omega_g \in H_+^2(X;\mathbf{R})$ with $\omega_g^2 = 1$ and corresponding to the positive orientation. Fix a characteristic homology class $k \in H_2(X;\mathbf{Z})$. Given a pair (A,ψ), where A is a connection in the complex line bundle whose first Chern class is the Poincaré dual $\widehat{k} = \frac{i}{2\pi}[F_A]$ of k and ψ a section of the bundle of self-dual spinors for the associated *spin*^{c} structure, the perturbed Seiberg-Witten equations are:

$$D_A\psi = 0, \quad F_A^+ = q(\psi) + i\eta,$$

where F_A^+ is the self-dual part of the curvature F_A, D_A is the twisted Dirac operator, η is a self-dual 2-form on X, and q is a quadratic function. Write $\mathrm{SW}_{X,g,\eta}(k)$ for the corresponding invariant. As the pair (g,η) varies, $\mathrm{SW}_{X,g,\eta}(k)$ can change only at those pairs (g,η) for which there are solutions with $\psi = 0$. These solutions occur for pairs (g,η) satisfying $(2\pi\widehat{k} + \eta)\cdot\omega_g = 0$. This last equation defines a wall in $H^2(X;\mathbf{R})$.

The point ω_g determines a component of the double cone consisting of elements of $H^2(X;\mathbf{R})$ of positive square. We prefer to work with $H_2(X;\mathbf{R})$. The dual component is determined by the Poincaré dual H of ω_g. An element $H' \in H_2(X;\mathbf{R})$ of positive square lies in the same component as H if $H'\cdot H > 0$. If $(2\pi\widehat{k}+\eta)\cdot\omega_g \neq 0$ for a generic η, $\mathrm{SW}_{X,g,\eta}(k)$ is well-defined, and its value depends only on the sign of $(2\pi\widehat{k}+\eta)\cdot\omega_g$. Write $\mathrm{SW}^+_{X,H}(k)$ for $\mathrm{SW}_{X,g,\eta}(k)$ if $(2\pi\widehat{k}+\eta)\cdot\omega_g > 0$ and $\mathrm{SW}^-_{X,H}(k)$ in the other case.

The invariant $\mathrm{SW}_{X,H}(k)$ is defined by $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^+_{X,H}(k)$ if $(2\pi\widehat{k})\cdot\omega_g > 0$, or dually, if $k\cdot H > 0$, and $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^-_{X,H}(k)$ if $k \cdot H < 0$. As in the case above, we work with the modified invariant $\mathrm{SW}'_{X,H}(k) = \sum_{c(\mathfrak{s})=k} \mathrm{SW}_{X,H}(\mathfrak{s})$.

We now proceed to the proof of the theorem.

Proof of Theorem 1. Set $X_0 = X_{\Lambda,\lambda}(0)$. Recall that Λ_0 is the torus in X_0 which is the core torus of the surgery. Note that $\Lambda_0\cdot\Lambda_0 = 0$, and that Λ_0 is essential (in fact primitive in H_2) because the surface formed from the the normal disk to Λ_0

together with the surface in $X_0 \smallsetminus N_{\Lambda_0} = X \smallsetminus N_\Lambda$ bounded by S^1_λ intersects Λ_0 in a single point.

Similarly, set $X_n = X_{\Lambda,\lambda}(1/n)$. Then $\Lambda_{1/n}$ is the core torus of the surgery in X_n. Its meridian $\mu_{\Lambda_{1/n}}$ represents the class of $n[\lambda]+[\mu_\Lambda]$, which in $X_n \smallsetminus N_{\Lambda_{1/n}} = X \smallsetminus N_\Lambda$ is homologous to $[\mu_\Lambda]$, a nontorsion class. This means that $\Lambda_{1/n}$ is nullhomologous in X_n.

Let $k_0 \in H_2(X_0;\mathbf{Z})$ be a basic class; *i.e.* $\mathrm{SW}'_{X_0}(k_0) \neq 0$. The adjunction inequality implies that k_0 is orthogonal to Λ_0. Thus, there are (unique, because Λ (resp. $\Lambda_{1/n}$) are nullhomologous) corresponding homology classes k_n and k in $H_2(X_n;\mathbf{Z})$ and $H_2(X;\mathbf{Z})$, respectively, where k agrees with the restriction of k_0 in $H_2(X \smallsetminus N_\Lambda, \partial; \mathbf{Z})$ in the diagram:

$$\begin{array}{ccc}
H_2(X;\mathbf{Z}) & \longrightarrow & H_2(X, N_\Lambda;\mathbf{Z}) \\
 & & \downarrow \cong \\
 & & H_2(X \smallsetminus N_\Lambda, \partial;\mathbf{Z}) \\
 & & \uparrow \cong \\
H_2(X_0;\mathbf{Z}) & \longrightarrow & H_2(X_0, N_T;\mathbf{Z})
\end{array}$$

and similarly for k_n.

It follows from [MMS] that

$$\mathrm{SW}'_{X_n}(k_n) = \mathrm{SW}'_X(k) + n\sum_i \mathrm{SW}'_{X_0}(k_0 + 2i[\Lambda_0])$$

and that these comprise all the basic classes of X_n. It is then clear that the integer invariants

$$S_n = \max\{|\mathrm{SW}'_{X_n}(k_n)|;\ k_n \text{ basic for } X_n\}$$

will distinguish an infinite family of pairwise nondiffeomorphic manifolds among the X_n.

In case $b^+_X = 1$, we need to check issues with chambers. The inclusions of $X \smallsetminus N_\Lambda$ in X and X_n induces isomorphisms on H_2 and thus an isomorphism of $H_2(X_n;\mathbf{Z})$ with $H_2(X;\mathbf{Z})$. The gluing formula of [MMS] relates chambers using this isomorphism. So, for example, if $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^+_{X,H}(k)$ this means that $k \cdot H > 0$. The isomorphism above gives a $k_n \in H_2(X_n;\mathbf{Z})$ and an $H \in H_2(X_n;\mathbf{R})$ (and $H_n \cdot H_n = H \cdot H > 0$), and it also gives $k_n \cdot H_n = k \cdot H > 0$; so $\mathrm{SW}_{X_n,H_n}(k_n) = \mathrm{SW}^+_{X_n,H_n}(k_n)$. Thus the gluing formula applies to the invariant $\mathrm{SW}_{X,H}$. This works for any choice of period point H. Hence the argument in the $b^+ > 1$ case applies directly to $b^+ = 1$ as well. □

Corollary 2. *Let X be a smooth closed oriented 4-manifold which contains a nullhomologous torus Λ and let λ be a simple loop on Λ so that S^1_λ is nullhomologous in $X \smallsetminus N_\lambda$. Suppose also that there is a square 0 torus $T \subset X_0$ that satisfies $T \cdot \Lambda_0 \neq 0$. If $\mathcal{SW}'_{X_0} \neq 0$, then among the manifolds $\{X_{\Lambda,\lambda}(1/n)\}$, infinitely many are pairwise nondiffeomorphic.*

Proof. If $T \subset X_0$ is a torus of square 0 that satisfies $T \cdot \Lambda_0 \neq 0$, then the adjunction inequality implies that in each collection $\{k_0 + 2i\Lambda_0\}$, there is at most one basic class. Thus $\mathcal{SW}'_{X_0} \neq 0$ implies that the hypothesis of Theorem 1 is satisfied. □

We now provide two examples to illustrate the reverse engineering procedure.

3. Fake $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$'s

Let $Y = Sym^2(\Sigma_3)$ be the 2-fold symmetric product of genus 3 surfaces. Recall that $\pi_1(Sym^2(\Sigma_3)) = H_1(\Sigma_3; \mathbf{Z})$. So $b_1(Y) = 6$ and also $\mathrm{e}(Y) = 6$, $\mathrm{sign}(Y) = -2$, and $b_2(Y) = 16$.

The symmetric product $Y = Sym^2(\Sigma_3)$ is the quotient of $\Sigma_3 \times \Sigma_3$ by the action of the involution $t : \Sigma_3 \times \Sigma_3 \to \Sigma_3 \times \Sigma_3$ given by $t(x, y) = (y, x)$. Let $\varphi : \Sigma_3 \times \Sigma_3 \to Y$ be the quotient map, and let $\{a_i, b_i\}$, $i = 1, 2, 3$ denote standard generators for $\pi_1(\Sigma_3)$. It follows from [P] that the natural singular Kähler form $Sym^2(\omega)$ on Y derived from the Kähler curve (Σ_3, ω) admits a cohomologous smoothing to a Kähler form which equals $Sym^2(\omega)$ away from a chosen neighborhood of the diagonal. (We thank Paul Kirk for pointing out the necessity of this reference.)

We obtain a basis for $H_2(Y; \mathbf{Z})$ as follows. The tori $a_i \times a_j$, $b_i \times b_j$, $a_i \times b_j$, and $b_i \times a_j$, $i < j$, in $\Sigma_3 \times \Sigma_3$ descend to twelve Lagrangian tori in Y, and we also denote these by $a_i \times a_j$, $b_i \times b_j$, $a_i \times b_j$, and $b_i \times a_j$. The three tori $a_i \times b_i$ in $\Sigma_3 \times \Sigma_3$ descend to tori T_i of square -1, and together with the image of $\{\mathrm{pt}\} \times \Sigma_3 \cup \Sigma_3 \times \{\mathrm{pt}\}$, a genus three surface which represents a homology class b with self-intersection $+1$, we get a basis for $H_2(Y; \mathbf{Z})$. The euler number $\mathrm{e}(Y) = 6$ and its signature $\mathrm{sign}(Y) = -2$, in agreement with the characteristic numbers for $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$.

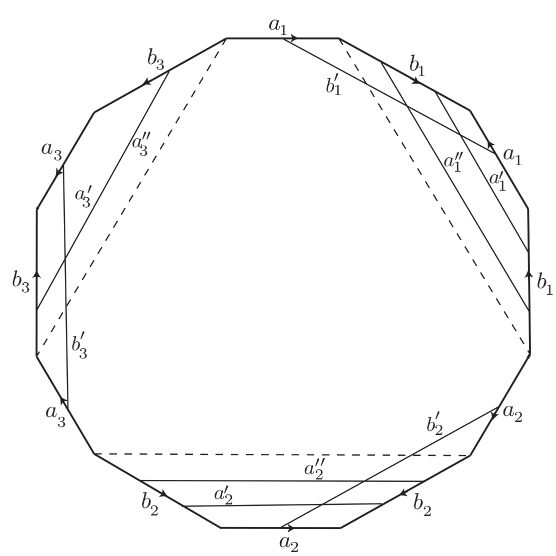

FIGURE 1.

To establish some notation, consider Figure 1. For example, we see loops a_i, a_i', and a_i''. We also have based loops (with basepoint x, the vertex) which we shall denote by the same symbols. The based loop a_2', for example, is the one which starts at the lower left vertex x proceeds backwards along b_2 to the initial point of a_2', then traverses a_2' and heads vertically downward back to x. The based loop a_2'' starts at the vertex at the initial point of b_2, travels along b_2, traverses a_2'', then heads upward back to x.

The abelian group $\pi_1(Y) = \mathbf{Z}^6$ is generated by the $a_i = a_i \times \{x\}$ and $b_j = b_j \times \{x\}$, where the basepoint x is the vertex. We will perform six surgeries on disjoint Lagrangian tori to kill these generators. We need some notation to describe our surgeries. For example, if we consider the torus $a_1' \times a_2'$ and we do n-framed surgery along the loop a_1' with respect to the Lagrangian framing, we denote this as the surgery $(a_1' \times a_2', a_1', n)$. We now perform surgeries along disjoint Lagrangian tori

$$(a_1' \times a_2', a_2', -1), \quad (a_1'' \times b_2', b_2', -1), \quad (a_1' \times a_3', a_1', -1),$$
$$(b_1' \times a_3'', b_1', -1), \quad (a_2' \times a_3', a_3', -1), \quad (a_2'' \times b_3', b_3', -1).$$

Lemma 1. *In the complement of the above six Lagrangian tori in $Y = Sym^2(\Sigma_3)$, the Lagrangian framings give the following product decomposition of the 3-torus boundaries of the tubular neighborhoods of these tori.*

$$\begin{aligned} &a_1 \times a_2 \times [b_1^{-1}, b_2^{-1}], \quad && (b_1 a_1 b_1^{-1}) \times b_2 \times [b_1, a_2^{-1}],\\ &a_1 \times a_3 \times [b_1^{-1}, b_3^{-1}], && b_1 \times (b_3 a_3 b_3^{-1}) \times [a_1^{-1}, b_3],\\ &a_2 \times a_3 \times [b_2^{-1}, b_3^{-1}], && (b_2 a_2 b_2^{-1}) \times b_3 \times [b_2, a_3^{-1}]. \end{aligned}$$

Proof. (cf. [BK1]) Let g be one of a_i, b_i. The Lagrangian pushoff of g' is easily seen to be g. The trapezoid in Figure 1 with the top side equal to g' and the opposite side the parallel copy of g gives the desired homotopy which has a Lagrangian image at each time t. Of course, these trapezoids overlap one another in Figure 1, but taking into consideration the points in the other factor we see that they are disjoint.

Just as an example, consider the torus $a_1' \times a_2'$, let $\delta(t)$ be the path from the basepoint x to the initial point of a_2' as discussed above, and let $\gamma(t)$ be a similar path starting at x and ending at the initial point of a_1'. For each $s \in [0,1]$, let $\gamma_s(t) = \gamma(st)$ and $\delta_s(t) = \delta(st)$. The trapezoid for a_1', thought of as a homotopy, is composed of parallel paths $a_1'(t)$, where $a_1'(0) = a_1'$, $a_1'(1) = a_1$, and the initial point of $a_1'(t)$ is $\gamma(1-t)$. This trapezoid gives rise to the based homotopy of a_1' to a_1 whose path at level s is the image in Y of the product of paths

$$(\gamma_{1-s} \times \delta_{1-s}) \cdot (a_1'(s) \times \{\delta(1-s)\}) \cdot (\gamma_{1-s}^{-1} \times \delta_{1-s}^{-1}).$$

The Lagrangian pushoffs of a_1'', a_2'', a_3'' are respectively $b_1 a_1 b_1^{-1}$, $b_2 a_2 b_2^{-1}$, and $b_3 a_3 b_3^{-1}$. These three pushoffs are represented by the dotted circles in Figure 1. The trapezoid with one side equal to g'' and opposite side equal to a dotted line gives the necessary homotopy (based in a similar fashion as above). The homotopies in Y between the dotted circles and $b_1a_1b_1^{-1}$, $b_2a_2b_2^{-1}$, and and $b_3a_3b_3^{-1}$ are *not* required to be Lagrangian pushoffs; so we may perform the homotopy after pushing off to be disjoint from the Lagrangian tori.

A meridian to each of these six Lagrangian tori comes from the orthogonal punctured torus that lies in the complement of the six tori. Each of the orthogonal tori are also Lagrangian, so the commutator of its π_1 generators indeed bounds a normal disk in the Lagrangian framing. This torus contains the basepoint x and also needs to contain the part of the base path $\varphi(\gamma \times \delta)$ which runs from the basepoint $\varphi(x, x)$ to the basepoint of the orthogonal torus which can be taken as the intersection point of the orthogonal torus with the original torus. (This is because in order to see that the based boundary curve of $\pi_1(T^2 \setminus D^2)$ is given by the commutator of the generators of $\pi_1(T^2)$, this based boundary must lie in the torus.)

For example, consider the torus $a_1' \times a_2'$. The based loop $\gamma \times \delta$ runs backwards along both b_1 and b_2 to the point of intersection of $a_1' \times a_2'$ with its orthogonal torus $b_1 \times b_2$. It follows that the path in $b_1 \times b_2$ which is discussed above is $\gamma^{-1} \times \delta^{-1}$, and that the meridian to $a_1 \times a_2$ is $[b_1^{-1}, b_2^{-1}]$. (There is a choice of orientation for this meridian. The opposite choice would cause us to change the signs of our surgeries. For definiteness, we choose the orientation implied by the statement of the lemma.)

For the torus $a_1'' \times b_2'$ we see that the appropriate path runs positively along b_1 and negatively along a_2, and the same reasoning as above gives us $[b_1, a_2^{-1}]$ as the meridian of $a_1'' \times b_2'$. $\square$

Denote the result of these surgeries by X. The result of each surgery is to reduce $b_1(Y)$ by one, reduce $b_2(Y)$ by two, and introduce a relation in π_1. For example, because of Lemma 1, the surgery $(a_1' \times a_2', a_2', -1)$ introduces the relation $a_2 = [b_1^{-1}, b_2^{-1}]$, and the surgery $(a_1'' \times b_2', b_2', -1)$ introduces the relation $b_2 = [b_1, a_2^{-1}]$. Note that $b_1(X) = 0$, and since the surgeries change neither the euler number nor signature, $b_2 = 4$, and X is a homology $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$.

The following relations hold in $\pi_1(X)$:

$$
\begin{gathered}
a_2 = [b_1^{-1}, b_2^{-1}], \quad b_2 = [b_1, a_2^{-1}], \quad a_1 = [b_1^{-1}, b_3^{-1}], \\
b_1 = [a_1^{-1}, b_3], \quad a_3 = [b_2^{-1}, b_3^{-1}], \quad b_3 = [b_2, a_3^{-1}], \\
[a_1, b_1] = 1, \quad [a_1, a_2] = 1, \quad [a_1, b_2] = 1, \quad [a_1, a_3] = 1, \quad [b_1, a_3] = 1, \\
[a_2, b_2] = 1, \quad [a_2, a_3] = 1, \quad [a_2, b_3] = 1, \quad [a_3, b_3] = 1.
\end{gathered}
$$

Thus we have $b_2 = [b_1, a_2^{-1}] = [[a_1^{-1}, b_3], a_2^{-1}] = 1$, using the commutativity relations $[a_2, b_3] = 1$ and $[a_1, a_2] = 1$. Now it follows from the other relations that $\pi_1(X) = 1$. The fact that the fundamental group of Y is abelian was essential for this simple computation.

Since the surgeries that we perform on the Lagrangian tori all have surgery coefficient ± 1 with respect to the Lagrangian framing, the resultant manifolds all have induced symplectic structures. One simple way to see that X is not diffeomorphic to $\mathbf{CP}^2\#\,3\overline{\mathbf{CP}}^2$ is to use the fact from [LL] that $\mathbf{CP}^2\#\,3\overline{\mathbf{CP}}^2$ has a unique symplectic form up to diffeomorphism and symplectic deformation. This means that for any symplectic form on $\mathbf{CP}^2\#\,3\overline{\mathbf{CP}}^2$, the canonical class must pair negatively with the symplectic form. On $Y = Sym^2(\Sigma_3)$, which is a surface of general type, the canonical class pairs positively with the symplectic form, and since we have constructed X by surgeries on Lagrangian tori of Y, the same is still true in X. (The point here is that if $\hat{Y}$ is the result of a Luttinger surgery on Y, then the complements of tubular neighborhoods of the respective Lagrangian tori in each can be identified, and the restrictions of the symplectic forms can as well. The canonical classes are supported in the complements of these tori and agree over the complements of the tubular neighborhoods. It follows that as elements of $H_2(\hat{Y};\mathbf{Z})$ the Poincaré duals satisfy $\hat{K} = K + n[\hat{T}]$. Cf. [ADK]. So $\hat{K}\cdot\hat{\omega} = K\cdot\omega$ since $\hat{T}$ is Lagrangian.) Hence X cannot be diffeomorphic to $\mathbf{CP}^2\#\,3\overline{\mathbf{CP}}^2$.

Theorem 2 (cf. [AP, BK1])**.** *The symplectic manifold X is irreducible and homeomorphic but not diffeomorphic to $\mathbf{CP}^2\#\,3\overline{\mathbf{CP}}^2$.*

The irreducibility of X follows from [HK] once we show that X is minimal. This follows from the Seiberg-Witten calculations below. It is interesting to ask whether X is actually diffeomorphic to the symplectic manifolds constructed in [AP, BK1].

In order to produce an infinite family of exotic $\mathbf{CP}^2\#\,3\overline{\mathbf{CP}}^2$'s, let X' denote the result of the first seven Luttinger surgeries on Y. Thus $b_1(X') = 1$ and $b_2(X') = 6$. We construct X by performing a surgery $(a_2''\times b_3', b_3', -1)$ in X'. In X, the surgery gives us a nullhomologous torus Λ, the "core" of the surgery. There is a loop λ on Λ so that surgery on (Λ,λ) gives X' back. The framing for this surgery must be the nullhomologous framing. We apply Theorem 1 to (X,Λ,λ). In fact, Corollary 1 will tell us that the manifolds $X_{\Lambda,\lambda}(1/n)$ are pairwise nondiffeomorphic once we see that X' has exactly two basic classes. Note that $X_n = X_{\Lambda,\lambda}(1/n)$ is the result of performing the surgery $(a_2''\times b_3', b_3', n+1)$ in X'. (These are *not* Luttinger surgeries.)

Theorem 3. *The manifolds $\mathbf{CP}^2\#\,3\overline{\mathbf{CP}}^2$, X, and X_n, $n\geq 2$, are pairwise homeomorphic and are minimal, but no two are diffeomorphic.*

Proof. The homeomorphism statement will follow once we see that each X_n is simply connected. A presentation for $\pi_1(X_n)$ is obtained from the one above for $\pi_1(X)$ by replacing the relation $b_3 = [b_2, a_3^{-1}]$ by $b_3 = [b_2, a_3^{-1}]^{-(n+1)}$, and $\pi_1(X_n) = 1$ follows as above.

Next, we need to show that the manifold X' has just two basic classes, $\pm$ its canonical class, and then call on Corollary 1. Since Y is a surface of general type, its only basic classes are $\pm$ its canonical class, i.e. $3b+T_1+T_2+T_3$. According to [MMS], each time we do a surgery, the Seiberg-Witten invariant of the result is calculated in terms of the Seiberg-Witten invariants of the original manifold and those of the result of the surgery that kills the curve on the torus. For example, if Y_1 is the result of the surgery $(a_1'\times a_2', a_2', -1)$ on Y, then let Z be the result of the surgery that kills a_2' directly (0-surgery). In Z, the surface $\varphi(\Sigma_3\times\{\mathrm{pt}\})$, which represents b, has its genus reduced by one. Applying the adjunction inequality to this situation, we see that any basic class of Z has the form $\pm b\pm T_1\pm T_2\pm T_3$. Since the square of a basic class must be $3\,\mathrm{sign}(Z)+2\,\mathrm{e}(Z)=6$, in fact none of these classes can be basic; so the Seiberg-Witten invariant of Z vanishes. The result of this argument is that the manifold Y_1 also has just two basic classes, $\pm$ its canonical class. The very same argument works for each surgery and finally shows that X' has just two basic classes.

Thus X and X_n have just two basic classes, $\pm k_n$, and the difference is a class of square $(2k_n)^2=24$. If one of these manifolds were minimal, it would have to have a pair of basic classes, $k\pm E$, whose difference has square -4. Thus X and X_n are minimal. $\square$

In order to obtain infinitely many smooth structures, we did not need to perform this last step which shows that X and X_n have just two basic classes. We did this to explicitly show that all the X_n are distinct. The hypothesis of Corollary 2 is satisfied because each of the Lagrangian tori on which surgery is performed has a dual Lagrangian torus.

Each X_n contains disjoint embeddings of a minimal genus 3 surface representing b (the image of $\Sigma_3\times\{\mathrm{pt}\}\cup\{\mathrm{pt}\}\times\Sigma_3$ in $Sym^2(\Sigma_3)$) and the three tori T_1, T_2, T_3 with self-intersection -1. More interestingly, each X_n contains a sphere of self-intersection -2 representing $b-[T_1]-[T_2]-[T_3]$ that is the image in $Sym^2(\Sigma_3)$ of a pushoff of the diagonal in $\Sigma_3\times\Sigma_3$. These surfaces can be useful for other constructions.

The symmetric product $Z_\ell=Sym^2(\Sigma_\ell)$ of a genus ℓ surface has $\pi_1(Sym^2(\Sigma_\ell))=H_1(\Sigma_\ell;\mathbf{Z})$ and that $\mathrm{e}(Z_\ell)=(\ell-1)(2\ell-3)=2\ell^2-5\ell+3$, $\mathrm{sign}(Z_\ell)=1-\ell$, and $b_2(Z_\ell)=2\ell^2-\ell+1$. Thus Z_ℓ is a model for $(\ell^2-3\ell+1)\mathbf{CP}^2\#(\ell^2-2\ell)\overline{\mathbf{CP}}^2$. A straightforward generalization of the above application of reverse engineering provides infinitely many distinct smooth structures on these manifolds, one of which is symplectic.

4. Fake homology $S^2\times S^2$'s

We now give an example to point out that the computation of fundamental groups in the reverse engineering procedure can be difficult.

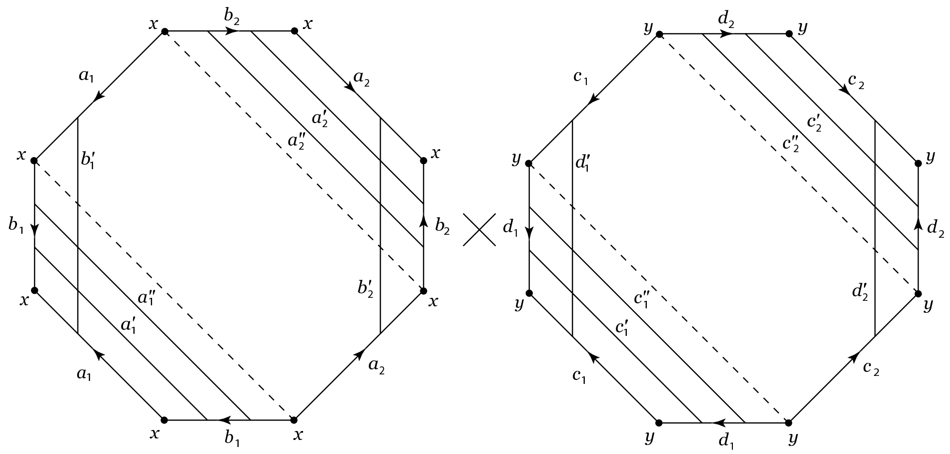

FIGURE 2.

Let $Y = \Sigma_2 \times \Sigma_2$, the product of two genus 2 surfaces, and denote the standard generators of π_1 by $\{a_i, b_i\}$ and $\{c_i, d_i\}$ for $i = 1, 2$. So $\pi_1(Y)$ has these eight generators with relations $[a_1, b_1][a_2, b_2] = 1$, $[c_1, d_1][c_2, d_2] = 1$ and all a_i and b_i commute with all c_j and d_j. The area forms on the two copies of Σ_2 induce a symplectic form on the product Y, all tori of the form $a_i \times c_j$, $a_i \times d_j$, $b_i \times c_j$, and $b_i \times d_j$ are Lagrangian, and the Lagrangian framing is the obvious one coming from the product structure. The euler number $\mathrm{e}(Y) = 4$ and its signature $\mathrm{sign}(Y) = 0$, in agreement with the characteristic numbers for $S^2 \times S^2$.

Perform eight Luttinger surgeries along the Lagrangian tori

$$\begin{array}{llll}(a_1' \times c_1', a_1', -1), & (b_1' \times c_1'', b_1', -1), & (a_2' \times c_2', a_2', -1), & (b_2' \times c_2'', b_2', -1), \\ (a_2' \times c_1', c_1', +1), & (a_2'' \times d_1', d_1', +1), & (a_1' \times c_2', c_2', +1), & (a_1'' \times d_2', d_2', +1).\end{array}$$

to obtain a symplectic manifold X. (See Figure 2.)

Arguing as in §3 we have

Lemma 2. *Let Y' be the complement of the above eight Lagrangian tori in $Y = \Sigma_2 \times \Sigma_2$. Inside Y', the Lagrangian framings give the following product decomposition of the 3-torus boundaries of the tubular neighborhoods of the above eight tori.*

$$\begin{array}{ll} a_1 \times c_1 \times [b_1^{-1}, d_1^{-1}], & b_1 \times (d_1 c_1 d_1^{-1}) \times [a_1^{-1}, d_1], \\ a_2 \times c_2 \times [b_2^{-1}, d_2^{-1}], & b_2 \times (d_2 c_2 d_2^{-1}) \times [a_2^{-1}, d_2], \\ a_2 \times c_1 \times [b_2^{-1}, d_1^{-1}], & (b_2 a_2 b_2^{-1}) \times d_1 \times [b_2, c_1^{-1}], \\ a_1 \times c_2 \times [b_1^{-1}, d_2^{-1}], & (b_1 a_1 b_1^{-1}) \times d_2 \times [b_1, c_2^{-1}]. \end{array}$$

The following relations hold in $\pi_1(X)$:

$$\begin{gathered}
[b_1^{-1}, d_1^{-1}] = a_1, \quad [a_1^{-1}, d_1] = b_1, \quad [b_2^{-1}, d_2^{-1}] = a_2, \quad [a_2^{-1}, d_2] = b_2, \\
[d_1^{-1}, b_2^{-1}] = c_1, \quad [c_1^{-1}, b_2] = d_1, \quad [d_2^{-1}, b_1^{-1}] = c_2, \quad [c_2^{-1}, b_1] = d_2, \\
[a_1, c_1] = 1, \quad [a_1, c_2] = 1, \quad [a_1, d_2] = 1, \quad [b_1, c_1] = 1, \\
[a_2, c_1] = 1, \quad [a_2, c_2] = 1, \quad [a_2, d_1] = 1, \quad [b_2, c_2] = 1, \\
[a_1, b_1][a_2, b_2] = 1, \quad [c_1, d_1][c_2, d_2] = 1.
\end{gathered}$$

Note that $b_1(X) = 0$, and since the surgeries change neither the euler number nor signature, $b_2 = 2$. In fact, the only homology classes that survive are those represented by $\Sigma_2 \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times \Sigma_2$; so X is a homology $S^2 \times S^2$.

We have been unable to determine if the perfect group $\pi_1(X)$ is trivial or not. Also, there are other surgeries that can be performed, and also other sets of eight Lagrangian tori that can be surgered, to obtain many presentations of perfect groups that we have not succeeded in showing are trivial. The fact that the fundamental group of our model manifold is not abelian makes these computations difficult.

We can produce an infinite family of distinct homology $S^2 \times S^2$'s, exactly as in §3. The presentation for π_1 is exactly the one given above for $\pi_1(X)$ except that the relation $[c_2^{-1}, b_1] = d_2$ is replaced by $[c_2^{-1}, b_1]^{n+1} = d_2$ compounding the difficulty of determining whether the group is trivial.

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