

Six Lectures on Four 4-Manifolds

Ronald Fintushel and Ronald J. Stern

Six Lectures on Four 4-Manifolds

Ronald Fintushel and Ronald J. Stern

Introduction

Despite spectacular advances in defining invariants for simply connected smooth and symplectic 4-dimensional manifolds and the discovery of effective surgical techniques, we still have been unable to classify simply connected smooth manifolds up to diffeomorphism. In these notes, adapted from six lectures given at the 2006 Park City Mathematics Institute Graduate Summer School on *Low Dimensional Topology*, we will review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply connected 4-manifolds. We will focus on those surgical techniques that have been effective in altering smooth and symplectic structures and the Seiberg-Witten invariants that are used to distinguish them. In the last lecture we will then pose a possible classification scheme and test it on a few examples.

Department of Mathematics, Michigan State University
East Lansing, Michigan 48824

E-mail address: ronfint@math.msu.edu

Department of Mathematics, University of California
Irvine, California 92697

E-mail address: rstern@math.uci.edu

The first author was partially supported NSF Grant DMS0305818 and the second author by NSF Grant DMS0505080

1991 *Mathematics Subject Classification*. Primary 57R55, 57R57, 14J26; Secondary 53D05

Key words and phrases.

4-manifold, Seiberg-Witten invariant, symplectic, Lagrangian

LECTURE 1

How to construct 4-manifolds

The classification of smooth 4-manifolds remains a fascinating black box that has yet to open. You shake it, probe it, and sometimes the box mystically falls open only to find a pair of glasses and a companion black box that itself cannot be opened in any obvious way. These newfound glasses, lately fitted for physicists, usually provide a different vantage point from which to view the new and now even more fascinating black box. Again, you shake it and probe it with renewed vigor and enthusiasm.

For example, you could look at smooth 4-manifolds through the glasses of a high dimensional topologist. There it is known that every closed manifold of dimension greater than four has only finitely many smooth structures [27]. It is also known that for $n > 2$ there exist infinitely many topological manifolds that have the homotopy type of \mathbf{CP}^n [59]. By contrast, in dimension four many (and perhaps all) 4-manifolds have infinitely many distinct smooth structures [13], and there exist only two topological manifolds homotopy equivalent to \mathbf{CP}^2 [19].

You could also look at smooth 4-manifolds through the glasses of a complex algebraic geometer. There it is known that except for the elliptic and Hirzebruch surfaces every simply connected complex surface has finitely many deformation types. However, as already mentioned, it is known that many (perhaps all) simply connected smooth 4-manifolds have infinitely many distinct smooth structures. In addition, there are simply connected smooth (and irreducible) manifolds not even homeomorphic to a complex surface [12].

Viewing 4-manifolds through the glasses of a symplectic geometer also misses many manifolds. While all known irreducible simply connected smooth 4-manifolds (other than S^4) are homeomorphic (possibly changing orientation) to a symplectic manifold, for these examples infinitely many of the smooth structures which they support admit no underlying symplectic structure [13]. Thus, viewing 4-manifolds through the glasses of either a symplectic or complex geometer will not recover the full story. However, the amalgam of ideas from complex and symplectic geometry has provided surgical techniques that are useful in the study of smooth 4-manifolds.

Since this is a summer school in low-dimensional topology, we will look at 4-manifolds from the vantage point of a 2- and 3-manifold topologist, consider operations successful in those dimensions, and then study their analogues in dimension 4. We then discuss their effect on the smooth invariants *du jour*, the Seiberg-Witten invariants.

In this first lecture we list the invariants that arise from algebraic topology and then those surgical constructions that have proven useful in constructing and altering smooth and symplectic structures on 4-manifolds. We finish the lecture by describing some important manifolds that will be useful in later lectures.

1. Algebraic Topology

The critical algebraic topological information for a closed, smooth, simply connected, oriented 4-manifold X is encoded in its Euler characteristic $e(X)$, its signature $\sigma(X)$, and its type $t(X)$ (0 if the intersection form of X is even and 1 if it is odd). These invariants completely classify the homeomorphism type of X ([7, 19]). We recast these algebraic topological invariants by defining $\chi_h(X) = (e(X) + \sigma(X))/4$ and $c(X) = 3\sigma(X) + 2e(X)$. When X is a complex surface these are the holomorphic Euler characteristic and the self-intersection of the first Chern class of X , respectively. If \overline{X} denotes X with the opposite orientation, then $(c, \chi_h, t)(\overline{X}) = (48\chi_h - 5c, 5\chi_h - \frac{c}{2}, t)(X)$

2. Techniques used for the construction of simply connected smooth and symplectic 4-manifolds

We now define surgical operations that mimic constructions in lower dimensions.

Surgery on a torus. This operation is the 4-dimensional analogue of Dehn surgery. Assume that X contains a homologically essential torus T with self-intersection zero. Let N_T denote a tubular neighborhood of T . Deleting the interior of N_T and regluing $T^2 \times D^2$ via a diffeomorphism $\varphi : \partial(T^2 \times D^2) \rightarrow \partial(X - \text{int } N_T) = \partial N_T$ we obtain a new manifold X_φ , the result of surgery on X along T . The manifold X_φ is determined by the homology class $\varphi_*[\partial D^2] \in H_1(\partial(X \setminus N_T); \mathbf{Z})$. Fix a basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial(X \setminus N_T); \mathbf{Z})$, then there are integers p, q, r , such that $\varphi_*[\partial D^2] = p\alpha + q\beta + r[\partial D^2]$. We sometimes write $X_\varphi = X_T(p, q, r)$. It is often the case that $X_\varphi = X_T(p, q, r)$ only depends upon r , e.g. T is contained in a cusp neighborhood, i.e. α and β can be chosen so that they bound vanishing cycles in $(X - \text{int } N_T)$. We will sometimes refer to this process as a *generalized logarithmic transform* or an *r-surgery* along T .

If the complement of T is simply connected and $t(X) = 1$, then $X_\varphi = X_T(p, q, r)$ is homeomorphic to X . If the complement of T is simply connected and $t(X) = 0$, then X_φ is homeomorphic to X if r is odd, otherwise X_φ has the same c and χ_h but with $t(X_\varphi) = 1$.

Knot surgery. This operation is the 4-dimensional analogue of sewing in a knot complement along a circle in a 3-manifold. Let X be a 4-manifold which contains a homologically essential torus T of self-intersection 0, and let K be a knot in S^3 . Let $N(K)$ be a tubular neighborhood of K in S^3 , and let $T \times D^2$ be a tubular neighborhood of T in X . Then the knot surgery manifold X_K is defined by

$$X_K = (X \setminus (T \times D^2)) \cup (S^1 \times (S^3 \setminus N(K)))$$

The two pieces are glued together in such a way that the homology class $[\text{pt} \times \partial D^2]$ is identified with $[\text{pt} \times \lambda]$ where λ is the class of a longitude of K . If the complement of T in X is simply connected, then X_K is homeomorphic to X .

Fiber sum. This operation is a 4-dimensional analogue of sewing together knot complements in dimension 3, where a knot in dimension 4 is viewed as an embedded surface. Assume that two 4-manifolds X_1 and X_2 each contain an embedded genus g surface $F_j \subset X_j$ with self-intersection 0. Identify tubular neighborhoods N_{F_j} of F_j with $F_j \times D^2$ and fix a diffeomorphism $f : F_1 \rightarrow F_2$. Then the fiber sum $X = X_1 \#_f X_2$ of (X_1, F_1) and (X_2, F_2) is defined as $X_1 \setminus N_{F_1} \cup_\varphi X_2 \setminus N_{F_2}$, where

φ is $f \times$ (complex conjugation) on the boundary $\partial(X_j \setminus N_{F_j}) = F_j \times S^1$. We have

$$(c, \chi_h)(X_1 \#_f X_2) = (c, \chi_h)(X_1) + (c, \chi_h)(X_2) + (8g - 8, g - 1)$$

Also $t(X_1 \#_f X_2) = 1$ unless F_j is characteristic in X_j , $j = 0, 1$.

Branched covers. A smooth proper map $f : X \rightarrow Y$ is a d -fold branched covering if away from the critical set $B \subset Y$ the restriction $f|_{X \setminus f^{-1}(B)} : X \setminus f^{-1}(B) \rightarrow Y \setminus B$ is a covering map of degree d , and for $p \in f^{-1}(B)$ there is a positive integer m so that the map f is $(z, x) \rightarrow (z^m, x)$ in appropriate coordinate charts around p and $f(p)$. The set B is called the *branch locus* of the branched cover $f : X \rightarrow Y$. In the case of *cyclic* branched covers, i.e. when the index- d subgroup $\pi_1(X \setminus f^{-1}(B)) \subset \pi_1(Y \setminus B)$ is determined by a surjection $\pi_1(Y \setminus B) \rightarrow \mathbf{Z}_d$, and B is a smooth curve in Y , then $e(X) = de(Y) - (d-1)e(B)$ and $\sigma(X) = d\sigma(Y) - \frac{d^2-1}{3d}B^2$, and it follows that

$$(c, \chi_h)(X) = d(c, \chi_h)(Y) - (d-1)e(B)(2, \frac{1}{4}) - \frac{(d^2-1)}{3d}B^2(3, \frac{1}{4})$$

Blowup. This operation is borrowed from complex geometry. Form $X \# \overline{\mathbf{CP}^2}$, where $\overline{\mathbf{CP}^2}$ is the complex projective plane \mathbf{CP}^2 with the opposite orientation.

Rational blowdown. Let C_p be the smooth 4-manifold obtained by plumbing $p-1$ disk bundles over the 2-sphere according to the diagram

$$\begin{array}{ccccccc} -(p+2) & -2 & & & & & -2 \\ \bullet & \bullet & \dots & \dots & \dots & \dots & \bullet \\ u_0 & u_1 & & & & & u_{p-2} \end{array}$$

Then the classes of the 0-sections have self-intersections $u_0^2 = -(p+2)$ and $u_i^2 = -2$, $i = 1, \dots, p-2$. The boundary of C_p is the lens space $L(p^2, 1-p)$ which bounds a rational ball B_p with $\pi_1(B_p) = \mathbf{Z}_p$ and $\pi_1(\partial B_p) \rightarrow \pi_1(B_p)$ surjective. If C_p is embedded in a 4-manifold X then the rational blowdown manifold $X_{(p)}$ of [12] is obtained by replacing C_p with B_p , i.e., $X_{(p)} = (X \setminus C_p) \cup B_p$.

Connected sum. Another operation is the *connected sum* $X_1 \# X_2$ of two 4-manifolds X_1 and X_2 . We call a 4-manifold *irreducible* if it cannot be represented as the connected sum of two manifolds except if one factor is a homotopy 4-sphere. Keep in mind that we do not know if there exist smooth homotopy 4-spheres not diffeomorphic to the standard 4-sphere S^4 and that we have very little understanding of the uniqueness of connect sum decompositions of a reducible 4-manifold.

3. Some examples: Horikawa Surfaces

Starting with the complex projective plane \mathbf{CP}^2 and $S^2 \times S^2$ we can now use these operations to construct many complex surfaces. For example, blowing up \mathbf{CP}^2 nine times we obtain the rational elliptic surface $E(1) = \mathbf{CP}^2 \# 9\overline{\mathbf{CP}^2}$. The reason this is called an elliptic surface is that there is a pencil of cubics in \mathbf{CP}^2 , and by blowing up each of the nine base points (*i.e.* intersection points) of this family of cubics we obtain a family of tori (elliptic curves). Let F be one of these smoothly embedded tori (fibers) with trivial normal bundle. We can then take the fiber sum of n copies $E(1)$ along parallel copies of F to obtain the elliptic surfaces $E(n)$. Note that

$$(c, \chi_h, t)(E(n)) = (0, n, n \pmod{2})$$

The elliptic surface $E(n)$ is also a double cover of $S^2 \times S^2$ branched over four disjoint copies of $S^2 \times \{\text{pt}\}$ together with $2n$ disjoint copies of $\{\text{pt}\} \times S^2$. The resultant branched cover has $8n$ singular points (corresponding to the double points in the branch set), whose neighborhoods are cones on \mathbf{RP}^3 . These are desingularized in the usual way, replacing their neighborhoods with cotangent bundles of S^2 . The result is $E(n)$. Equivalently, $E(n)$ is the 2-fold cover of $S^2 \times S^2$ branched over a complex curve of bidegree $(4, 2n)$ in $S^2 \times S^2$. The horizontal and vertical fibrations of $S^2 \times S^2$ pull back to give (Lefschetz) fibrations of $E(n)$ over \mathbf{CP}^1 . A generic fiber of the vertical fibration is the double cover of S^2 branched over 4 points — a torus. This describes an elliptic fibration on $E(n)$. The generic fiber of the horizontal fibration is the double cover of S^2 branched over $2n$ points, and this gives a genus $n-1$ fibration on $E(n)$. This genus $n-1$ fibration has four singular fibers which are the preimages of the four $S^2 \times \{\text{pt}\}$'s in the branch set together with the spheres of self-intersection -2 arising from desingularization. The generic fiber F of the elliptic fibration meets a generic fiber Σ_{n-1} of the horizontal fibration in two points, $\Sigma_{n-1} \cdot F = 2$.

Likewise, we can consider the the 2-fold cover of $S^2 \times S^2$ branched over a connected curve of bidgree $(6, 2n)$ to obtain the Horikawa surfaces $H(2n-1)$. Note that

$$(c, \chi_h, t)(H(2n-1)) = (4n-8, 2n-1, 1)$$

Horikawa [23] showed that these complex surfaces support at most two deformation inequivalent complex structures. A complex surface in the other deformation type is obtained as follows. Consider the Hirzebruch surfaces F_{2n} . These can be formed by taking the double of the euler class $2n$ disk bundle over the sphere. While they are all diffeomorphic to $S^2 \times S^2$, n classifies the deformation type of the induced complex structure on $S^2 \times S^2$. Each F_{2n} contains a sphere S_+ of self-intersection $2n$ and another sphere S_- of self-intersection $-2n$ disjoint from S_+ . Let B_n be a curve in F_{2n} with two components, B_{n+} and S_- , with B_{n+} representing $5[S_+]$. The 2-fold cover of F_{2n} branched over B_n is the complex surface $H'(4n-1)$ with

$$(c, \chi_h)(H(4n-1)) = (8n-8, 4n-1) = (c, \chi_h)(H'(4n-1))$$

Horikawa has shown that $H(4n-1)$ and $H'(4n-1)$ are the only distinct deformation types of complex surfaces with $c = 2\chi_h - 6$ [23].

Further, note that $t(H'(4n-1)) = n \pmod{2}$, so that if n is odd, $H(4n-1)$ and $H'(4n-1)$ are homeomorphic (and are not homeomorphic if $n = 0 \pmod{2}$ and hence represent two distinct deformation types). It is an open question whether for $n = 1 \pmod{2} > 1$ these surfaces are either diffeomorphic or symplectomorphic. Weak evidence that they are not diffeomorphic is provided by the $n = 1$ case where $H(3)$ is, by construction, the elliptic surface $E(3)$ and is minimal. But $H'(3)$ contains a sphere of self-intersection -1 (the lift of S_-). Hence $H'(3)$ is not minimal and cannot be diffeomorphic to the elliptic surface $H(3)$. Unfortunately for $n > 1$ both $H(2n-1)$ and $H'(2n-1)$ have the same Seiberg-Witten invariants and we are unable to distinguish them. We will discuss these surfaces in the last lecture.

Of course, this construction extends to the 2-fold covers $H(m, n)$ of $S^2 \times S^2$ branched over curves of bidegree $(2m, 2n)$. Then

$$(c, \chi_h)(H(m, n)) = (4(m-2)(n-2), (m-1)(n-1) + 1)$$

It is a fun exercise to determine manifolds homeomorphic to some of the $H(m, n)$ as covers of the Hirzebruch surfaces F_r with branch set a disconnected

curve in the spirit of the Horikawa surfaces that result in different deformation types. As for the Horikawa surfaces, It is an open problem to determine if these homeomorphic complex surfaces are diffeomorphic. We will see in the last lecture, however, that they are all related by a single ± 1 generalized logarithmic transform along homologically essential tori.

4. Complex Surfaces

There are restrictions on $(c, \chi_h)(X)$ for a simply connected complex surface X that is *minimal*, i.e. is not of the form $Y \# \mathbf{CP}^2$. They either have $c = 0$, i.e. the elliptic surfaces $E(n)$ and their logarithmic transforms (which we will further discuss in Lecture 2), or they have $\chi_h = 1$ and $c \geq 0$, i.e. \mathbf{CP}^2 , $S^2 \times S^2$, or minimal complex manifolds homeomorphic to their blowups (which we will discuss in Lecture 5), or are surfaces of general type for which $2\chi_h - 6 \leq c < 9\chi_h$ (which we will discuss in Lecture 6). If we remove the simple connectivity condition, then then the restrictions on (c, χ_h) remain except there are surfaces with $c = 9\chi_h$ all of whom must be ball quotients and hence not simply connected (cf. [21, 46]). At the other extreme we have the Horikawa surfaces with $c = 2\chi_h - 6$. In Lecture 6 we will further discuss surfaces with $c = 9\chi_h$ and $c = 2\chi_h - 6$.

5. More Symplectic Manifolds

To motivate the utility of our list of operations, note that the elliptic surface $E(4)$ has a sphere with self-intersection -4 . We can rationally blow down this sphere to obtain a (symplectic) manifold $E'(4)$ with $(c, \chi_h, t)(E'(4)) = (1, 4, 1)$. More generally, it is easy to see that $E(n)$ contains the configuration C_{n-2} whose lead sphere is a section of $E(n)$. This can be rationally blown down do obtain a simply-connected 4-manifold $E'(n)$ with $(c, \chi_h, t)(E'(n)) = (n - 3, n, 1)$, which according to the above restrictions on complex surfaces, cannot support a complex structure. It turns out that these 4-manifolds are also irreducible and symplectic [12, 53]. However, we first need some invariants that will detect when a smooth manifold is irreducible.

A User's Guide to Seiberg-Witten Theory

1. The Set-up

Let S be a smooth complex surface and T^*S its cotangent bundle. An important associated line bundle is the *canonical bundle* of S , $K_S = \det T^*S$, the determinant line bundle. (This is the top exterior power.) We have $c_1(K_S) = -c_1(TS) = -c_1(S)$. We often identify K_S with $c_1(K_S) \in H^2(S; \mathbf{Z})$ and then, using Poincaré duality, with an element of $H_2(S; \mathbf{Z})$. For example, for the elliptic surface $E(n)$ with F the homology class of an elliptic fiber, we have $K_{E(n)} = (n-2)F$.

An important property of K_S is the adjunction formula: If C is an embedded holomorphic curve in S and the genus of C is g then

$$2g - 2 = C^2 + K_S \cdot C$$

Gauge theory gives us a way to mimic this important class in the setting of oriented smooth 4-manifolds. The critical analytical information for a smooth, closed, oriented 4-manifold X is encoded in its Seiberg-Witten invariant [58]. The goal of this lecture is to provide a ‘user’s guide’ to these invariants. For more detailed explanations one should see [29, 32, 35, 58]. Consider a smooth compact oriented 4-manifold X with tangent bundle TX . The choice of a Riemannian metric on X reduces the structure group of TX to $\mathrm{SO}(4)$, which may be equivalently taken as the structure group of PX , the bundle of tangent frames of X . $\pi_1(\mathrm{SO}(4)) = \mathbf{Z}_2$ and its double covering group is $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. A *spin structure* on X is a lift of PX to a principal $\mathrm{Spin}(4)$ -bundle $\tilde{P}X$ over X such that in the diagram

$$\begin{array}{ccc} \tilde{P}X & \rightarrow & PX \\ & \searrow & \swarrow \\ & X & \end{array}$$

the horizontal map is a double cover on each fiber of PX .

A spin structure gives rise to spinor bundles $S^\pm = \tilde{P}X \times_{\mathrm{SU}(2)} \mathbf{C}^2$, where the action of $\mathrm{SU}(2)$ on $\tilde{P}X$ arises from one of the two factors of $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. From the point of view of algebraic topology, one can think of a spin structure on X as a lift

$$\begin{array}{ccc} & & B\mathrm{Spin}(4) \\ & \nearrow \text{dotted} & \downarrow \\ X & \rightarrow & B\mathrm{SO}(4) \end{array}$$

The obstruction to finding such a lift is the second Stiefel-Whitney class $w_2(X) \in H_2(X; \mathbf{Z}_2)$. One may alternatively think in terms of the transition functions

$$\{\varphi_{i,j} : U_i \cap U_j \rightarrow \mathrm{SO}(4)\}$$

of PX . A spin structure on X consists of lifts $\tilde{\varphi}_{i,j} : U_i \cap U_j \rightarrow \text{Spin}(4)$. In order to give a bundle $\tilde{P}X$, these lifts must satisfy the cocycle condition $\tilde{\varphi}_{i,j} \circ \tilde{\varphi}_{j,k} = \tilde{\varphi}_{i,k}$. From this point of view, $\tilde{P}X$ corresponds to an element $\tilde{\xi}$ of the Čech cohomology group $H^1(X; \text{Spin}(4))$ such that in the sequence

$$\dots \rightarrow H^1(X; \mathbf{Z}_2) \xrightarrow{i_*} H^1(X; \text{Spin}(4)) \xrightarrow{p_*} H^1(X; \text{SO}(4)) \xrightarrow{\delta} H^2(X; \mathbf{Z}_2) \rightarrow \dots$$

$p_*\tilde{\xi} = \xi$, the element which corresponds to PX . Note that $\delta\xi = w_2(X)$, affirming our comment above. Also note that if X admits a spin structure (i.e. a lift of ξ), then all such lifts are in 1-1 correspondence with $H^1(X; \mathbf{Z}_2)$. To each spin structure there is associated a Dirac operator $D : \Gamma(S^+) \rightarrow \Gamma(S^-)$, an elliptic operator which plays an important role in topology and geometry.

In case $w_2(X) \neq 0$, X admits no spin structure, but it can still admit a spin^c structure. A spin^c structure is given by a pair of rank 2 complex vector bundles W^\pm over X with isomorphisms $\det(W^+) = \det(W^-) = L$, some complex line bundle over X , so that locally $W^\pm = S^\pm \otimes L^{\frac{1}{2}}$. To make sense of this, consider the transition maps $\{\varphi_{i,j} : U_i \cap U_j \rightarrow \text{SO}(4)\}$ for PX . We can assume that our charts have overlaps $U_i \cap U_j$ which are contractible, so that we can always get lifts $\tilde{\varphi}_{i,j} : U_i \cap U_j \rightarrow \text{Spin}(4)$. However, if $w_2(X) \neq 0$, we can never find lifts satisfying the cocycle condition.

Similarly, suppose that we are given a complex line bundle L with transition functions $\{g_{i,j} : U_i \cap U_j \rightarrow \text{U}(1)\}$. Locally these functions have square roots $(g_{i,j})^{\frac{1}{2}}$. The obstruction to finding a system of square roots which satisfy the cocycle condition, i.e. to finding a global bundle $L^{\frac{1}{2}}$ over X such that $L^{\frac{1}{2}} \otimes L^{\frac{1}{2}} \cong L$ is $c_1(L) \pmod{2}$ in $H^2(X; \mathbf{Z}_2)$. Now suppose that L is characteristic, i.e. that $w_2(X) = c_1(L) \pmod{2}$. The statement that W^\pm should locally be $S^\pm \otimes L^{\frac{1}{2}}$ means that the tensor products $\tilde{\varphi}_{i,j} \otimes (g_{i,j})^{\frac{1}{2}}$ should satisfy the cocycle condition. This function has values in $(\text{U}(1) \times \text{SU}(2) \times \text{SU}(2))/\{\pm 1\} = \text{Spin}^c(4)$, and the corresponding obstruction is $2w_2(X) = 0$; so spin^c structures exist provided we can find characteristic line bundles L over X . A theorem of Hirzebruch and Hopf states that these exist on any oriented 4-manifold [22]. Spin^c structures on X are classified by lifts of $w_2(X)$ to $H^2(X; \mathbf{Z})$ up to the action of $H^1(X; \mathbf{Z}_2)$. (Spin structures correspond to $0 \in H^2(X, \mathbf{Z})$ up to this action.)

The group $\text{Spin}^c(4) \cong (\text{U}(1) \times \text{SU}(2) \times \text{SU}(2))/\{\pm 1\}$ fibers over $\text{SO}(4) \cong (\text{SU}(2) \times \text{SU}(2))/\{\pm 1\}$ with fiber $S^1 \cong \text{U}(1)$. A *spin^c structure* s on X is a lift of PX to a principal $\text{Spin}^c(4)$ bundle \hat{P}_X over X . Since $\text{U}(2) \cong (\text{U}(1) \times \text{SU}(2))/\{\pm 1\}$, we get representations $s^\pm : \text{Spin}^c(4) \rightarrow \text{U}(2)$, and associated rank 2 complex vector bundles

$$W^\pm = \hat{P}_X \times_{s^\pm} \mathbf{C}^2$$

called spinor bundles, and referred to above, and $L = \det(W^\pm)$. We sometimes write $c_1(s)$ for $c_1(L)$.

As for ordinary spin structures, one has Clifford multiplication

$$c : T^*X \otimes W^\pm \rightarrow W^\mp$$

written $c(v, w) = v.w$ and satisfying $v.(v.w) = -|v|^2w$. Thus c induces a map

$$c : T^*X \rightarrow \text{Hom}(W^+, W^-)$$

A connection A on L together with the Levi-Civita connection on the tangent bundle of X forms a connection $\nabla_A : \Gamma(W^+) \rightarrow \Gamma(T^*X \otimes W^+)$ on W^+ . This

connection, followed by Clifford multiplication, induces the Dirac operator

$$D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$$

Thus D_A depends both on the connection A and the Riemannian metric on X . The case where $L = \det(W^+)$ is trivial corresponds to a usual spin structure on X , and in this case we may choose A to be the trivial connection and then $D_A = D : \Gamma(S^+) \rightarrow \Gamma(S^-)$, the usual Dirac operator.

Fix a spin^c structure s on X with determinant line bundle L , and let \mathcal{A}_L denote the affine space of connections on the line bundle L . Let $F_A \in \Omega^2(X)$ denote the curvature of a connection A on L . The Hodge star operator acts as an involution on $\Omega^2(X)$. Its ± 1 eigenspaces are $\Omega_{\pm}^2(X)$, the spaces of self-dual and anti-self-dual 2-forms. We have $F_A = F_A^+ + F_A^-$. The bundle of self-dual 2-forms $\Omega_+^2(X)$ is also associated to $\hat{P}X$ by $\Omega_+^2(X) \cong \hat{P}X \times_{\text{SU}(2)} \mathfrak{su}(2)$ where $\text{SU}(2)$ acts on its Lie algebra $\mathfrak{su}(2) \cong \mathbf{C} \oplus \mathbf{R}$ via the adjoint action. The map

$$\mathbf{C} \oplus \mathbf{C} \rightarrow \mathbf{C} \oplus \mathbf{R} \quad (z, w) \rightarrow (z\bar{w}, |z|^2 - |w|^2)$$

is $\text{SU}(2)$ -equivariant, and so it induces a map

$$q : \Gamma(W^+) \rightarrow \Omega_+^2(X)$$

2. The Equations

Given a pair $(A, \psi) \in \mathcal{A}_X(L) \times \Gamma(W^+)$, i.e. A a connection in $L = \det(W^{\pm})$ and ψ a section of W^+ , the Seiberg-Witten equations [58] are:

$$\begin{aligned} D_A \psi &= 0 \\ F_A^+ &= iq(\psi) \end{aligned}$$

The gauge group $\text{Aut}(L) = \text{Map}(X, S^1)$ acts on the space of solutions to these equations via

$$g \cdot (A, \psi) = (A - g^{-1}dg, g\psi)$$

and its orbit space is the *Seiberg-Witten moduli space* $M_X(s)$.

Some important features of the Seiberg-Witten equations are

- (1) If (A, ψ) is a solution to the Seiberg-Witten equations with $\psi \neq 0$ then its stabilizer in $\text{Aut}(L)$ is trivial. Such solutions are called *irreducible*. The stabilizer of a *reducible* solution $(A, 0)$ consists of the constant maps in $\text{Map}(X, S^1)$. This is a copy of S^1 .
- (2) $(A, 0)$ is a reducible solution if and only if A is an anti-self-dual connection on the complex line bundle L (i.e. if its curvature $F_A = F_A^-$, is anti-self-dual). If $b_X^+ > 0$ and $c_1(L)$ is nontorsion, a generic metric on X admits no such connections.
- (3) The formal dimension of the Seiberg-Witten moduli space is calculated by the Atiyah-Singer theorem to be

$$\dim M_X(s) = d(s) = \frac{1}{4}(c_1(L)^2 - (3\sigma(X) + 2e(X)))$$

where $e(X)$ is the Euler number of X and $\sigma(X)$ is its signature. Especially interesting is the case where $\dim M_X(s) = 0$, since this is precisely the condition for X to admit an almost-complex structure with first Chern class equal to $c_1(L)$.

- (4) An anti-self-dual 2-form η on X gives us a perturbation of the Seiberg-Witten equations:

$$\begin{aligned} D_A \psi &= 0 \\ F_A^+ &= iq(\psi) + i\eta, \end{aligned}$$

and for a generic perturbation η , the corresponding moduli space of solutions $M_X(s, \eta)$ is an orientable manifold whose dimension is $\dim M_X(s)$, provided $M_X(s, \eta)$ contains at least one irreducible solution. (As in (2), if $b^+(X) > 0$ and $c_1(L) \neq 0$, all solutions will be irreducible for a generic choice of metric or perturbation η .) For simplicity we let the notation ignore this perturbation and write $M_X(s)$ for $M_X(s, \eta)$. An orientation is given to $M_X(s)$ by fixing a ‘homology orientation’ for X , that is, an orientation of $H^1(X) \oplus H_+^2(X)$.

- (5) There is a Lichnerowicz-type theorem, proved, as usual, with an application of the Weitzenböck formula [29, 58]: If X carries a metric of positive scalar curvature, then the only solutions of the Seiberg-Witten equations are reducible (of the form $(A, 0)$). Hence, if $b_X^+ > 0$, for a generic metric of positive scalar curvature, $M_X(s) = \emptyset$.
- (6) For each s , the Seiberg-Witten moduli space $M_X(s)$ is compact.
- (7) There are only finitely many characteristic line bundles L on X for which both $M_X(s) \neq \emptyset$ and $\dim M_X(s) \geq 0$.

Items (6) and (7) are also proved by using the Weitzenböck formula [29, 58].

In case $\dim M_X(s) = 0$, items (4) and (6) imply that that generically, $M_X(s)$ is a finite set of signed points (once a homology orientation has been chosen). In this case one defines the *Seiberg-Witten invariant* $SW_X(s)$ to be the signed count of these points. Generally, $(\mathcal{A}_L \times \Gamma(W^+))/\text{Aut}(L)$ is homotopy equivalent to $\mathbf{CP}^\infty \times \mathbf{T}^{b_1(X)}$, and its homology can be utilized to define $SW_X(s)$. The Seiberg-Witten invariant is a diffeomorphism invariant provided $b^+ > 1$. (The $b^+ = 1$ case will be discussed in a later lecture.)

3. Adjunction Inequality

One of the most important consequences of the Seiberg-Witten equations is the Adjunction Inequality.

The Adjunction Inequality. [29] *Suppose $b^+(X) > 1$ and $SW_X(s) \neq 0$. Let k be the Poincaré dual of $c_1(s)$. If Σ is an embedded closed surface in X with self-intersection ≥ 0 and genus $g \geq 1$ then $2g - 2 \geq \Sigma \cdot \Sigma + |k \cdot \Sigma|$.*

We often view SW_X as a map from characteristic homology classes to the integers by defining $SW_X(k) = \sum_{PD(c_1(s))=k} SW_X(s)$. (Recall that spin^c structures on X are classified by integral lifts of $w_2(X)$ up to the action of $H^1(X; \mathbf{Z}_2)$. So if $H_1(X; \mathbf{Z})$ has no 2-torsion, and in particular, if X is simply connected, then for each characteristic $k \in H_2(X; \mathbf{Z})$ there is a unique spin^c structure s satisfying $PD(c_1(s)) = k$.)

A class k for which $SW_X(k) \neq 0$ is called a *basic class* of X . The adjunction inequality shows that basic classes satisfy a property similar to that of the canonical class of a complex surface. Property (7) above shows that (for $b^+(X) > 1$), X has at most finitely many basic classes.

4. Kähler manifolds

A Kähler surface is a complex surface with a metric g such that $g(Jx, y) = \omega(x, y)$ is a symplectic form. Each simply connected complex surface admits a Kähler structure. A Kähler surface has a distinguished spin^c structure s_K with $c_1(s_K) = K_X$, the canonical class of X . ($K_X = -c_1(TX)$.)

Theorem ([58]). *If X is a minimal Kähler surface with $b^+(X) > 1$ then for its canonical class $|SW_X(\pm s_K)| = 1$. Furthermore, if $c_1^2(X) > 0$ then $SW_X(s) = 0$ for all other spin^c structures.*

(‘Minimal’ means that X contains no embedded holomorphic 2-spheres with self-intersection equal to -1 .)

Another important basic fact is that $SW_X(-s) = (-1)^{(e+\sigma)/4} SW_X(s)$. It is useful to combine all the Seiberg-Witten invariants of X into a single invariant in the integral group ring $\mathbf{Z}H_2(X; \mathbf{Z})$, where for each $\alpha \in H_2(X; \mathbf{Z})$ we let t_α denote the corresponding element in $\mathbf{Z}H_2(X; \mathbf{Z})$. (Note that $t_\alpha^{-1} = t_{-\alpha}$ and $t_0 = 1$.) We then view the Seiberg-Witten invariant of X as the Laurent polynomial

$$SW_X = \sum SW_X(k) t_k$$

where the sum is taken over all characteristic elements k of $H_2(X; \mathbf{Z})$.

For example, for a minimal Kähler surface with $b^+ > 1$ and $c_1^2 > 0$ we have $SW_X = t_K \pm t_K^{-1}$. The $K3$ -surface, $E(2)$, is a Kähler surface with $b^+ = 3$ and $c_1 = 0$. Hence $SW_{K3}(0) = 1$. Adjunction inequality arguments can be used to show that there are no other nontrivial Seiberg-Witten invariants of the $K3$ -surface; so $SW_{K3} = 1$.

5. Blowup Formula

The blowup of a manifold X is $X \# \overline{\mathbf{CP}}^2$. We may identify $H_2(X \# \overline{\mathbf{CP}}^2; \mathbf{Z})$ with $H_2(X; \mathbf{Z}) \oplus H_2(\overline{\mathbf{CP}}^2; \mathbf{Z})$, and let E denote the generator of $H_2(\overline{\mathbf{CP}}^2; \mathbf{Z}) = \mathbf{Z}$ which is represented by a sphere, also called E , the exceptional curve. This sphere is oriented so that if S is a complex surface with canonical class K_S , then the canonical class of its blowup is $K_S + E$. The ‘Blowup Formula’ relates the Seiberg-Witten invariant of $X \# \overline{\mathbf{CP}}^2$ to that of X . It takes an especially straightforward form when X has *simple type*, that is, for each basic class k , the dimension, $d(k)$, of the associated Seiberg-Witten moduli space is 0. Then

$$SW_{X \# \overline{\mathbf{CP}}^2} = SW_X \cdot (e + e^{-1})$$

where $e = t_E$. The assumption of simple type is true for all symplectic manifolds with $b^+ > 1$ by a theorem of Taubes [56].

6. Gluing Formula

Our goal will be to use Seiberg-Witten invariants to study constructions of 4-manifolds. The techniques will involve cutting and pasting along 3-tori. Seiberg-Witten invariants can be defined for 4-manifolds whose boundary is a disjoint union of 3-tori under the extra condition that there is an $\varpi \in H^2(X; \mathbf{R})$ which restricts nontrivially to each boundary component [57]. In this case the invariant is an element of $\mathbf{Z}[[H_2(X; \mathbf{Z})]]$, the ring of formal power series. For example, $SW_{T^2 \times D^2} = (t_T^{-1} - t_T)^{-1} = t_T + t_T^3 + t_T^5 + \dots$. There is an important gluing

theorem due to Morgan, Mrowka, and Szabó [33], B.D. Park [37], and in its most general form to Taubes [57]:

Theorem (Taubes). *Suppose that $\partial X_1 = \partial X_2 = T^3$, and that $X = X_1 \cup_{T^3} X_2$ has $b^+ \geq 1$. Also suppose that there is a class $\varpi \in H^2(X; \mathbf{R})$ restricting nontrivially to $H^2(T^3; \mathbf{R})$. Let $j_i : X_i \rightarrow X$ be the inclusions. Then*

$$\mathcal{SW}_X = (j_1)_* \mathcal{SW}_{X_1} \cdot (j_2)_* \mathcal{SW}_{X_2}$$

When $b_X^+ = 1$, one gets an orientation of $H_+^2(X; \mathbf{R})$ from ϖ : Since the restriction $i^*(\varpi) \in H^2(T^3; \mathbf{R})$ is nonzero, there is a nonzero class $v \in H_2(T^3; \mathbf{R})$ such that $\langle i^*(\varpi), v \rangle > 0$. Then the condition $\langle \alpha, i_*(v) \rangle > 0$ orients $H_+^2(X; \mathbf{R})$. Now (as we shall discuss below) it makes sense to speak of \mathcal{SW}_X^\pm , and in Taubes' theorem, one takes \mathcal{SW}_X^- .

As an example of the use of Taubes' theorem, let T be a homologically nontrivial torus of self-intersection 0 in X with a tubular neighborhood $N_T = T \times D^2$; then

$$\mathcal{SW}_X = \mathcal{SW}_{X \setminus N_T} \cdot \frac{1}{t_T^{-1} - t_T}$$

Hence $\mathcal{SW}_{X \setminus N_T} = \mathcal{SW}_X \cdot (t_T^{-1} - t_T)$.

7. Seiberg-Witten Invariants of Elliptic Surfaces

We apply this gluing theorem to calculate the Seiberg-Witten invariants of the elliptic surfaces $E(n)$. Recall that $E(1) = \mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$ which admits a holomorphic map to S^2 whose generic fiber is a self-intersection 0 torus, F , and that $E(n)$ is the fiber sum $E(n-1) \#_F E(1)$. This means that $E(n) = (E(n-1) \setminus N_F) \cup_{T^3} (E(1) \setminus N_F)$. In this case, each inclusion j_i is the identity. We have seen that $\mathcal{SW}_{E(2)} = 1$. Hence

$$1 = \mathcal{SW}_{E(2)} = (\mathcal{SW}_{E(1) \setminus N_F})^2$$

so $\mathcal{SW}_{E(1) \setminus N_F} = \pm 1$; we choose a homology orientation so that $\mathcal{SW}_{E(1) \setminus N_F} = -1$. Also $\mathcal{SW}_{E(2) \setminus N_F} \cdot \mathcal{SW}_{N_F} = \mathcal{SW}_{E(2)}$; so $\mathcal{SW}_{E(2) \setminus N_F} = t^{-1} - t$. We then get $\mathcal{SW}_{E(3)} = \mathcal{SW}_{E(2) \setminus N_F} \cdot \mathcal{SW}_{E(1) \setminus N_F} = t - t^{-1}$. Inductively, we see that $\mathcal{SW}_{E(n)} = (t - t^{-1})^{n-2}$, where $t = t_F$. Note that the top term t^{n-2} corresponds to $K_{E(n)} = (n-2)F$.

8. Nullhomologous tori

Taubes' formula does not apply to neighborhoods of nullhomologous tori. We need a formula for the effect of surgery on the Seiberg-Witten invariants which applies also to surgery on nullhomologous tori. Let T be a self-intersection 0 torus embedded in X with tubular neighborhood $N_T = T \times D^2$. Given a diffeomorphism $\varphi : \partial(T \times D^2) \rightarrow \partial(X \setminus N_T)$ form $X_\varphi = (X \setminus N_T) \cup_\varphi (T \times D^2)$. The manifold X_φ is determined by the homology class $\varphi_*[\partial D^2] \in H_1(\partial(X \setminus N_T); \mathbf{Z})$. Fix a basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial(X \setminus N_T); \mathbf{Z})$, then there are integers p, q, r , such that $\varphi_*[\partial D^2] = p\alpha + q\beta + r[\partial D^2]$. We write $X_\varphi = X_T(p, q, r)$. (With this notation, note that $X_T(0, 0, 1) = X$.) We have the following important formula of Morgan, Mrowka, and Szabó:

Theorem. [33] *Given a class $k \in H_2(X)$:*

$$\begin{aligned} \sum_i SW_{X_T(p,q,r)}(k_{(p,q,r)} + i[T]) &= p \sum_i SW_{X_T(1,0,0)}(k_{(1,0,0)} + i[T]) + \\ &+ q \sum_i SW_{X_T(0,1,0)}(k_{(0,1,0)} + i[T]) + r \sum_i SW_{X_T(0,0,1)}(k_{(0,0,1)} + i[T]) \end{aligned}$$

and there are no other nontrivial Seiberg-Witten invariants of $X_T(p, q, r)$.

In this formula, T denotes the torus $T_{(a,b,c)}$ which is the core $T^2 \times 0 \subset T^2 \times D^2$ in each specific manifold $X_T(a, b, c)$ in the formula, and $k_{(a,b,c)} \in H_2(X_T(a, b, c))$ is any class which agrees with the restriction of k in $H_2(X \setminus T \times D^2, \partial)$ in the diagram:

$$\begin{array}{ccc} H_2(X_T(a, b, c)) & \longrightarrow & H_2(X_T(a, b, c), T \times D^2) \\ & & \downarrow \cong \\ & & H_2(X \setminus T \times D^2, \partial) \\ & & \uparrow \cong \\ H_2(X) & \longrightarrow & H_2(X, T \times D^2) \end{array}$$

Furthermore, unless the homology class $[T]$ is 2-divisible, in each term, each i must be even since the classes $k_{(a,b,c)} + i[T]$ must be characteristic in $H_2(X_T(a, b, c))$.

Often this formula simplifies. For example, suppose that $\gamma = \varphi_*[\partial D^2]$ is indivisible in $H_1(X \setminus N_T)$. Then there is a dual class $A \in H_3(X \setminus N_T, \partial)$ such that $A \cdot \gamma = 1$. This means that ∂A generates $H_2(N_{T(p,q,r)})$; so

$$H_3(X_T(p, q, r), N_{T(p,q,r)}) \xrightarrow{\text{onto}} H_2(N_{T(p,q,r)}) \xrightarrow{0} H_2(X_T(p, q, r))$$

So $T(p, q, r)$ is nullhomologous in $X_T(p, q, r)$. Hence in the Morgan, Mrowka, Szabó formula, the left hand side has just one term.

A second condition which simplifies the formula uses the adjunction inequality. Suppose that there is an embedded torus Σ of self-intersection 0, such that $\Sigma \cdot T(1, q, r) = 1$ and $\Sigma \cdot k_{(1,q,r)} = 0$. Then the adjunction inequality implies that at most one of the classes $k_{(1,q,r)} + iT_{(1,q,r)}$ can have SW nonzero.

9. Seiberg-Witten invariants for log transforms

These gluing formulas can often be used to quickly calculate invariants, for example, for log transforms. Consider an elliptic surface with a smooth fiber F with tubular neighborhood $N_F \cong F \times D^2$. Choose a basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_F; \mathbf{Z})$ so that $\{\alpha, \beta\}$ is a basis for $H_1(F; \mathbf{Z})$, and hence is in the kernel of π , the projection of the elliptic fibration. A *log transform of multiplicity r* is a surgery on F of type (p, q, r) . Note that r , the multiplicity of the log transform is the degree of the map

$$\text{pt} \times \partial D^2 \xrightarrow{\varphi|} \partial(X \setminus N_F) = \partial N_F \xrightarrow{\pi} \partial D^2$$

A *vanishing cycle* of an elliptic fibration (or for a Lefschetz fibration) is the homology class on a fiber of a loop which is collapsed in a nearby singular fiber. For example, a nodal fiber is a sphere with a transverse positive double point. Such a fiber occurs when a nonseparating loop on a smooth elliptic fiber collapses to a point. The trace of this ‘collapse’ is a disk with relative self-intersection -1 , often called a vanishing disk (or ‘Lefschetz thimble’). In $E(n)$, the homology of a fiber

is spanned by vanishing cycles. It can be proved as a result the a log transform on $E(n)$ depends only on the multiplicity r .

Here is a short argument using the Morgan,-Mrowka-Szabó formula that shows why the Seiberg-Witten invariants only depend upon r . Suppose that α and β are the vanishing cycles. In $E(n)_F(1, 0, 0)$, the union of a vanishing disk bounded by α and the surgery disk form a sphere E of self-intersection -1 , and $E \cdot T_{(1,0,0)} = 1$ (where $T_{(1,0,0)}$ is the core torus of the surgery). Blow down E to get a 4-manifold Y . (In other words, notice that the existence of E forces $E(n)_F(1, 0, 0)$ to equal $Y \# \overline{\mathbf{CP}^2}$.) It can be easily seen that Y contains a torus Λ of square $+1$, and in fact $T_{(1,0,0)} = \Lambda - E$. The adjunction inequality then implies that $\mathcal{SW}_Y = 0$; so the blowup formula says that $\mathcal{SW}_{E(n)_F(1,0,0)} = 0$, and we see that only the “ r ”-term of the Morgan-Mrowka-Szabó formula is nontrivial.

We now use the Taubes formula for the full calculation. Suppose we perform a log transform of order r on $E(n)$; then the result is $E(n; r) = (E(n) \setminus N_F) \cup_j (T^2 \times D^2)$. Let t be the class in the integral group ring corresponding to $T_r = T_{(p,q,r)} \in H_2(E(n; r))$. T_r is a multiple torus in the sense that in $H_2(E(n; r))$, the generic fiber is $F = r T_r$; so $j_*(t_F) = t^r$. Hence

$$\mathcal{SW}_{E(n; r)} = j_*(\mathcal{SW}_{E(n) \setminus N_F}) \cdot \frac{1}{t - t^{-1}} = \frac{(t^r - t^{-r})^{n-1}}{t - t^{-1}} = \mathcal{SW}_{E(n)} \cdot (t^{r-1} + t^{r-3} + \dots + t^{1-r})$$

Each simply connected elliptic surface is the result of 0, 1, or 2 log transforms on $E(n)$, $n \geq 1$, of relatively prime orders. (In this notation, $r = 1$ corresponds to no log transform.) Thus we complete the calculation of Seiberg-Witten invariants of simply connected elliptic surfaces with $b^+ > 1$ by noting that a similar argument shows (for $n > 1$)

$$\mathcal{SW}_{E(n; r, s)} = \frac{(t^{rs} - t^{-rs})^n}{(t^r - t^{-r})(t^s - t^{-s})}$$

Knot Surgery

1. The Knot Surgery Theorem

Let X be a 4-manifold containing an embedded torus T of self-intersection 0, and let K be a knot in S^3 . *Knot surgery* on T is the result of replacing a tubular neighborhood $T \times D^2$ of T with S^1 times the exterior $S^3 \setminus N_K$ of the knot [13]:

$$X_K = (X \setminus (T \times D^2)) \cup (S^1 \times (S^3 \setminus N_K))$$

where ∂D^2 is identified with a longitude of K . This description doesn't necessarily determine X_K up to diffeomorphism; however, under reasonable hypotheses, all manifolds obtained from the same (X, T) and $K \subset S^3$ will have the same Seiberg-Witten invariant. Knot surgery is a homological variant of torus surgery in the sense that a torus surgery is the process that removes a $T^2 \times D^2$ and reglues it, whereas knot surgery removes a $T^2 \times D^2$ and replaces it with a homology $T^2 \times D^2$.

Here is an alternative description of knot surgery: Consider a knot K in S^3 , and let m denote a meridional circle to K . Let M_K be the 3-manifold obtained by performing 0-framed surgery on K . The effect of such a surgery is to span a longitude of K with a disk. The meridian m can also be viewed as a circle in M_K . In $S^1 \times M_K$ we have the smooth torus $T_m = S^1 \times m$ of self-intersection 0. Let X_K denote the fiber sum

$$X_K = X \#_{T=T_m} (S^1 \times M_K) = (X \setminus (T \times D^2)) \cup ((S^1 \times M_K) \setminus (T_m \times D^2))$$

As above, the two pieces are glued together so as to preserve the homology class $[\text{pt} \times \partial D^2]$. Because M_K has the homology of $S^2 \times S^1$ with the class of m generating H_1 , the complement $(S^1 \times M_K) \setminus (T \times D^2)$ has the homology of $T^2 \times D^2$. Thus X_K has the same homology (and intersection pairing) as X .

Let us make the additional assumption that $\pi_1(X) = 1 = \pi_1(X \setminus T)$. Then, since the class of m normally generates $\pi_1(M_K)$; the fundamental group of $M_K \times S^1$ is normally generated by the image of $\pi_1(T)$, and it follows from Van Kampen's Theorem that X_K is simply connected. Thus X_K is homotopy equivalent to X . Also, in order to define Seiberg-Witten invariants, the oriented 4-manifold X must also be equipped with an orientation of $H_+^2(X; \mathbf{R})$. The manifold X_K inherits an orientation as well as an orientation of $H_+^2(X_K; \mathbf{R})$ from X .

For example, consider knot surgery on a fiber F of the elliptic surface $E(2)$ (the $K3$ -surface). Recall that $\mathcal{SW}_{E(2)} = 1$. The elliptic fibration $E(2) \rightarrow S^2$ has a section S which is a sphere of square -2 . The homology class $S + F$ is represented by a torus of square 0 which intersects a generic fiber once. Apply the adjunction inequality to this class to see that mF cannot have a nonzero Seiberg-Witten invariant unless $m = 0$. (Of course we already knew this, but the point is that it is the apparatus of the adjunction inequality that is forcing $\mathcal{SW}_{E(2)} = 1$.) Now do knot surgery on F with a knot K of genus g . In $E(2)_K$ we no longer have

the section S ; the normal disk D to F has been removed from S . In its place there is a surface S' of genus g formed from $S \setminus D$ together with a Seifert surface of the knot K . The class S' still has self-intersection -2 , and $S' + F$ is a class represented by a genus $g + 1$ surface of self-intersection 0. The fiber F still intersects $S' + F$ once. Apply the adjunction inequality to this class to test whether we can now have $\text{SW}_{E(2)_K}(mF) \neq 0$:

$$2(g+1) - 2 \geq (S' + F) \cdot (S' + F) + |mF \cdot (S' + F)| = |m|$$

Thus mF has the possibility of having a nonzero Seiberg-Witten invariant if $|m| \leq 2g$ (and is even since $E(2)_K$ is spin). Thus performing knot surgery gives us the possibility of constructing 4-manifolds with interesting Seiberg-Witten invariants. In fact:

Theorem. [13] *Suppose that $b^+(X) > 1$ and $\pi_1(X) = 1 = \pi_1(X \setminus T)$ and that T is a homologically essential torus of self-intersection 0. Then X_K is homeomorphic to X and*

$$\text{SW}_{X_K} = \text{SW}_X \cdot \Delta_K(t^2)$$

where $t = t_T$ and Δ_K is the symmetrized Alexander polynomial of K .

In particular, $\text{SW}_{E(2)_K} = \Delta_K(t_F^2)$. It was shown by Seifert that any symmetric Laurent polynomial $p(t) = a_0 + \sum_{i=1}^n a_i(t^i + t^{-i})$ whose coefficient sum $p(1) = \pm 1$ is the Alexander polynomial of some knot in S^3 . It follows that the family of smooth 4-manifolds homeomorphic to the $K3$ -surface is at least as rich as this family of Alexander polynomials. Also note that since $\text{SW}_{E(2)_K}(1) = \pm 1$ and $\text{SW}_{E(2;r)}(1) = r$, these manifolds are not diffeomorphic to a log transform, or any number of log transforms, of $K3$.

Note that if \bar{K} is the mirror image knot to K in S^3 then $S^1 \times (S^3 \setminus N_K) \cong S^1 \times (S^3 \setminus N_{\bar{K}})$ since we may view this construction as revolving the knot exterior about an axis. At 180° in $S^1 \times (S^3 \setminus N_K)$ we see $S^3 \setminus N_{\bar{K}}$. Thus $X_{\bar{K}} \cong X_K$. There are currently no other known examples of inequivalent knots which give diffeomorphic manifolds via knot surgery.

2. Proof of Knot Surgery Theorem: the role of nullhomologous tori

The rest of this section will be devoted to a presentation of the proof of the knot surgery theorem as given in [13]. This proof depends on the description of the Alexander polynomial of a knot in terms of the ‘knot theory macarena’:

$$\Delta_{K_+}(t) = \Delta_{K_-}(t) + (t^{1/2} - t^{-1/2}) \cdot \Delta_{K_0}(t)$$

where K_+ is an oriented knot or link, K_- is the result of changing a single oriented positive (right-handed) crossing in K_+ to a negative (left-handed) crossing, and K_0 is the result of resolving the crossing as shown in Figure 1.

Note that if K_+ is a knot, then so is K_- , and K_0 is a 2-component link. If K_+ is a 2-component link, then so is K_- , and K_0 is a knot.

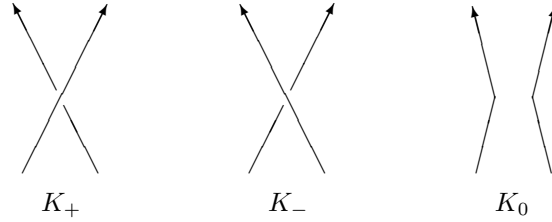


Figure 1

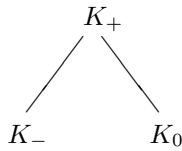
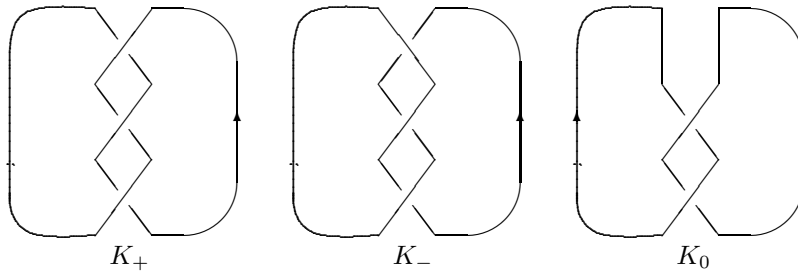


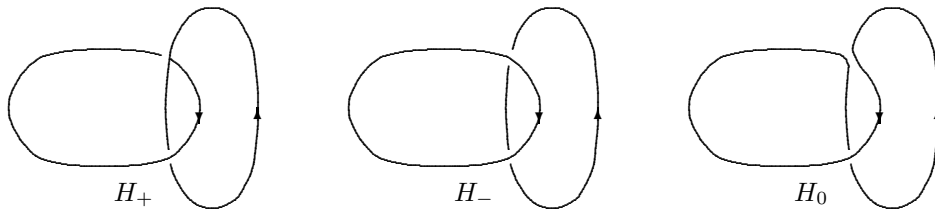
Figure 2

It is proved in [13] that one can start with a knot K and perform macareña moves so as to build a tree starting from K and at each stage adding the bifurcation of Figure 2, where each K_+ , K_- , K_0 is a knot or 2-component link, and so that at the bottom of the tree we obtain only unknots and split links. Then, because for an unknot U we have $\Delta_U(t) = 1$, and for a split link S (of more than one component) we have $\Delta_S(t) = 0$, we can work backwards using the macareña relation to calculate $\Delta_K(t)$.

For example, we compute the Alexander polynomial of the trefoil knot:



In the figure above, $K_+ = K$ is the trefoil knot, K_- is the unknot, and $K_0 = H$ is the Hopf link. Thus we have $\Delta_K = 1 + (t^{1/2} - t^{-1/2}) \cdot \Delta_H$. We see from the figure below that H_- is the unlink and H_0 is the unknot; hence $\Delta_H = 0 + (t^{1/2} - t^{-1/2}) \cdot 1$, and $\Delta_K(t) = 1 + (t^{1/2} - t^{-1/2})^2 = t - 1 + t^{-1}$.



We next need to describe a method for constructing 3-manifolds which was first studied by W. Brakes [5] and extended by J. Hoste [24]. Let L be a link in S^3 with two oriented components C_1 and C_2 . Fix tubular neighborhoods $N_i \cong S^1 \times D^2$ of C_i with $S^1 \times (\text{pt on } \partial D^2)$ a longitude of C_i , i.e. nullhomologous in $S^3 \setminus C_i$. For $n \in \mathbf{Z}$, let $A_n = \begin{pmatrix} -1 & 0 \\ n & 1 \end{pmatrix}$. Note that A_n takes a meridian to a meridian. We get a 3-manifold

$$s(L; n) = (S^3 \setminus (N_1 \cup N_2)) / A_n$$

called a ‘sewn-up link exterior’ by identifying ∂N_1 with ∂N_2 via a diffeomorphism inducing A_n in homology. A simple calculation shows that $H_1(s(L; n); \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}_{2\ell-n}$ where ℓ is the linking number in S^3 of the two components C_1, C_2 , of L . (See [5].) The second summand is generated by the meridian to either component.

J. Hoste [24, p.357] has given a recipe for producing Kirby calculus diagrams for $s(L; n)$. Consider a portion of L consisting of a pair of strands, oriented in opposite directions, and separated by a band B as in Figure 3.

Proposition. [24] *Let $L = C_1 \cup C_2$ be an oriented link in S^3 . Consider a portion of L consisting of a pair of strands as in Figure 3. The band sum of C_1 and C_2 is a knot K , and the sewn-up link exterior $s(L; n)$ is obtained from the framed surgery on the the 2-component link on the right hand side of Figure 3.*

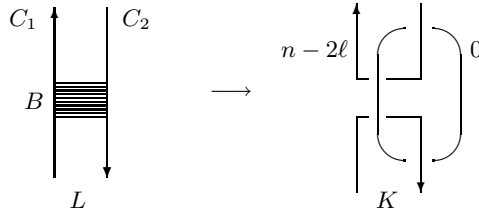


Figure 3

Now we can outline a proof of the knot surgery theorem. Begin with the resolution tree for a given oriented knot K . Each vertex of the tree corresponds to an oriented knot or oriented 2-component link. Replace each knot K' in the tree with the 4-manifold $X_{K'}$, and replace each 2-component link L with the fiber sum

$$X_L = X \#_{T=S^1 \times m} (S^1 \times s(L; 2\ell))$$

where m is a meridian to either component.

Suppose first that K_- is a knot (and therefore so is K_+). We see in Figure 4 that K_+ is the result of $+1$ surgery on the circle C . The circle C is nullhomologous; it bounds a punctured torus. In Figure 4 there is an obvious disk which is punctured twice by K_- . The punctured torus bounded by C consists of this disk, punctured at the points of intersection with K_- together with an annulus running ‘halfway around K_- ’.

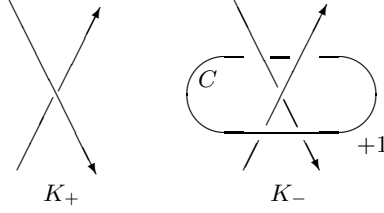


Figure 4

Take the product of this with S^1 , and glue into X along $S^1 \times m$ to obtain, on the one hand, X_{K_+} , and on the other, the result of $(+1)$ -surgery on the nullhomologous torus $T_C = S^1 \times C$.

$$X_{K_+} = (X_{K_-})_{T_C}(0, 1, 1) \equiv X_{K_-}(1)$$

where the basis for $H_1(\partial(T_C \times D^2))$ consists of $S^1 \times \text{pt}$, a pushoff of C in the punctured torus, and $\partial D^2 = m_C$. The Morgan-Mrowka-Szabó formula implies that

$$\text{SW}_{X_{K_+}}(\alpha) = \text{SW}_{X_{K_-}}(\alpha) + \sum_i \text{SW}_{X_{K_-}(0)}(\alpha + 2i[T_0])$$

where $X_{K_-}(0) = (X_{K_-})_{T_C}(0, 1, 0)$ is the result of 0 -surgery on T_C . (Note that T_C is also nullhomologous in X_{K_+} .) As in the concluding comments of §1 of this lecture, only one of the terms $\text{SW}_{X_{K_-}(0)}(\alpha + 2i[T_0])$ in the sum can be nonzero. To see this, we show that there is a torus Λ of self-intersection 0 in $X_{K_-}(0)$ such that $\Lambda \cdot T_0 = 1$ (note $T_0 = S^1 \times m_C$), and such that $\Lambda \cdot \alpha = 0$ for all $\alpha \in H_2(X \setminus T)$. In fact, Λ is formed from the union of the punctured torus bounded by C and the core disk of the 0 -framed surgery giving $X_{K_-}(0)$. We thus have

$$\text{SW}_{X_{K_+}} = \text{SW}_{X_{K_-}} + \text{SW}_{X_{K_-}(0)}$$

The manifold $X_{K_-}(0) = X \#_{T=S^1 \times m}(S^1 \times Y)$ where Y is the 3 -manifold obtained from 0 -framed surgery on both components of the link $K_- \cup C$ in S^3 as in Figure 5.

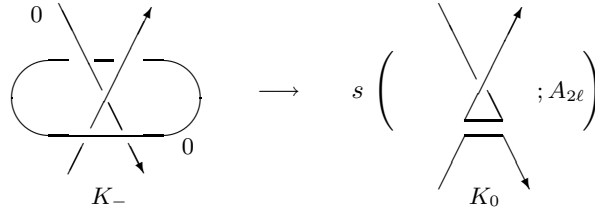


Figure 5

Hoste's recipe tells us that Y is the sewn-up manifold $s(K_0; 2\ell)$. Hence, by definition, $X_{K_-}(0) = X_{K_0}$. We thus get

$$\text{SW}_{X_{K_+}} = \text{SW}_{X_{K_-}} + \text{SW}_{X_{K_0}}$$

The other case to consider is where L_- is an oriented 2 -component link (so also L_+ is a 2 -component link, and L_0 is a knot.). We get L_+ from L_- by a single surgery on a loop U as in Figure 6.

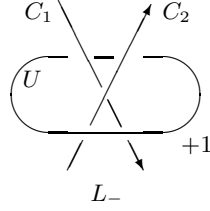


Figure 6

Let ℓ_- denote the linking number of the two components C_1 and C_2 of L_- . In the sewn-up manifold $s(L_-; 2\ell_-)$, the loop U becomes nullhomologous, because according to Hoste's recipe $s(L_-; 2\ell_-)$ is:

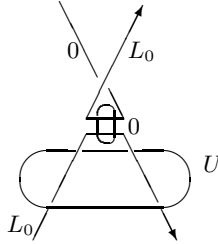


Figure 7

Thus

$$X_{L_+} = X \#_{T=S^1 \times m} (S^1 \times s(L_+; 2\ell_+)) = X_{L_-}(1)$$

where $X_{L_-}(1)$ is shorthand for the surgery manifold $(X_{L_-})_{S^1 \times U}(0, 1, 1)$. The first equality is the definition of X_{L_+} and the last equality is an exercise in Kirby calculus. Similarly we let $X_{L_-}(0)$ denote $(X_{L_-})_{S^1 \times U}(0, 1, 0) = X \#_{T=S^1 \times m} (S^1 \times s(L_-; 2\ell_-)_0)$ where $s(L_-; 2\ell_-)_0$ stands for 0 (rather than +1) surgery on U in Figure 6.

Proposition. $\mathcal{SW}_{X_{L_-}(0)} = \mathcal{SW}_{X_{L_0}} \cdot (t - t^{-1})^2$

PROOF. Cut open $s(L_-; 2\ell_-)$ to get $S^3 \setminus N_{L_-}$ (where N_{L_-} denotes a tubular neighborhood of L_-). Similarly we can cut open $s(L_-; 2\ell_-)_0$ to get a link exterior in $S^1 \times S^2$. Take a product with S^1 and glue into X giving $X \#_{T=S^1 \times m} (S^1 \times (S^1 \times S^2))$ with a pair of tori of self-intersection 0 removed. If we sew up the boundary of this manifold using the map $(1) \oplus A_{2\ell_-}$, we re-obtain $X_{L_-}(0)$.

Instead, fill in the boundary components with copies of $T^2 \times D^2$ to get a new manifold, Z . We wish to do this in such a way that when we remove a neighborhood of the new link $T^2 \times \{0\} \cup T^2 \times \{0\} \subset Z$ and sew up the boundaries using $(1) \oplus A_0$, we get $X_{L_-}(0)$. (We want to be able to sew up with this particular matrix because A_0 identifies $S^1 \times C_1$ with $S^1 \times C_2$.) We can accomplish this by gluing each $T^2 \times D^2$ to a boundary component using the matrix $(1) \oplus \begin{pmatrix} 0 & 1 \\ 1 & -\ell_- \end{pmatrix}$. This matrix corresponds to $S^1 \times ((-\ell_-)$ -framed surgery). Then, using the internal fiber sum formula of §1,

$$\mathcal{SW}_{X \#_{T=S^1 \times m} S^1 \times s(L_-; 2\ell_-)_0} = \mathcal{SW}_Z \cdot (t - t^{-1})^2$$

Now $Z = X \#_{T=S^1 \times m} (S^1 \times Y)$ where Y is the 3-manifold of Figure 8

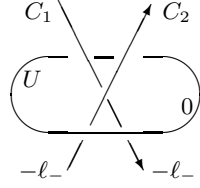


Figure 8

Now slide C_1 over C_2 . We get 0-surgery on L_0 together with a cancelling pair of handles. Thus $Y = M_{L_0}$, and the proposition is proved. \square

It follows from the proposition and the Morgan-Mrowka-Szabó theorem that

$$\mathcal{SW}_{X_{L_+}} = \mathcal{SW}_{X_{L_-}} + \mathcal{SW}_{X_{L_0}} \cdot (t - t^{-1})^2$$

We are now able to finish the proof of the knot surgery theorem. For a knot K , or an oriented 2-component link L and fixed X , we define a formal Laurent series Θ . For a knot K , define Θ_K to be the quotient, $\Theta_K = \mathcal{SW}_{X_K} / \mathcal{SW}_X$, and for a 2-component link define $\Theta_L = (t^{1/2} - t^{-1/2})^{-1} \cdot \mathcal{SW}_{X_L} / \mathcal{SW}_X$. It follows from our calculations that in either case Θ satisfies the relation

$$\Theta_{K_+} = \Theta_{K_-} + (t - t^{-1}) \cdot \Theta_{K_0}.$$

Furthermore, for the unknot U , the manifold X_U is just $X \#_T(S^2 \times T^2) = X$, and so $\Theta_U = 1$. If L is a 2 component oriented split link, construct from L the knots K_+ and K_- as shown in Figure 9. Note that in this situation, $K_+ = K_-$ and $K_0 = L$. It follows from $\Theta_{K_+} = \Theta_{K_-} + (t - t^{-1}) \cdot \Theta_L$ that $\Theta_L = 0$.

Subject to these initial values, the resolution tree and the macarena relation determine $\Delta_K(t)$ for any knot K . It follows that Θ_K is a Laurent polynomial in a single variable t , and $\Theta_K(t) = \Delta_K(t)$, completing the proof of the knot surgery theorem.

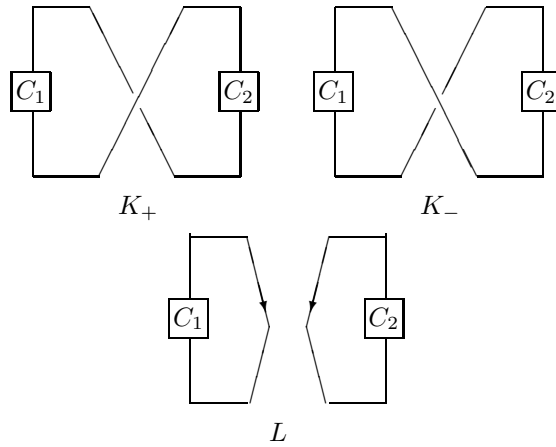


Figure 9

Rational blowdowns

1. Configurations of Spheres and Associated Rational Balls

Suppose that X contains an embedded sphere of self-intersection -1 , and let N be its tubular neighborhood. Then $\partial N \cong S^3$ and $N \cup B^4 \cong \overline{\mathbf{CP}^2}$ hence $X \cong Y \# \overline{\mathbf{CP}^2}$ for $Y = (X \setminus N) \cup B^4$. Y is called the *blowdown* of X . Similarly, since the complement of a nonsingular quadric curve in \mathbf{CP}^2 is a tubular neighborhood B_2 of \mathbf{RP}^2 , if X contains an embedded sphere of self-intersection -4 , its neighborhood, C_2 can be ‘rationally blown down’ by replacing it with B_2 . Note that $H_*(B_2; \mathbf{Z}) = H_*(\mathbf{RP}^2; \mathbf{Z})$; so $H_*(B_2; \mathbf{Q}) = H_*(B^4; \mathbf{Q})$; *i.e.* B_2 is a rational ball. Note that this process, like blowing down, reduces b^- by one, while leaving b^+ unchanged.

More generally, as was mentioned in Lecture 1, neighborhoods C_p , $p = 2, 3, \dots$, of the configurations of embedded spheres

$$\begin{array}{ccccccc} -(p+2) & -2 & & & & & -2 \\ \bullet & \bullet & \text{---} & \dots & \text{---} & & \bullet \\ u_0 & u_1 & & & & & u_{p-2} \end{array}$$

also bound rational balls and can be blown down. The boundary of C_p is the lens space $L(p^2, 1-p)$ which bounds a rational ball B_p , and

$$\pi_1(L(p^2, 1-p)) = \mathbf{Z}_{p^2} \rightarrow \pi_1(B_p) = \mathbf{Z}_p$$

is onto. If $C_p \subset X$, then its rational blowdown $X_{(p)} = (X \setminus C_p) \cup B_p$. This operation reduces b^- by $p-1$ while leaving b^+ unchanged. It follows that $c(X_{(p)}) = c(X)$ while $\chi_h(X_{(p)}) = \chi_h(X) - (p-1)$. Also, if X and $X \setminus C_p$ are simply connected, then so is $X_{(p)}$.

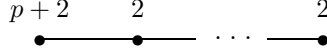
To see that $L(p^2, 1-p)$ bounds a rational ball B_p with $\pi_1(B_p) = \mathbf{Z}_p$ we can argue in several ways:

1. Let \mathbf{F}_{p-1} , $p \geq 2$, be the simply connected ruled surface whose negative section s_- has square $-(p-1)$. Let s_+ be a positive section (with square $(p-1)$) and f a fiber. Then the homology classes $s_+ + f$ and s_- are represented by embedded 2-spheres which intersect each other once and have intersection matrix

$$\begin{pmatrix} p+1 & 1 \\ 1 & -(p-1) \end{pmatrix}$$

It follows that the regular neighborhood of this pair of 2-spheres has boundary $L(p^2, p-1)$. Its complement in \mathbf{F}_{p-1} is the rational ball B_p .

2. Consider the configuration of $(p-1)$ 2-spheres



in $\#(p - 1)\mathbf{CP}^2$ where the spheres (from left to right) represent

$$-2h_1 - h_2 - \dots - h_{p-1}, \quad h_1 - h_2, \quad h_2 - h_3, \dots, h_{p-2} - h_{p-1}$$

where h_i is the hyperplane class in the i th copy of \mathbf{CP}^2 . Note that two adjacent spheres in this configuration have intersection number -1 . This is the configuration \bar{C}_p , *i.e.* C_p with the opposite orientation. Its boundary is $L(p^2, p - 1)$, and the classes of the configuration span $H_2(\mathbf{CP}^2; \mathbf{Q})$. The complement is the rational ball B_p .

3. A Kirby calculus description of B_p consists of a 1-handle and an unknotted 2-handle that links it p times and has framing $p - 1$.

Lemma. *Any diffeomorphism of $L(p^2, 1 - p)$ extends over B_p .*

PROOF. Bonahon has showed that $\pi_0(\text{Diff}(L(p^2, 1 - p))) = \mathbf{Z}_2$. The Kirby calculus description (3) of B_p gives a symmetric link in S^3 . Rotation by 180° gives the nontrivial element of $\pi_0(\text{Diff}(L(p^2, 1 - p)))$ and extends over B_p . \square

This lemma implies that rational blowdown is well-defined. Here a few examples:

The elliptic surface $E(4)$ has nine disjoint and homologically independent sections arising from fiber summing the nine sections of $E(1)$ which come from the exceptional curves $E_1, \dots, E_9 \subset E(1) = \mathbf{CP}^2 \# 9 \overline{\mathbf{CP}^2}$ in $E(4)$, the fiber sum of four copies of $E(1)$. Each of these sections is a sphere of square -4 which may be rationally blown down. We obtain manifolds with $\chi_h = \chi_h(E(4)) = 4$ and $c = c(E(4)) + m = m$ where we have blown down m of the sections. It can be checked that for $m = 1, \dots, 8$, these manifolds are simply connected.

The next example, whose discovery motivated our interest in rational blowdowns, shows that a logarithmic transform of order p can be obtained by a sequence of $(p - 1)$ blowups and one rational blowdown of a natural embedding of the configuration C_p .

The homology class F of the fiber of $E(n)$ can be represented by a nodal curve, an immersed sphere with one positive double point. Blow up this double point (*i.e.* take the proper transform of F) so that the class $F - 2e_1$ (where e_1 is the homology class of the exceptional divisor) is represented by an embedded sphere with square -4 in $E(n) \# \overline{\mathbf{CP}^2}$. This is the configuration C_2 . In $E(n) \# \overline{\mathbf{CP}^2}$, the exceptional divisor intersects the -4 -sphere in two positive points. Blow up one of these points, *i.e.* again take a proper transform. One obtains the homology classes $u_2 = F - 2e_1 - e_2$ and $u_1 = e_1 - e_2$ which form the configuration C_3 in $E(n) \# 2 \overline{\mathbf{CP}^2}$. Continuing in this fashion, C_p naturally embeds in $E(n) \#_{p-1} \overline{\mathbf{CP}^2} \subset E(n) \#_{p-1} \overline{\mathbf{CP}^2}$.

Theorem ([12]). *The rational blowdown of the above configuration*

$$C_p \subset E(n) \#_{p-1} \overline{\mathbf{CP}^2}$$

is $E(n)_p$, the multiplicity p log transform of $E(n)$.

2. Effect on Seiberg-Witten Invariants

We next wish to study the effect of rational blowdowns on Seiberg-Witten invariants. Before undertaking this, we need a lemma, whose proof we leave as an exercise.

Lemma. *In the configuration C_p*

$$\begin{array}{ccccccc} -(p+2) & -2 & & & & & -2 \\ \bullet & \bullet & \cdots & \cdots & \cdots & \cdots & \bullet \\ U_0 & U_1 & & & & & U_{p-2} \end{array}$$

take the meridian of U_0 to be the generator γ of the homology of $L(p^2, 1-p) = \partial C_p$. Then the meridian of U_j represents $(1+j(p+1))\gamma$.

Suppose that we know the Seiberg-Witten invariants \mathcal{SW}_X and wish to calculate $\mathcal{SW}_{X(p)}$, the Seiberg-Witten invariant of the rational blowdown. To facilitate this discussion, it will be convenient to now view basic classes as characteristic cohomology classes. (Of course Poincaré duality allows us to switch between H_2 and H^2 at will.)

Let $k \in H^2(X(p); \mathbf{Z})$ be characteristic. Write $X(p) = X' \cup B_p$ where $X' \cap B_p = L(p^2, 1-p) = L$; so $X = X' \cup C_p$. The restriction $H^2(B_p; \mathbf{Z}) \rightarrow H^2(L(p^2, 1-p); \mathbf{Z})$ is multiplication, $\mathbf{Z}_p \xrightarrow{\cdot p} \mathbf{Z}_{p^2}$, and the restriction k_L of k to L is in the image; so $k_L = mp$, for some integer m . Description (3) above of B_p , shows that B_p has even type if and only if p is odd. Since k is characteristic, if p is even, m must be odd. If p is odd, we can assume that m is even by taking $-p < m$ even $< p$.

We claim that k_L extends over C_p as a characteristic cohomology class. Recall from description (2) of B_p that the connected sum $(p-1)\mathbf{CP}^2 = B_p \cup \bar{C}_p$. Hence $(p-1)\overline{\mathbf{CP}}^2 = \bar{B}_p \cup C_p$, where the classes in C_p are $u_0 = -2E_1 - E_2 - \dots - E_{p-1}$, $u_1 = E_1 - E_2 - \dots$, $u_{p-2} = E_{p-2} - E_{p-1}$. (Again, view these as cohomology classes.) Define cohomology classes $\lambda'_m \in H^2((p-1)\overline{\mathbf{CP}}^2; \mathbf{Z})$ by:

$$\lambda'_m = - \sum_{i=1}^{(m+p-1)/2} E_i + \sum_{(m+p+1)/2}^{p-1} E_i$$

(Thus, for example, for $p=5$, $\lambda'_{-4} = E_1 + E_2 + E_3 + E_4$, $\lambda'_{-2} = -E_1 + E_2 + E_3 + E_4$, $\lambda'_0 = -E_1 - E_2 + E_3 + E_4$, $\lambda'_2 = -E_1 - E_2 - E_3 + E_4$, and $\lambda'_4 = -E_1 - E_2 - E_3 - E_4$.) Finally, set $\lambda_m = \lambda'_m|_{C_p} \in H^2(C_p; \mathbf{Z})$.

Note that the intersection vector of E_1 with C_p is

$$(E_1 \cdot U_0, E_1 \cdot U_1, \dots, E_1 \cdot U_{p-2}) = (2, -1, 0, \dots, 0)$$

Thus, using the lemma above, $E_1|_L = (2 - (p+2))\gamma = -p\gamma$. Similarly the intersection vector of E_2 with C_p is $(1, 1, -1, 0, \dots, 0)$; so again $E_2|_L = -p\gamma$. Similarly, one sees that each E_i , $1 \leq i \leq p-1$ satisfies $E_i|_L = -p\gamma$. Thus λ_m restricts to L as $m p \gamma \in H^2(L(p^2, 1-p); \mathbf{Z})$. This proves our claim that each characteristic cohomology class of X' extends over C_p .

Let $k \in H^2(X(p); \mathbf{Z})$ be characteristic, and let \tilde{k} be the (characteristic) class in $H^2(X; \mathbf{Z})$ obtained by restricting k to X' and extending over C_p by using the appropriate λ_m . Note that $\tilde{k}^2 = k^2 + \lambda_m^2 = k^2 + (1-p)$. It follows easily that the dimensions of the corresponding Seiberg-Witten moduli spaces are equal, $d_X(\tilde{k}) = d_{X(p)}(k)$. Call such a \tilde{k} a *lift* of k .

The main theorem on rational blowdowns, proved in [12], is:

Theorem. *Let k be a characteristic (co)homology class of $X_{(p)}$, and let \tilde{k} be any lift. Then $\mathcal{SW}_{X_{(p)}}(k) = \mathcal{SW}_X(\tilde{k})$.*

3. Taut Configurations

As a first example, we consider taut configurations. The configuration $C_p \subset X$ is called *taut*, if for each basic class \bar{k} of X , $|\bar{k} \cdot U_0| \leq p$, and $\bar{k} \cdot U_i = 0$ for $i > 0$. That is to say, the configuration is taut if its spheres satisfy the adjunction inequality with respect to all basic classes of X . Recall that, in general, we do not expect the adjunction inequality to hold for spheres.

For a taut configuration and basic class \bar{k} of X , $k|_{C_p} = (\bar{k} \cdot U_0)\text{PD}(U_0)$ where $\text{PD}(U_0) \in H^2(C_p, \partial; \mathbf{Z})$. Thus $\bar{k}|_L = (\bar{k} \cdot U_0) \in \mathbf{Z}_{p^2}$. Thus, to extend over B_p we must have $\bar{k} \cdot U_0 = mp \in \mathbf{Z}_{p^2}$, and so $\bar{k} \cdot U_0 = \pm p$ or 0 , since $|\bar{k} \cdot U_0| \leq p$. It is an exercise (see [12]) to show that $\bar{k} \cdot U_0 = 0$ is not a possibility. Hence for a taut configuration, precisely the basic classes \bar{k} which satisfy $\bar{k} \cdot U_0 = \pm p$ descend.

Consider the result of rationally blowing down a section S of $E(4)$. Recall that S is a sphere of square -4 . We obtain a simply connected manifold $Y(4)$ with $\chi_h = 4$ and $c = 1$. Since $\mathcal{SW}_{E(4)} = (t_F - t_F^{-1})^2$, the basic classes of $E(4)$ are $\pm 2F$ and 0 . Since $S \cdot F = 1$, the configuration $C_2 = \{S\}$ is taut and the classes $\pm 2F$ descend to give basic classes of $Y(4)$. It follows that $\mathcal{SW}_{Y(4)} = t + t^{-1}$. Comparing this with the blowup formula shows that $Y(4)$ is minimal; *i.e.* it contains no 2-spheres of square -1 . It follows from general algebraic surface theory that a minimal algebraic surface with $c > 0$ satisfies the Noether inequality, $c \geq 2\chi_h - 6$. This means that $Y(4)$ can admit no complex structure. (And with its opposite orientation, $Y(4)$ cannot have a complex structure either. It is an easy exercise to use the adjunction inequality to see this.) In fact, a theorem of Symington [53] implies that $Y(4)$ does have a symplectic structure.

The other eight sections of $E(4)$ are disjoint from S and so they give disjoint spheres of square -4 in $Y(-4)$. One of these gives yet another taut configuration which can be rationally blown down. This gives a simply connected manifold with $\chi_h = 4$ and $c = 2$, with $\mathcal{SW} = t_K + t_K^{-1}$. Note that $c = 2\chi_h - 6$ realizing equality in the Noether inequality. Minimal complex surfaces satisfying $c = 2\chi_h - 6$ are called Horikawa surfaces. These were discussed in Lecture 1, and it can be shown that the manifold obtained by rationally blowing down two sections of $E(4)$ is $H(4)$. More generally, each $E(n)$, $n \geq 4$, contains two disjoint taut configurations C_{n-2} in which the lead sphere U_0 is a section. Blowing down one of the configurations gives $Y(n)$, a symplectic 4-manifold which admits no complex structure, and blowing down both gives the Horikawa surface $H(n)$.

We conclude our study of rational blowdowns by recalculating the Seiberg-Witten invariant of log transforms on $E(n)$. (In fact, in the days of Donaldson invariants, this is how the original calculation was accomplished.) For the sake of simplicity we will work out the case of a multiplicity 5 log transform on $E(2)$. As above, we get C_5 embedded in $E(2) \# 4 \overline{\mathbf{CP}}^2$, and the spheres of the configuration are $U_0 = F - 2e_1 - e_2 - e_3$, $U_1 = e_1 - e_2$, $U_2 = e_2 - e_3$, and $U_3 = e_3 - e_4$. Since $\mathcal{SW}_{E(2)} = 1$, the blowup formula gives $\mathcal{SW}_{E(2) \# 4 \overline{\mathbf{CP}}^2} = (\epsilon_1 + \epsilon_1^{-1}) \dots (\epsilon_4 + \epsilon_4^{-1})$ where $\epsilon_i = t_{e_i}$. So the basic classes of $E(2) \# 4 \overline{\mathbf{CP}}^2$ are $\sum \pm e_i$.

When C_5 is rationally blown down, each e_i descends to the class F_5 of the multiple fiber in $E(2)_5$; $F = 5F_5$. Each class $\sum \pm e_i$ descends and has Seiberg-Witten invariant 1. Thus

$$\mathcal{SW}_{E(2)_5} = t^4 + t^2 + 1 + t^{-2} + t^{-4}$$

$t = t_{F_5}$ (as it should).

Manifolds with $b^+ = 1$

1. Seiberg-Witten Invariants

Let X be a smooth simply connected oriented 4-manifold with $b^+ = 1$. As simple examples we have \mathbf{CP}^2 , $\mathbf{CP}^2 \# k \overline{\mathbf{CP}^2}$, and $S^2 \times S^2$. As we mentioned in §2, there are some complications in defining Seiberg-Witten invariants in this case. Suppose that we are given a Riemannian metric g for X . Recall that the input for the Seiberg-Witten equations consists of a pair of complex 2-plane bundles W^\pm over X and a complex line bundle L satisfying $\det(W^\pm) = L$. Given (ψ, A) where ψ is a section of W^+ and A is a connection on L , we have the perturbed Seiberg-Witten equations:

$$\begin{aligned} D_A \psi &= 0 \\ F_A^+ &= iq(\psi) + i\eta, \end{aligned}$$

where η is a self-dual 2-form on X . Suppose also that we are given a homology orientation for X , that is, a given orientation of $H^1(X; \mathbf{R}) \oplus H_+^2(X; \mathbf{R})$. The Seiberg-Witten invariant depends on g and η as follows.

Recall that the curvature F_A represents the cohomology class $-2\pi ic_1(L)$, and that $H^2(X; \mathbf{R})$ is the orthogonal direct sum of $H_g^+(X)$ and $H_g^-(X)$, the spaces of self-dual and antiself-dual harmonic 2-forms for g . The Seiberg-Witten invariant for (g, η) is well-defined provided there are no reducible solutions to the equations, *i.e.* solutions with $\psi = 0$, and hence $F_A^+ = i\eta$. For a fixed L and g , this occurs only if $-2\pi c_1(L)^+ = \eta \in H_g^+(X)$ or equivalently if $2\pi c_1(L) + \eta \in H_g^-(X)$. This happens if and only if $2\pi c_1(L) + \eta$ is orthogonal to $H_g^+(X)$, a dimension b^+ subspace. Thus the condition that there should exist a reducible solution cuts out a codimension b^+ affine subspace of $H^2(X; \mathbf{R})$.

It follows that if $b^+ > 0$, $\text{SW}_{X,g,\eta}(L)$ is well-defined for a generic (g, η) . If $b^+ > 1$ a generic path of (g, η) contains no reducible solutions; and this is why the Seiberg-Witten invariant is an oriented diffeomorphism invariant in this case. (Its sign is determined by the homology orientation.) In case $b^+ = 1$, there can be reducible solutions in one-parameter families. $H_g^-(X)$ has dimension 1 and it is spanned by a single g harmonic 2-form ω_g of norm 1 agreeing with the homology orientation. This form is often called the *period point* for (X, g) .

The condition for admitting a reducible solution now becomes $(2\pi c_1(L) + \eta) \cdot \omega_g = 0$. This defines a codimension 1 affine subspace or *wall*. The Seiberg-Witten invariant is well-defined for families of (g, η) satisfying the condition that any two can be connected by a path not crossing the wall. Let $H \in H_2(X; \mathbf{R})$ be the Poincaré dual of ω_g . Then there are two well-defined Seiberg-Witten invariants $\text{SW}_{X,H}^\pm(L)$ defined by: $\text{SW}_{X,H}^\pm(L) = \text{SW}_{X,g,\eta}(L)$ for any (g, η) such that

$(2\pi c_1(L) + \eta) \cdot \omega_g > 0$, and $\mathcal{SW}_{X,H}^-(L) = \mathcal{SW}_{X,g,\eta}(L)$ for any (g, η) such that $(2\pi c_1(L) + \eta) \cdot \omega_g < 0$.

These invariants are related via a “wall-crossing formula” [29, 30] which states that

$$\mathcal{SW}_{X,H}^+(L) - \mathcal{SW}_{X,H}^-(L) = (-1)^{1+\frac{1}{2}d(k)}$$

Thus for any characteristic homology class k , at least one of $\mathcal{SW}_{X,H}^\pm(k)$ is nonzero. For example, on the ray given by multiples of the fiber class F of $E(1)$ (and with $t_F = t$) we have

$$\mathcal{SW}_{E(1),H}^- = \sum_{m=0}^{\infty} t^{2m+1}, \quad \mathcal{SW}_{E(1),H}^+ = - \sum_{m=0}^{\infty} t^{-(2m+1)}$$

where H is the class of a line in \mathbf{CP}^2 viewing $E(1) = \mathbf{CP}^2 \# 9 \overline{\mathbf{CP}^2}$. (The way to do this calculation is to note that in the “chamber” of the positive scalar curvature metric, the invariant is 0, then use the wall-crossing formula.)

The small-perturbation Seiberg-Witten invariant is defined by

$$\mathcal{SW}_{X,H}(k) = \begin{cases} \mathcal{SW}_{X,H}^+(k) & \text{if } k \cdot H > 0 \\ \mathcal{SW}_{X,H}^-(k) & \text{if } k \cdot H < 0 \end{cases}$$

There is again a wall-crossing formula which can be deduced from the one above. If H', H'' are elements of positive square in $H_2(X; \mathbf{R})$ with $H' \cdot H > 0$ and $H'' \cdot H > 0$, then if $k \cdot H' < 0$ and $k \cdot H'' > 0$,

$$\mathcal{SW}_{X,H''}(k) - \mathcal{SW}_{X,H'}(k) = (-1)^{1+\frac{1}{2}d(k)}$$

Furthermore, in case $b^- \leq 9$, one can see that $\mathcal{SW}_{X,H}(k)$ is independent of H , and we denote it as $\mathcal{SW}_X(k)$.

The small-perturbation invariant satisfies all the usual properties of the Seiberg-Witten invariant. In particular, there are only finitely many basic classes; so we get a Laurent polynomial $\mathcal{SW}_{X,H}$ or \mathcal{SW}_X , in case $b^- \leq 9$. Furthermore, the invariant vanishes in the presence of a positive scalar curvature metric. So, for example, $\mathcal{SW}_{\mathbf{CP}^2 \# m \overline{\mathbf{CP}^2}, H} = 0$, for H the class of a line in \mathbf{CP}^2 and if $m \leq 9$, $\mathcal{SW}_{\mathbf{CP}^2 \# m \overline{\mathbf{CP}^2}} = 0$.

$E(1) = \mathbf{CP}^2 \# 9 \overline{\mathbf{CP}^2}$ is an elliptic surface. Consider the result of a log transform of multiplicity p on $E(1)$. The log transform formula gives us (along the ray of a fiber):

$$\mathcal{SW}_{E(1)_p,H}^- = \mathcal{SW}_{E(1),H}^- \cdot (\tau^{p-1} + \tau^{p-3} + \dots + \tau^{1-p}) = \sum_{m=0}^{\infty} \tau^{2m+1}$$

where τ corresponds to the multiple fiber, so $t = \tau^p$. This means that we can't distinguish $\mathcal{SW}_{E(1)_p,H}^-$ from $\mathcal{SW}_{E(1),H}^-$ (at least along the ray under consideration). In fact, it can be shown that $E(1)_p$ is actually diffeomorphic to $E(1)$.

The situation is quite different when we perform two log transforms. (In order that the result be simply connected, and hence homeomorphic to $E(1)$, there can be at most two log transforms, and their multiplicities p, q must be relatively prime.) The result $E(1)_{p,q}$ is called a *Dolgachev surface*. A calculation of Seiberg-Witten invariants shows that its diffeomorphism type depends on the pair of relatively

prime integers p, q . In fact, $E(1)_{2,3}$ and $E(1)$ were the first known pair of 4-manifolds which are homeomorphic but not diffeomorphic. This was discovered in a ground-breaking paper of Simon Donaldson [6] (1985).

Knot surgery also gives examples of this phenomenon. For example, for the twist knots K_n :

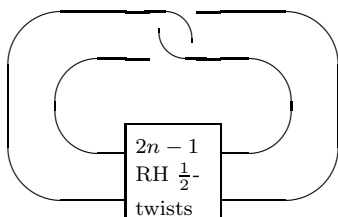


Figure 10: $K_n =$ twist knot

we have $\Delta_{K_n}(t) = nt - (2n - 1) + nt^{-1}$, and one can calculate that $\mathcal{SW}_{E(1)_{K_n}} = -nt + nt^{-1}$.

The blowup formula shows that all these ‘exotic’ manifolds remain exotic after connect summing with $\overline{\mathbf{CP}^2}$, however, it is still unknown whether there are examples of minimal exotic simply connected 4-manifolds with $b^+ = 1$ and $b^- > 9$.

2. Smooth structures on blow-ups of \mathbf{CP}^2

Examples of interesting simply connected smooth 4-manifolds with $b^+ = 1$ and $b^- < 9$ are harder to come by. In the late 1980’s, Dieter Kotschick [28] proved that the Barlow surface, which is homeomorphic to $\mathbf{CP}^2 \# 8\overline{\mathbf{CP}^2}$, is not diffeomorphic to it. However, in following years the subject of simply connected smooth 4-manifolds with $b^+ = 1$ languished because of a lack of suitable examples. In 2004, Jongil Park reignited interest in this topic by producing an example of a smooth simply connected 4-manifold P which is homeomorphic to $\mathbf{CP}^2 \# 7\overline{\mathbf{CP}^2}$ but not diffeomorphic to it. A year later, Stipsicz and Szabó produced a similar example with $b^- = 6$ [51].

Our goal in this lecture is to describe our technique (introduced in [18]) for producing infinite families of simply connected manifolds with $b^+ = 1$ and $b^- = 6, 7, 8$. This technique was also used in [43] (and [18]) to obtain examples with $b^- = 5$.

The starting point for this construction is a specific elliptic fibration on $E(1)$. A simply connected elliptic surface is fibered over S^2 with smooth fiber a torus and with singular fibers. The most generic type of singular fiber is a nodal fiber (an immersed 2-sphere with one transverse positive double point). Singular fibers are isolated; so the boundary of a small tubular neighborhood of a singular fiber is a torus bundle over the circle. It is therefore described by a diffeomorphism of T^2 called the *monodromy* of the singularity. The monodromy of a nodal fiber is D_a , a Dehn twist around the ‘vanishing cycle’ $a \in H_1(F; \mathbf{Z})$, where F is a smooth fiber of the elliptic fibration. As we discussed earlier, the vanishing cycle a is represented by a nonseparating loop on the smooth fiber, and the nodal fiber is obtained by collapsing this vanishing cycle to a point to create a transverse self-intersection. The vanishing cycle bounds a ‘vanishing disk’, a disk of relative self-intersection -1

with respect to the framing of its boundary given by pushing the loop off itself on the smooth fiber.

An I_m -singularity consists of a cycle of spheres $\{A_i | i \in \mathbf{Z}_m\}$ of self-intersection -2 . These spheres are arranged so that A_i intersects $A_{i\pm 1}$ once, positively and transversely, and $A_i \cap A_j = \emptyset$ for $j \neq i, i \pm 1$. The monodromy of an I_m -fiber is D_a^m . In particular, an I_2 fiber consists of a pair of 2-spheres of self-intersection -2 which are plumbed at two points. The monodromy of an I_2 -fiber is D_a^2 , which is also the monodromy of a pair of nodal fibers with the same vanishing cycle. This means that an elliptic fibration which contains an I_2 -fiber can be perturbed to contain two nodal fibers with the same vanishing cycle.

The elliptic fibration on $E(1)$ which we use has an I_8 -fiber, and I_2 -fiber, and two other nodal fibers. (The existence of such a fibration can be found, for example, in [45].) A *double node neighborhood* D is a fibered neighborhood of an elliptic fibration which contains exactly two nodal fibers with the same vanishing cycle. If F is a smooth fiber of D , there is a vanishing class a that bounds vanishing disks in the two different nodal fibers, and these give rise to a sphere V of self-intersection -2 in D . It follows from our comments about I_2 -singularities that we have a double node neighborhood D in our copy of $E(1)$.

Perform knot surgery inside the neighborhood D using the twist knot K_n of Figure 10. These twist knots have genus one, and an obvious genus one Seifert surface Σ can be seen in Figure 10. Consider the loop Γ on Σ which runs once through the clasp on top. This loop satisfies the following two properties:

- (i) Γ bounds a disk in S^3 which intersects K in exactly two points.
- (ii) The linking number in S^3 of Γ with its pushoff on Σ is $+1$.

It follows from these properties that Γ bounds a punctured torus in $S^3 \setminus K$.

In the knot surgery construction, one has a certain freedom in choosing the gluing of $S^1 \times (S^3 \setminus N(K))$ to $D \setminus N(F)$. We are free to make any choice as long as a longitude of K is sent to the boundary circle of a normal disk to F . We choose the gluing so that the class of a meridian m of K is sent to the class of $a \times \{pt\}$ in $H_1(\partial(D \setminus N(F)); \mathbf{Z}) = H_1(F \times \partial D^2; \mathbf{Z})$, where a is the vanishing cycle for the double node neighborhood.

Before knot surgery, the fibration of D has a section which is a disk. The result of knot surgery is to remove a smaller disk in this section and to replace it with the Seifert surface of K_n . Call the resulting relative homology class in $H_2(D_{K_n}, \partial; \mathbf{Z})$ the *pseudosection*. Thus in D_{K_n} , our knot-surgered neighborhood, there is a punctured torus Σ representing the pseudosection. The loop Γ sits on Σ and by (i) it bounds a twice-punctured disk Δ in $\{pt\} \times \partial(S^3 \setminus N(K))$ where $\partial\Delta = \Gamma \cup m_1 \cup m_2$ where the m_i are meridians of K . The meridians m_i bound disjoint vanishing disks Δ_i in $D \setminus N(F)$ since they are identified with disjoint loops each of which represents the class of $a \times \{pt\}$ in $H_1(\partial(D \setminus N(F)); \mathbf{Z})$. Hence in D_K the loop $\Gamma \subset \Sigma$ bounds a disk $U = \Delta \cup \Delta_1 \cup \Delta_2$. By construction, the self-intersection number of U relative to the framing given by the pushoff of Γ in Σ is $+1 - 1 - 1 = -1$. (This uses (ii).)

If we had an embedded disk V in D_K with boundary Γ , with $V \cap \Sigma = \Gamma$, and with self-intersection number 0 with respect to the framing given by Σ , we could perform ambient surgery on $\Sigma \subset D_{K_n}$, surgering it to a 2-sphere. Instead of V , we actually have an embedded disk U such that $U \cap \Sigma = \Gamma$, and with self-intersection number -1 with respect to the framing given by Σ . We can perform

ambient surgery to turn Σ in to a sphere, but now it is immersed with one positive double point. Viewing $D_{K_n} \subset E(1)_{K_n}$, we get a pseudosection S' of $E(1)_{K_n}$ and it is represented by an immersed sphere of square -1 . We are then in the situation of Figure 11 where we have the configuration consisting of the immersed 2-sphere S' with a pair of disjoint nodal fibers, each intersecting S' once transversely. Also, S' intersects the I_8 -fiber transversely in one point.

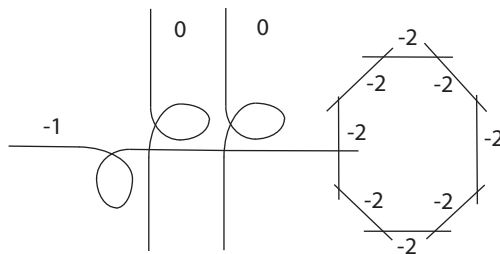


Figure 11

At this stage we have three possibilities:

1. If we blow up the double point of S' , then in $E(1)_{K_n} \# \overline{\mathbf{CP}}^2$ we obtain a configuration consisting of the total transform S'' of S' , which is a sphere of self-intersection -5 , and the sphere of self-intersection -2 at which S' intersects I_8 . (See Figure 12.) This is the configuration C_3 which can be rationally blown down to obtain a manifold Y_n with $b^+ = 1$ and $b^- = 8$. It is easy to see that Y_n is simply connected; so it is homeomorphic to $\mathbf{CP}^2 \# 8 \overline{\mathbf{CP}}^2$.

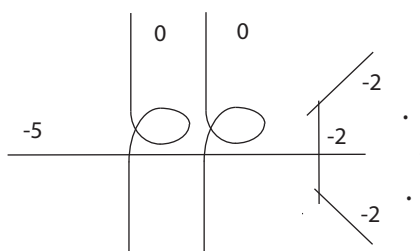


Figure 12

2. If we blow up at the double point of S' as well as at the double point of one of the nodal fibers, in $E(1)_{K_n} \# 2 \overline{\mathbf{CP}}^2$ we get a configuration of 2-spheres consisting of S'' , a transverse sphere F' of self-intersection -4 , and three spheres from the I_8 -fiber. (See Figure 13.) We can smooth the intersection of S'' and F' by replacing a neighborhood of the intersection point (a cone on the Hopf link) with an annulus (a Seifert surface for the Hopf link). This gives a sphere of self-intersection -7 , and we obtain the configuration C_5 . Rationally blowing down C_5 gives a manifold X_n homeomorphic to $\mathbf{CP}^2 \# 7 \overline{\mathbf{CP}}^2$.

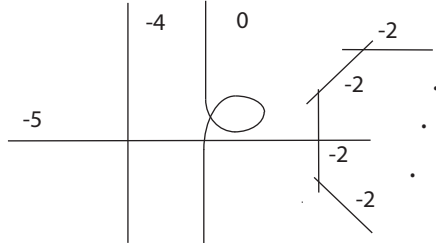


Figure 13

3. If we blow up at the double point of S' as well as at the double points of both nodal fibers, then in $E(1)_{K_n} \# 3 \overline{\mathbf{CP}}^2$ we get a configuration of 2-spheres consisting of S'' , two disjoint transverse spheres F', F'' of self-intersection -4 , and five spheres from the I_8 -fiber. (See Figure 14.) Smoothing the intersections of S'', F' and F'' as above gives a sphere of self-intersection -9 and we obtain the configuration C_7 . Rationally blowing down C_7 gives a manifold Z_n homeomorphic to $\mathbf{CP}^2 \# 6 \overline{\mathbf{CP}}^2$.

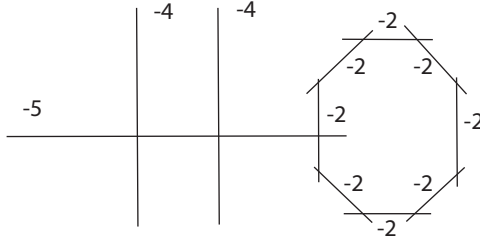


Figure 14

Theorem. *No two of the manifolds Y_n, X_n, Z_n are diffeomorphic, and for $n \geq 2$ they are minimal. In particular, there are infinite families of mutually nondiffeomorphic 4-manifolds homeomorphic to $CP \# k \overline{\mathbf{CP}}^2$, for $k = 6, 7, 8$.*

We will discuss the $k=7$ case. The other cases are similar. Since $\mathcal{SW}_{E(1)_{K_n}} = -nt + nt^{-1}$, the blowup formula easily calculates $\mathcal{SW}_{E(1)_{K_n} \# 2 \overline{\mathbf{CP}}^2}$. If we let $\pm T$ denote the basic classes of $E(1)_{K_n}$, then $|\mathcal{SW}_{E(1)_{K_n} \# 2 \overline{\mathbf{CP}}^2, h}(\pm T \pm E_1 \pm E_2)| = n$, and $\mathcal{SW}_{Z_n, h}(L) = 0$ for all other classes L . Here h is the class of square $+1$ obtained as follows.

In $E(1)$ a sphere Λ representing H intersects the fiber F in 3 points. After knot surgery, this gives rise to a class h of genus 3 that has $h^2 = 1$ and $h \cdot T = 3$. (The three normal disks to a fiber that lie in the sphere Λ are replaced by genus one Seifert surfaces of K_n .) Since the Seiberg-Witten invariant of $E(1)_{K_n}$ is well-defined, $\mathcal{SW}_{E(1)_{K_n}, h}(L) = \mathcal{SW}_{E(1)_{K_n}}(L)$ for all characteristic $L \in H_2(E(1)_{K_n}; \mathbf{Z})$. The upshot of this and the wall-crossing formula is the paragraph above.

One can now apply the criterion of §4 to see that of the classes $\pm T \pm E_1 \pm E_2$ only $\pm(T + E_1 + E_2)$ descends to the rational blowdown X_n . So these give classes $\pm K_n \in H_2(X_n; \mathbf{Z})$, and $|\mathcal{SW}_{X_n}(\pm K_n)| = n$. (See [18] for details.) For all other characteristic homology classes L of X_n , our rational blowdown theorem tells us that $\mathcal{SW}_{X_n}(L) = 0, \pm 1$. This proves that no two $X_m, X_n, m \neq n$, are diffeomorphic. It is an exercise in the use of the blowup formula to see that X_n is minimal for $n \geq 2$.

From this vantage point it is now relatively easy to construct infinitely many distinct smooth structures on $\mathbf{CP}^2 \# 5 \overline{\mathbf{CP}^2}$, as is done in [18]. We outline the construction. It begins by noting that there is an elliptic fibration on $E(1)$ containing two double node neighborhoods and one I_6 singularity. Perform a knot surgery in each double node neighborhood; say using the twist knot $T(1)$ in one neighborhood and $T(n)$ in the other. In each neighborhood we need to be careful to perform the knot surgery so that the meridian of the knot is identified with the vanishing cycle. Let V_n be the resultant manifold. It is an exercise to show that V_n is simply-connected.

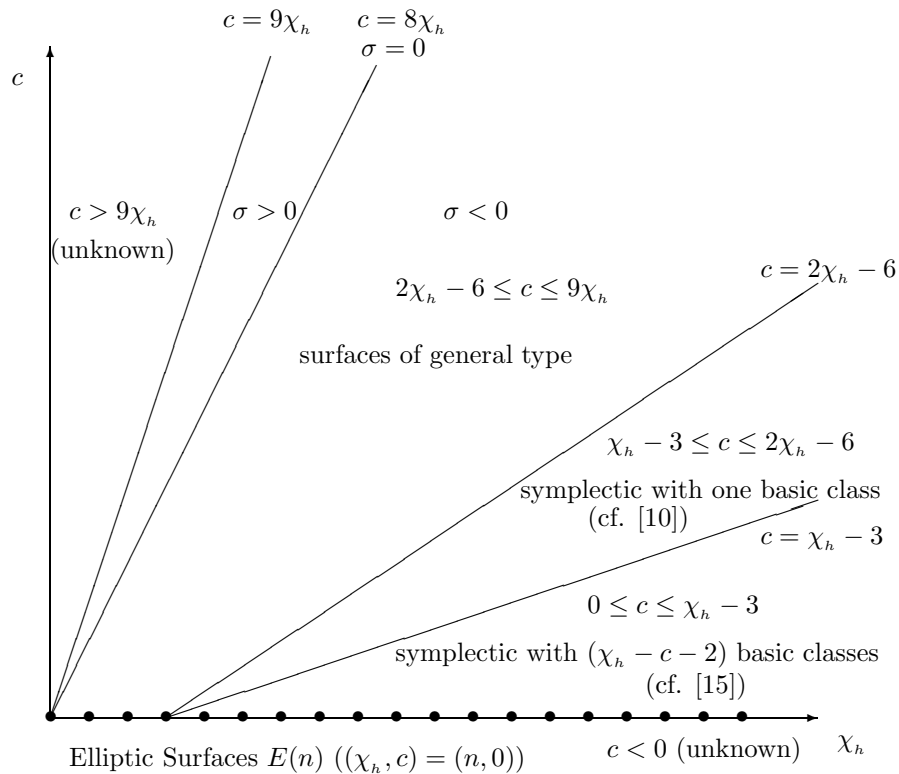
Using [13], one computes the Seiberg-Witten invariants of V_n : Up to sign the only classes in $H_2(V_n; \mathbf{Z})$ with nontrivial Seiberg-Witten invariants are T and $3T$, and $|\text{SW}_{V_n}(\pm 3T)| = n$, $|\text{SW}_{V_n}(\pm T)| = 2n - 1$.

To finish the construction, use the two double node neighborhoods to get a representative of the pseudo-section of V_n which is an immersed sphere with two positive double points. If we blow up these two points we get an embedded sphere S of self-intersection -9 in $W_n = V_n \# 2 \overline{\mathbf{CP}^2}$. By removing one of the spheres in the I_6 configuration, we get 5 spheres which, along with S form the configuration C_7 . As above, we can see that the only classes $L \in H_2(W_n; \mathbf{Z})$ which have $\text{SW}_{W_n, h}(L) \neq 0$ and whose restriction to C_7 has self-intersection -6 are $L = \pm(3T + E_0 + E_1)$. If we let Q_n denote the result of rationally blowing down C_7 in W_n , then an argument exactly as above shows that the manifolds Q_n ($n > 0$) are all homeomorphic to $\mathbf{CP}^2 \# 5 \overline{\mathbf{CP}^2}$, and no two of these manifolds are diffeomorphic. Furthermore, for $n \geq 2$, the Q_n are all minimal.

Putting it all together: The Geography and Botany of 4-manifolds

1. Existence: The Geography Problem

The existence part of a classification scheme for irreducible smooth (or symplectic) 4-manifolds could take the form of determining which $(\chi_h, c, t) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_2$ can occur as $(\chi_h, c, t)(X)$ for some smooth (symplectic) 4-manifold X . This is referred to as *the geography problem*. Our current understanding of the geography problem is given by Figure 15 where all known simply connected smooth irreducible 4-manifolds with nontrivial Seiberg-Witten invariants are plotted as lattice points in the (χ_h, c) -plane.



Simply connected complex surfaces are of three types:

- homeomorphic to blowups of the complex projective plane \mathbf{CP}^2 or the Hirzebruch surfaces F_n , i.e. are rational surfaces,
- the elliptic surfaces and their logarithmic transforms $E(n)_{p,q}$, i.e. $c = 0$,
- are of general type, i.e. $2\chi_h - 6 \leq c < 9\chi_h$.

We discussed the construction of the elliptic surfaces and their logarithmic transforms in Lectures 1 and 2 and discussed manifolds homeomorphic to rational surfaces in Lecture 5. We also showed how the operation of rational blowdown can give examples of irreducible simply connected symplectic manifolds with $0 < c < 2\chi_h - 6$ in Lectures 1 and 4. In fact every lattice point in the $0 < c < 2\chi_h - 6$ region can be realized by an irreducible simply connected symplectic manifold [10, 15, 21, 39, 49, 50]. The techniques used are an artful application of either the generalized fiber sum or the rational blowdown constructions. Within this region there appears to be an interesting relationship between the minimal number of Seiberg-Witten basic classes and the pair (χ_h, c) . In particular, all known smooth 4-manifolds with $0 \leq c \leq \chi_h - 3$ have at least $\chi_h - c - 2$ Seiberg-Witten basic classes [10]. There is a physics, but yet no mathematical, proof that this must be the case [31].

It is generally believed that all invariants in the range $2\chi_h - 6 \leq c \leq 9\chi_h$ can be realized by a complex surface. It is a more delicate question to require these manifolds to be simply connected. It is known that all complex surfaces with $c = 9\chi_h$ and $\chi_h > 1$ are ball quotients and hence cannot be simply connected. So the question remains if every lattice point in the $2\chi_h - 6 \leq c < 9\chi_h$ can be realized by a minimal (irreducible) simply connected complex manifold. There remain gaps, but this appears to be limited by the construction techniques rather than any intrinsic reason. In [38] Jongil Park uses the fiber sum and rational blowdown constructions to show that all but at most finitely many lattice points in the $c \leq 8\chi_h$ region can be realized by simply connected irreducible symplectic 4-manifolds. For complex manifolds, the result is less sharp. For example Persson [44] shows that the region $2\chi_h - 6 \leq c \leq 8(\chi_h - C\chi_h^{2/3})$ can be realized by a minimal simply connected complex surface where C can be taken to be $9/12^{1/3}$.

The so called *arctic region* where $8\chi_h < c < 9\chi_h$ has many gaps, again most likely due to a lack of sufficiently creative constructions. For simply connected complex surfaces see [46]. For symplectic manifolds Andras Stipsicz [49] and Jongil Park [39] have constructions of simply connected symplectic 4-manifolds in this region that start with nonsimply connected complex surfaces on or near the $c = 9\chi_h$ line and then use the fiber sum construction to kill the fundamental group.

Later in this lecture we will concentrate on some interesting problems and constructions at the extremes of the region for surfaces of general type, i.e the Horikawa surfaces with $c = 2\chi_h - 6$ and manifolds with $c = 9\chi_h$. Before we do there are some open problems that should not go unmentioned.

An irreducible 4-manifold need not lie on a lattice point, for example just change the orientation of any manifold with odd c . Note that $\chi_h(X) \in \mathbf{Z}$ iff X admits an almost-complex structure. In addition, the Seiberg-Witten invariants are only defined for manifolds with $\chi_h \in \mathbf{Z}$. Since our only technique to determine if a 4-manifold is irreducible is to use Seiberg-Witten invariants, all known irreducible 4-manifolds have (with one orientation) $\chi_h \in \mathbf{Z}$.

Now irreducibility does not depend upon orientation. So the obvious question is whether there exist irreducible smooth 4-manifolds with $\chi_h \notin \mathbf{Z}$ for both orientations? Here the work of Bauer and Furuta [4] on stable homotopy invariants derived from the Seiberg-Witten equations may be useful. As a simple candidate for an irreducible manifold with $\chi_h \notin \mathbf{Z}$ with both orientations consider two copies of the elliptic surface $E(2)$. Remove the neighborhood of a sphere with self-intersection -2 from each and glue together the resulting manifolds along their boundary $\mathbf{R}\mathbf{P}^3$ using the orientation reversing diffeomorphism of $\mathbf{R}\mathbf{P}^3$. The result has $\chi_h = \frac{7}{2}$ (or $\chi_h = \frac{37}{2}$ with the opposite orientation) and it is unknown if it is irreducible. See [21] for a handlebody description of this manifold.

As mentioned earlier, the work of Taubes [56] shows that $c \geq 0$ for an irreducible symplectic 4-manifold. Also, all of the known simply connected irreducible 4-manifolds are homeomorphic (possibly changing orientation) to symplectic manifolds. There may be an interesting class of 4-irreducible manifolds yet to be discovered, i.e. those with $c < 0$. (The above example has $c = -2$.) Note that if every simply connected irreducible smooth 4-manifold has $c \geq 0$ regardless of orientation, then $c \leq \frac{48}{5}\chi_h$.

Figure 15 contains no information about the geography of spin 4-manifolds, i.e. manifolds with $t = 0$. For a spin 4-manifold there is the relation $c = 8\chi_h \pmod{16}$. Almost every lattice point with $c = 8\chi_h \pmod{16}$ and $0 \leq c < 9\chi_h$ can be realized by an irreducible spin 4-manifold [40]. Surprisingly not all of the lattice points with $2\chi_h \leq 3(\chi_h - 5)$ can be realized by complex manifolds with $t = 0$ [46], so spin manifolds with $2\chi_h \leq 3(\chi_h - 5)$ provide more examples of smooth irreducible 4-manifolds with $2\chi_h - 6 \leq c < 9\chi_h$ that support no complex structure (cf. [16, 40]).

2. Uniqueness: The Botany Problem

The uniqueness part of the classification scheme, sometimes referred to as *the botany problem*, is to determine all smooth (symplectic) 4-manifolds with a fixed (χ_h, c, t) and to discover invariants that would distinguish them. Here is where we begin to lose control of the classification of smooth 4-manifolds. As we have seen in our lecture on knot surgery, if a topological 4-manifold admits a smooth (symplectic) structure with nontrivial Seiberg-Witten invariants and that also contains a smoothly (symplectically) embedded minimal genus torus with self-intersection zero and with simply connected complement, then it also admits infinitely many distinct smooth (symplectic) structures and also admits infinitely many distinct smooth structures with no compatible symplectic structure. There are no known examples of (simply connected) smooth or symplectic 4-manifolds with $\chi_h > 1$ that do not admit such tori. Hence, there are no known smooth or symplectic 4-manifolds with $\chi_h > 1$ that admit finitely many smooth or symplectic structures. We also showed in our last lecture that even for manifolds with $\chi_h = 1$, for which there are no essential minimal genus tori with trivial normal bundle, there are no known manifolds with finitely many smooth structures.

While knot surgery appears to be a new operation, we saw in an Lecture 3 that the knot surgery construction is actually a series of ± 1 surgeries on *nullhomologous tori*. The underlying observation is that any knot can be unknotted via a sequence of crossing changes, which in turn can be realized as a sequence of ± 1 surgeries on unknotted curves $\{c_1, \dots, c_n\}$ that link the knot algebraically zero

times and geometrically twice. When crossed with S^1 this translates to the fact that X can be obtained from X_K via a sequence of ± 1 surgeries on the *nullhomologous tori* $\{S^1 \times c_1, \dots, S^1 \times c_n\}$ in X . So the hidden mechanism behind the knot surgery construction is surgeries on nullhomologous tori. The calculation of the Seiberg-Witten invariants is then reduced to understanding how the Seiberg-Witten invariants change under a surgery on a nullhomologous torus.

So if we expand the notion of torus surgery to include both homologically essential and nullhomologous tori, then we can eliminate the knot surgery construction from our list of the essential operations we have used to both construct smooth 4-manifolds and to alter their smooth structures. Thus our list of essential operations is reduced to

- generalized fiber sum
- surgery on a torus with trivial normal bundle
- blowup
- rational blowdown

There are further relationships between these operations. In [12] it is shown that if T is contained in a node neighborhood, then a torus surgery can be obtained via a sequence of blowups and rational blowdowns. However, it is not clear, and probably not true, that a rational blowdown is always the result of blowups and logarithmic transforms.

As an aside, this fact together with work of Margaret Symington [53] shows that logarithmic transformations ($p \neq 0$) on a symplectic torus results in a symplectic manifold. We do not know of any general proof that a logarithmic transformation on a symplectic torus in a node neighborhood results in a symplectic manifold. However, there is an interesting proof that $E(1)_{p,q}$ is symplectic by observing that $E(1)_{p,q}$ is diffeomorphic to $E(1)_{T(p,q)}$ where $T(p,q)$ is the (p,q) torus knot. The proof uses the fact that every diffeomorphism of the boundary of $E(1) \setminus F$ extends. A simple comparison of Seiberg-Witten invariants show that $E(n)_{p,q}$ is not diffeomorphic to $E(n)_{T(p,q)}$ for $n > 1$.

Rational blowdown changes the topology of the manifold X ; while χ_h remains the same, c is decreased by $p - 3$. So, an obvious conjecture would be

Conjecture 6.1. Two homeomorphic simply connected smooth 4-manifolds are related via a sequence of surgeries on tori of square 0.

A slight weakening would be

Conjecture 6.2. Two homeomorphic simply connected smooth 4-manifolds are related via a sequence of of surgeries on tori of square 0 and rational blowups and rational blowdowns.

Of course there are no counterexamples to these conjectures since these operations are the only ones used so far to change smooth structures. However, there are many examples of complex structures on a given manifold that are deformation inequivalent that are unknown whether they are diffeomorphic. A good test of these conjectures would be to determine if these complex structures are related via a sequence of surgeries on tori.

3. Horikawa Surfaces: How to go from one deformation type to another

We mentioned in our first lecture that it is still unknown whether the two deformation types of Horikawa surfaces $H(4n - 1)$ and $H'(4n - 1)$ with $(c, \chi_h, t) =$

$(8n - 8, 4n - 1)$ are, for $n > 1$, diffeomorphic or even symplectomorphic. However, since both these deformation types are branched covers of $S^2 \times S^2$ we can use techniques of [2, 14] to braid the branched set to show that indeed $H(4n - 1)$ can be obtained from $H'(4n - 1)$ via a ± 1 surgery on an essential torus as follows.

The basic idea is that the only difference between $H(4n - 1)$ and $H'(4n - 1)$ is that both are covers of $S^2 \times S^2$ branched over a curve of bidegree $(6, 4n)$, with the former a connected curve and the latter a disconnected curve. By braiding the later we make the curve connected. This braiding process is a procedure in $S^2 \times S^2$ which removes an (Lagrangian) annulus whose boundary components lie on each component of the branch set for $H'(4n - 1)$ and sews it back in so that the branch set is now connected. This annulus lifts to a torus in $H'(4n - 1)$ and the surgery operation lifts to a ± 1 surgery on this torus. Of course, in order to know that the resulting manifold is $H'(4n - 1)$ we need to know that the resulting connected branch curve is isotopic to the connected complex curve of bidegree $(6, 4n)$. This is precisely the situation covered by the work of Siebert and Tian [47] which guarantees that any two such symplectic curves in $S^2 \times S^2$ are isotopic.

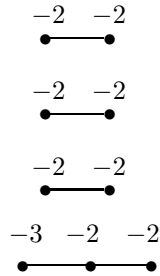
For a more general discussion of this phenomena and for other intriguing relationships between these deformation types see Auroux's fascinating article [1].

4. Manifolds with $c = 9\chi_h$: A fake projective plane

One would like to state that every lattice point (χ_h, c) in the $c \leq 9\chi_h$ can be realized by an irreducible simply connected 4-manifold. This is almost true; however, there are open problems near the $c = 9\chi_h$ line, called the *Bogomolov-Miyaoka-Yau (BMY) line*. Part of the problem is that it is known that all complex surfaces with $c = 9\chi_h > 9$ are ball quotients and hence not simply connected. As one approaches the $c = 9\chi_h$ line it is not known that there exist complex surfaces, simply connected or not. Most of the constructions have the property that the closer you get to the $c = 9\chi_h$ line the larger the fundamental group. For more detailed information of this phenomena see [46] or the papers [21, 39, 48, 49] that construct simply connected symplectic manifolds with large c that asymptotically approach the $9\chi_h$ line. It would be very interesting to find simply connected symplectic manifolds with $c = 9\chi_h > 9$ or with $c > 9\chi_h$.

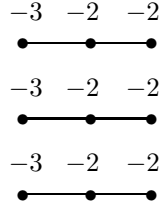
Part of the issue here is that the all the constructions of manifolds with positive signature, except for \mathbf{CP}^2 , rely on branched covering constructions with singular branched set and the closer one gets to the $c = 9\chi_h$ line the more singular the branched set. These singularities increase the complexity of the fundamental group of the cover. In the remainder of this lecture we would like to give the simplest construction that we know of a fake manifold with $c = 9\chi_h$. This construction is due to Keum [26] and starts with the existence of a particular configuration of spheres on a Doglachev surface shown to exist by Ishida [25]. The result will be a complex surface with the integral homology of \mathbf{CP}^2 .

In [25] Ishida constructs a Dolgachev surface $E(1; 2, 3)$ with a remarkable property. This elliptic fibration has two multiple fibers of multiplicity 2 and 3 respectively, and has an elliptic fibration with four singularities of type I_3 (see Lecture 5). It is not difficult to show that $E(1; 2, 3)$ has a sextuple section which is a sphere of self-intersection -3 . The remarkable property of Ishida's manifold is that it contains the following four disjoint configuration of spheres:



The first three configurations sit inside three of the four I_3 singularities and the -3 sphere in the last configuration is the sextuple section with the -2 spheres coming from the fourth I_3 singularity. It is remarkable to find such a configuration of nine spheres inside the Dolgachev surface $E(1; 2, 3)$ since they capture all its b^- .

With this starting point Keum then observes that the complement of the first three configurations has euler characteristic 3 and that it has a 3-fold cover with three boundary components that are S^3 each of which can be capped with a 4-ball to obtain a manifold X with euler characteristic 12. He shows that this is again a Dolgachev surface with four singular fibers, three of type I_1 and one of type I_9 . The important feature is that X now contains the following three disjoint configuration of spheres:



each a lift from $E(1; 2, 3)$. Again, this is a remarkable configuration to find in a Dolgachev surface since it again captures all of b^- . To finish Keum's construction he observes that the complement of this configuration has euler characteristic zero and has a 7-fold cover with three boundary components each a 3-sphere S^3 which can again be capped off with a 4-ball to obtain a manifold Y with Euler characteristic 3 and with $b_1 = 0$. This is Keum's construction of a fake (homology) projective plane. Of course there is some heavy lifting in showing the existence of the Ishida manifold for which we do not know of a direct (topological) construction.

5. Small 4-manifolds

As we have already mentioned, finding simply connected manifolds close to the $c = 9\chi_h$ line remains a challenge. This challenge is exposed in our inability to find irreducible simply connected 4-manifolds with small Euler characteristic. We have discussed in Lecture 5 the case when $\chi_h = 1$, i.e. are there irreducible simply connected smooth 4-manifolds homeomorphic to n blowups of \mathbf{CP}^2 with $n < 5$? For $\chi_h = 2$ the best that one can do to date is $c = 11$ [42, 52], i.e. there are irreducible

simply connected smooth 4-manifolds homeomorphic to $3\mathbf{CP}^2 \# n\overline{\mathbf{CP}^2}$ for $8 \leq n \leq 19$. The constructions are similar to those discussed in Lecture 5 applied to $E(2)$. What about $n < 8$? The inability to get close to the $c = 9\chi_h$ line permeates for $\chi_h > 2$. Further progress on constructing simply connected 4-manifolds with small Euler characteristic should yield further insight into constructing manifolds in the arctic region.

6. What were the four 4-manifolds?

Our four 4-manifolds represent examples of simply connected complex surfaces with Kodaira dimension $-\infty, 0, 1$, and 2 , *i.e.* \mathbf{CP}^2 and its blowups, the K3 surface $E(2)$, the elliptic fibrations $E(n)$, $n > 2$, and $H(4n - 1)$, $n > 1$.

Bibliography

- [1] Denis Auroux, *The canonical pencils on Horikawa surfaces* (2006), preprint, available at <http://front.math.ucdavis.edu/math.GT/0605692>.
- [2] Denis Auroux, Simon K. Donaldson, and Ludmil Katzarkov, *Singular Lefschetz pencils*, *Geom. Topol.* **9** (2005), 1043–1114 (electronic). MR **2140998** (**2005m**:53164)
- [3] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* [Results in Mathematics and Related Areas (3)], vol. 4, Springer-Verlag, Berlin, 1984. MR **749574** (**86c**:32026)
- [4] Stefan Bauer and Mikio Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants. I*, *Invent. Math.* **155** (2004), no. 1, 1–19. MR 2025298
- [5] W. R. Brakes, *Sewing-up link exteriors*, *Low-dimensional topology* (Bangor, 1979), London Math. Soc. Lecture Note Ser., vol. 48, Cambridge Univ. Press, Cambridge, 1982, pp. 27–37. MR **662425** (**83m**:57009)
- [6] Igor Dolgachev, *Algebraic surfaces with $q = p_g = 0$* , C.I.M.E. Algebraic Surfaces, Cortona 1977 (Naples, 1981).
- [7] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, *J. Differential Geom.* **18** (1983), no. 2, 279–315. MR710056 (85c:57015)
- [8] ———, *Irrationality and the h-cobordism conjecture*, *J. Differential Geom.* **26** (1987), no. 1, 141–168. MR **892034** (**88j**:57035)
- [9] Ronald Fintushel and Ronald J. Stern, *The blowup formula for Donaldson invariants*, *Ann. of Math. (2)* **143** (1996), no. 3, 529–546. MR **1394968** (**97i**:57036)
- [10] Ronald J. Fintushel, Jongil Park, and Ronald J. Stern, *Rational surfaces and symplectic 4-manifolds with one basic class*, *Algebr. Geom. Topol.* **2** (2002), 391–402 (electronic). MR **1917059** (**2003f**:57063)
- [11] Ronald Fintushel and Ronald J. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, *Turkish J. Math.* **19** (1995), no. 2, 145–157. MR **1349567** (**96j**:57036)
- [12] ———, *Rational blowdowns of smooth 4-manifolds*, *J. Differential Geom.* **46** (1997), no. 2, 181–235. MR **1484044** (**98j**:57047)
- [13] ———, *Knots, links, and 4-manifolds*, *Invent. Math.* **134** (1998), no. 2, 363–400. MR **1650308** (**99j**:57033)
- [14] ———, *Symplectic surfaces in a fixed homology class*, *J. Differential Geom.* **52** (1999), no. 2, 203–222. MR **1758295** (**2001j**:57036)
- [15] ———, *The canonical class of a symplectic 4-manifold*, *Turkish J. Math.* **25** (2001), no. 1, 137–145. MR **1829084** (**2002g**:57053)
- [16] ———, *Families of simply connected 4-manifolds with the same Seiberg-Witten invariants*, *Topology* **43** (2004), no. 6, 1449–1467. MR 2081432
- [17] ———, *Invariants for Lagrangian Tori*, *Geometry and Topology* **8** (2004), 949–968.
- [18] ———, *Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$* , *J. Amer. Math. Soc.* **19** (2006), no. 1, 171–180 (electronic). MR **2169045** (**2006h**:57024)
- [19] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, *J. Differential Geom.* **17** (1982), no. 3, 357–453. MR **679066** (**84b**:57006)
- [20] Robert Friedman and John W. Morgan, *On the diffeomorphism types of certain algebraic surfaces. I*, *J. Differential Geom.* **27** (1988), no. 2, 297–369. MR **925124** (**89d**:57046)
- [21] Robert E. Gompf and András I. Stipsicz, *4-manifolds and Kirby calculus*, *Graduate Studies in Mathematics*, vol. 20, American Mathematical Society, Providence, RI, 1999. MR **1707327** (**2000h**:57038)
- [22] Friedrich and Hopf Hirzebruch Heinz, *Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten*, *Math. Ann.* **136** (1958), 156–172. MR 0100844 (20 #7272)

- [23] Eiji Horikawa, *Algebraic surfaces of general type with small C_1^2 . I*, Ann. of Math. (2) **104** (1976), no. 2, 357–387. MR 0424831 (54 #12789)
- [24] Jim Hoste, *Sewn-up r -link exteriors*, Pacific J. Math. **112** (1984), no. 2, 347–382. MR **743990** (86f:57004)
- [25] Masa-Nori Ishida, *An elliptic surface covered by Mumford's fake projective plane*, Tohoku Math. J. (2) **40** (1988), no. 3, 367–396. MR **957050** (90a:14056)
- [26] Jonghae Keum, *A Fake Projective Plane with an order 7 automorphism* (2006), preprint, available at <http://arxiv.org/pdf/math.AG/0505339.pdf?front>.
- [27] Robion C. Kirby and Laurence C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J., 1977. With notes by John Milnor and Michael Atiyah; Annals of Mathematics Studies, No. 88. MR 0645390 (58 #31082)
- [28] Dieter Kotschick, *On manifolds homeomorphic to $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$* , Invent. Math. **95** (1989), no. 3, 591–600. MR **979367** (90a:57047)
- [29] P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994), no. 6, 797–808. MR **1306022** (96a:57073)
- [30] T. J. Li and A. Liu, *Symplectic structure on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett. **2** (1995), no. 4, 453–471. MR **1355707** (96m:57052)
- [31] Marcos Mariño, Gregory Moore, and Grigor Peradze, *Superconformal invariance and the geography of four-manifolds*, Comm. Math. Phys. **205** (1999), no. 3, 691–735. MR **1711332** (2001h:57037)
- [32] John W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Mathematical Notes, vol. 44, Princeton University Press, Princeton, NJ, 1996. MR **1367507** (97d:57042)
- [33] John W. Morgan, Tomasz S. Mrowka, and Zoltán Szabó, *Product formulas along T^3 for Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), no. 6, 915–929. MR **1492130** (99f:57039)
- [34] John W. Morgan and Zoltán Szabó, *On h -cobordisms and Seiberg-Witten invariants*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 1998, pp. 117–124. MR **1635699** (99h:57061)
- [35] Liviu I. Nicolaescu, *Notes on Seiberg-Witten theory*, Graduate Studies in Mathematics, vol. 28, American Mathematical Society, Providence, RI, 2000. MR **1787219** (2001k:57037)
- [36] Peter Ozsváth and Zoltán Szabó, *On Park's exotic smooth four-manifolds*, Geometry and topology of manifolds, Fields Inst. Commun., vol. 47, Amer. Math. Soc., Providence, RI, 2005, pp. 253–260. MR **2189937** (2006i:57060)
- [37] B. Doug Park, *A gluing formula for the Seiberg-Witten invariant along T^3* , Michigan Math. J. **50** (2002), no. 3, 593–611. MR **1935154** (2003i:57051)
- [38] Jongil Park, *The geography of irreducible 4-manifolds*, Proc. Amer. Math. Soc. **126** (1998), no. 8, 2493–2503. MR **1487335** (98j:57034)
- [39] ———, *Exotic smooth structures on 4-manifolds. II*, Topology Appl. **132** (2003), no. 2, 195–202. MR **1991809** (2004d:57033)
- [40] ———, *The geography of Spin symplectic 4-manifolds*, Math. Z. **240** (2002), no. 2, 405–421. MR **1900318** (2003c:57030)
- [41] ———, *Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$* , Invent. Math. **159** (2005), no. 3, 657–667. MR **2125736** (2006c:57024)
- [42] ———, *Exotic smooth structures on $3\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$* (2005), preprint, available at <http://front.math.ucdavis.edu/math.GT/0507085>.
- [43] Jongil Park, András I. Stipsicz, and Zoltán Szabó, *Exotic smooth structures on $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$* , Math. Res. Lett. **12** (2005), no. 5-6, 701–712. MR **2189231** (2006i:57059)
- [44] Ulf Persson, *Chern invariants of surfaces of general type*, Compositio Math. **43** (1981), no. 1, 3–58. MR **631426** (83b:14012)
- [45] ———, *Configurations of Kodaira fibers on rational elliptic surfaces*, Math. Z. **205** (1990), no. 1, 1–47. MR **1069483** (91f:14035)
- [46] Ulf Persson, Chris Peters, and Gang Xiao, *Geography of spin surfaces*, Topology **35** (1996), no. 4, 845–862. MR **1404912** (98h:14046)
- [47] Bernd Siebert and Gang Tian, *On the holomorphicity of genus two Lefschetz fibrations*, Ann. of Math. (2) **161** (2005), no. 2, 959–1020. MR **2153404** (2006g:53141)

- [48] András I. Stipsicz, *Simply connected symplectic 4-manifolds with positive signature*, Proceedings of 6th Gökova Geometry-Topology Conference, 1999, pp. 145–150. MR **1671185** (**2000h**:57047)
- [49] ———, *Simply connected 4-manifolds near the Bogomolov-Miyaoka-Yau line*, Math. Res. Lett. **5** (1998), no. 6, 723–730. MR **1671185** (**2000h**:57047)
- [50] András Stipsicz, *Geography of irreducible 4-manifolds*, European Congress of Mathematics, Vol. II (Budapest, 1996), 1998, pp. 221–233. MR **1645829** (**99j**:57026)
- [51] András I. Stipsicz and Zoltán Szabó, *An exotic smooth structure on $\mathbf{CP}^2\#6\overline{\mathbf{CP}}^2$* , Geom. Topol. **9** (2005), 813–832 (electronic). MR **2140993** (**2006f**:57030)
- [52] ———, *Small exotic 4-manifolds with $b_2^+ = 3$* (2005), preprint, available at <http://front.math.ucdavis.edu/math.GT/0501273>.
- [53] Margaret Symington, *Symplectic rational blowdowns*, J. Differential Geom. **50** (1998), no. 3, 505–518. MR **1690738** (**2000e**:57043)
- [54] Clifford Henry Taubes, *The Seiberg-Witten and Gromov invariants*, Math. Res. Lett. **2** (1995), no. 2, 221–238. MR **1324704** (**96a**:57076)
- [55] Clifford H. Taubes, *The geometry of the Seiberg-Witten invariants*, Surveys in differential geometry, Vol. III (Cambridge, MA, 1996), Int. Press, Boston, MA, 1998, pp. 299–339. MR **1677891** (**2000j**:57067)
- [56] Clifford Henry Taubes, *Seiberg-Witten and Gromov invariants for symplectic 4-manifolds*, First International Press Lecture Series, vol. 2, International Press, Somerville, MA, 2000. Edited by Richard Wentworth. MR **1798809** (**2002j**:53115)
- [57] ———, *The Seiberg-Witten invariants and 4-manifolds with essential tori*, Geom. Topol. **5** (2001), 441–519 (electronic). MR **1833751** (**2002d**:57025)
- [58] Edward Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), no. 6, 769–796. MR **1306021** (**96d**:57035)
- [59] C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, London, 1970. London Mathematical Society Monographs, No. 1. MR 0431216 (55 #4217)