How efficiently do 3-manifolds bound 4-manifolds?

Dylan Thurston
Joint work with Francesco Costantino

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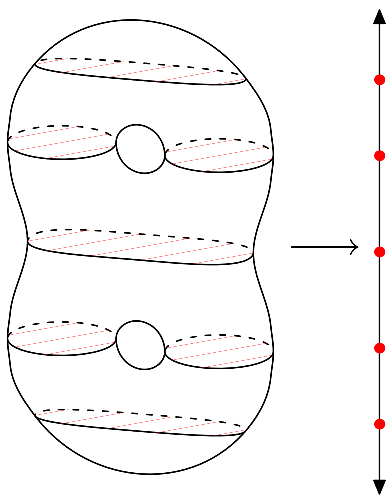


Part 1: Context

Representing 3-manifolds

Triangulations:

- Natural representation.
- Easy to construct from other representations.
- Compute some invariants (Turaev-Viro).
- Difficult to visualise.

Surgery diagrams:

- Compute invariants (Casson, Witten-Reshetikhin-Turaev).
- In practice, gives simple representations of small manifolds.
- How much do you lose in principle?
- Nearly the same as giving a 4-manifold bounded by the 3-manifold.

Problem statement

Definition. The *complexity* of an (oriented) d-manifold is the minimal number of simplices in a triangulation.

$$C(M^d) = \min_{\text{Triang. } \Delta \text{ of } M} \# \text{ of } d\text{-simplices in } \Delta$$

The *3-dimensional isoperimetric function* gives the minimal complexity of 4-manifolds bounding 3-manifolds of a given complexity.

$$G_3(n) = \max_{M^3 | C(M) \leq n} \ \min_{N^4 | \partial N \cong M} C(N)$$

Question. What is the asymptotic growth rate of $G_3(n)$?

Theorem (Costantino-T.). *There is a constant $k > 0$ so that*

$$G_3(n) \leq kn^2$$

Remark. A related question would require that the triangulation of the 4-manifold on the boundary agree with a given triangulated 3-manifold. We also get a quadratic bound in this case.

Previous work

Constructive proofs that 3-manifolds bound 4-manifolds give estimates for $G_3(n)$.

Proofs we're aware of:

- Rohlin (1951): Based on a generic map $f : M^3 \to \mathbb{R}^5$. Probably gives $G_3(n) \leq kn^4$.
- Thom (1954): Homotopy-theoretic. Hard to get explicit bounds on $G_3(n)$.
- Lickorish (1962), Rourke (1985), Matveev-Polyak (1994): Inductive proofs, based on mapping class group. Use the inductive hypothesis twice, so get exponential bounds for $G_3(n)$ at best.
- Costantino-T.: Based on a generic map $f : M^3 \to \mathbb{R}^2$. Gives $G_3(n) \leq kn^2$.

Analogous problems

We could also consider:

Question. What is the isoperimetric function G_{surf} for polygonal curves bounding triangulated surfaces in $\mathbb{R}^3$?

Theorem (Hass-Lagarias). $\frac{1}{2}n^2 \leq G_{\mathrm{surf}}(n) \leq 7n^2$.

Question. What is the isoperimetric function G_{disk} for *unknotted* polygonal curves bounding triangulated *disks* in $\mathbb{R}^3$?

Theorem (Hass-Lagarias-Snoeyink-W. Thurston). $c_1 A^n \leq G_{\mathrm{disk}}(n) \leq c_2 B^{n^2}$.

Question. What is the bound G_{Pachner} of the number of Pachner moves required to turn a triangulation with n tetrahedra into a standard one?

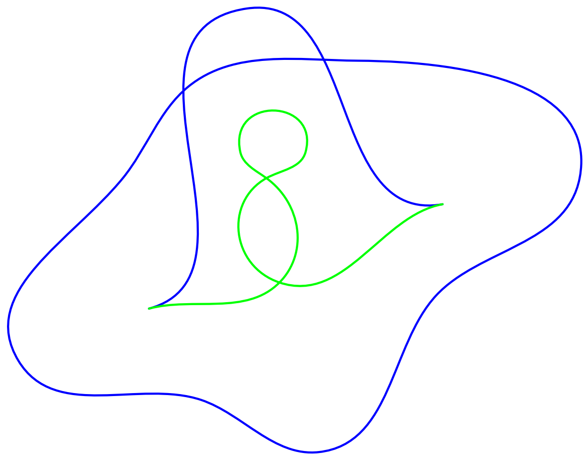
Theorem (King, Mijatović). $G_{\mathrm{Pachner}}(n) \leq c_1 A^{n^2}$.

Note that a sequence of Pachner moves gives a triangulation of the 4-ball. On the other hand, coning to a point gives a triangulation with many fewer 4-simplices. Perhaps bounding the geometry of the triangulation of the 4-ball in some way gives an appropriate analogous question to the growth of G_{disk}.

Part 2: The proof

Proof idea

Throw your 3-manifold at the screen.



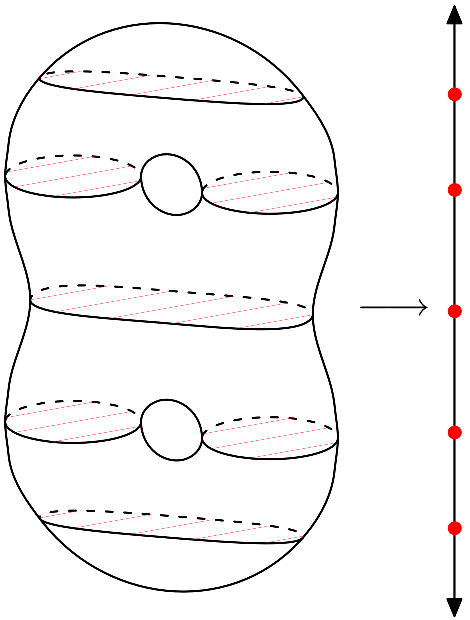
That is, take a generic smooth map from M^3 to $\mathbb{R}^2$.

The result is a blotch with some singularities. The singularities were first analysed by H. Levine (1988); we will look at them later.

The inverse image of a regular value is a 1-manifold, a disjoint union of circles. Idea: Glue in disks to these circles, and extend across the singularities.

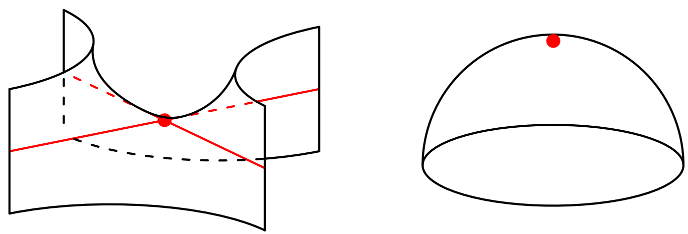
Warmup: 2-manifolds bound 3-manifolds
(idea suggested by Hatcher-W. Thurston)

One dimension down, let's consider a generic map from a surface Σ^2 to $\mathbb{R}^1$, i.e., a Morse function. The inverse image of a regular value is again a union of circles. Glue in disks to each of these circles.

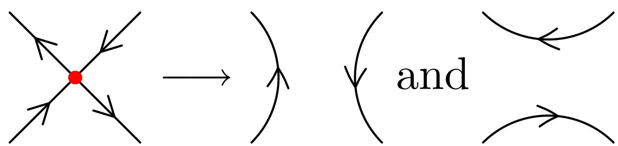


Singularites of maps from surfaces to $\mathbb{R}$

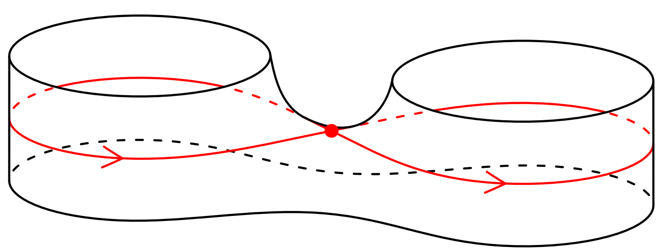
A critical point is either a saddle point or a maximum/minimum locally in the domain.



The inverse image of each regular point is an oriented 1-manifold, so the orientations into a saddle point must alternate:



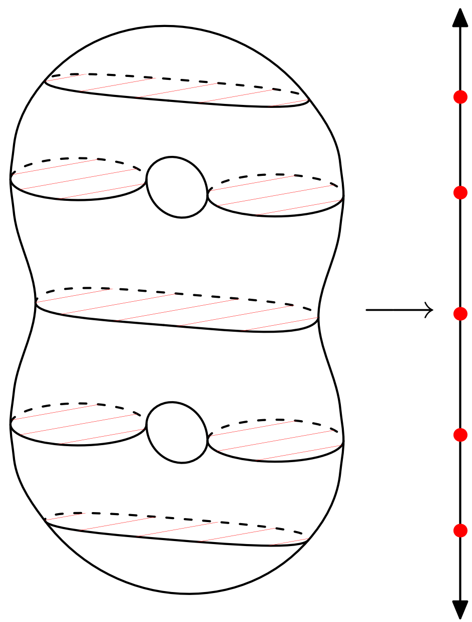
Therefore the inverse image of a saddle value is a figure 8, and the inverse image of its neighborhood is a pair of pants.



Filling in the 3-manifold

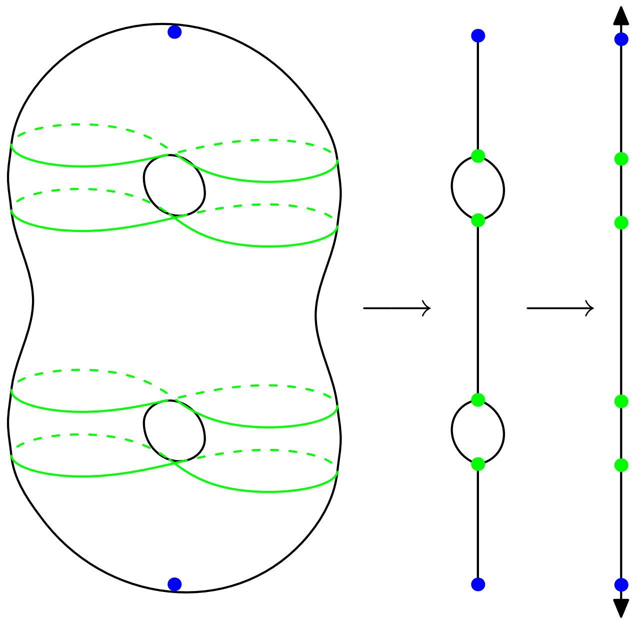
We can now finish constructing the 3-manifold. Take the surface cross an interval and glue in a 2-handle to the each circle in the inverse image of a regular point.

By the analysis above, the boundary remaining near each critical value is a sphere, in one of two ways. Glue in a ball to each one.



The Stein factorization

Let's view this construction from a more global point of view. For any map f from a compact manifold, we can consider the *Stein factorization* $f = g \circ h$ where the fibers of g are connected and h is finite-to-one.



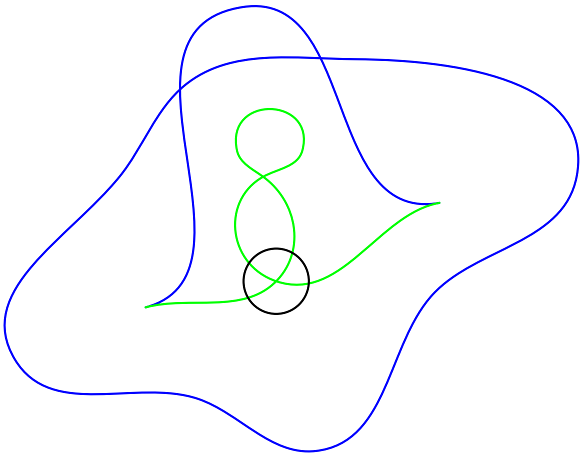
At generic points, the surface Σ is a circle bundle over the Stein graph and the 3-manifold is a disk bundle. Alternatively, the 3-manifold collapses onto the Stein graph.

3-manifolds

In the case of 3-manifolds mapping to $\mathbb{R}^2$, in codimension 1 the singularities look like the previous singularities (surface to $\mathbb{R}$), crossed with an interval.

In codimension 2 locally in the domain, the only new singularity is a kind of cusp. It can be viewed as two critical points of index 0 and 1 meeting and cancelling.

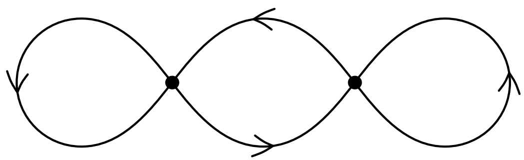
However, this singularity turns out to play little role. More interesting is the crossing of two codimension 1 saddle-type singularities.



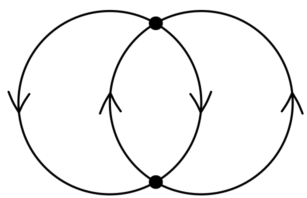
Crossing singularities

There are two singular points in the inverse image of a crossing of two saddle type singularities. Following the orientations as before, there are two ways to connect up these two singularities to get a connected fiber.

(a)



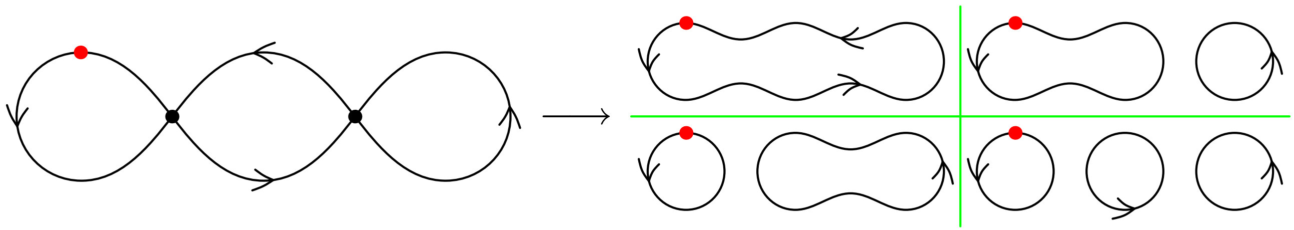
(b)



We will assume that our generic map has no singularities of type (b), since:

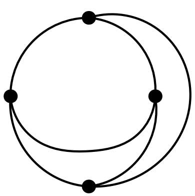
- Type (b) is similar to type (a), only slightly more complicated;
- A singularity of type (b) can be perturbed a little to get two singularities of type (a).

In a neighborhood of a singularity of type (a), the inverse image of a generic point is one of the 4 ways of resolving the singular graph into a 1-manifold.



We want to find the inverse image N of a neighborhood of this point, which is a 3-manifold with boundary.

The boundary has a map to S^1. The Stein graph is



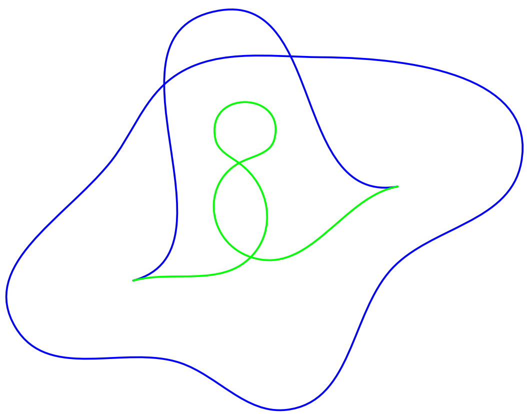
Therefore the boundary is a genus 3 surface.

Each regular point in the singular fiber gives us a disk in the boundary that bounds in N. Considering enough of these, we see that N is S^3 minus an unknotted tetrahedron graph.

Constructing the 4-manifold

Now assemble the pieces to construct a 4-manifold W^4 bounded by our 3-manifold M^3.

- Start with $M \times [0,1]$. We want to kill one boundary component.
- Attach a 2-handle along each circle in the inverse image of a regular point.
- Attach a 3-handle transverse to each codimension 1 singularity.
- The remaining boundary components are all 3-spheres, by the analysis of the singularities. Attach 4-balls to them.



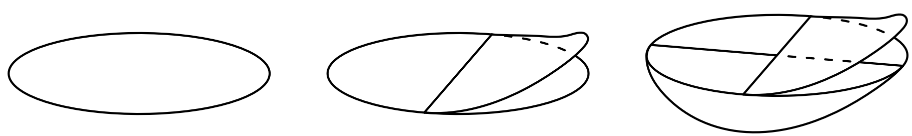
A more global view

As in the case of surfaces mapping to $\mathbb{R}$, we can consider the Stein factorization of our map from M^3 to $\mathbb{R}^2$.

The resulting Stein surface is a 2-complex. It has a number of local models, including

- the plane $\mathbb{R}^2$ at regular points;
- a 3-page book from codimension 1 singularities; and
- the cone over the graph we found around the crossing of singularities.

among a few others.



The 4-manifold collapses onto the Stein surface. As before, the 4-manifold is generically a disk bundle over the surface and the 3-manifold is generically a circle bundle.

We will say more on these surfaces later.

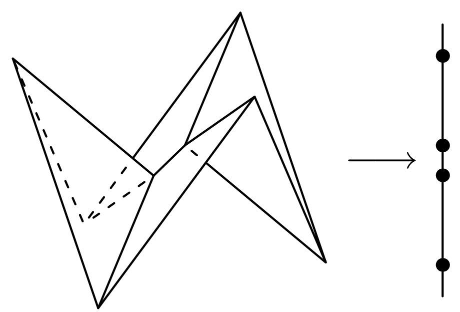
Starting from a triangulation

So far, we have been working with generic smooth maps. This has two problems:

- It involves analysis to find the generic singularities;
- To bound the complexity, we want to start from a triangulation.

So let's start with a (proper) triangulation and pick a generic piecewise linear map to the plane: Pick an arbitrary map from the vertices of the triangulation and extend linearly over the simplices.

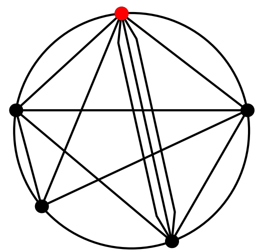
Problem: A generic piecewise linear map is not generic smooth. For example:



In codimension 1, we can see how to perturb these *monkey saddles* a little bit to get generic smooth map. But codimension 2 is harder...

Starting from a triangulation, continued

To avoid codimension 2 singularities, map the vertices of the triangulation to generic, distinct points on the circle $S^1 \subset R^2$.



Codimension 1 singularities occur along the image of an edge. Make them simple by perturbing along the length of the edge, effectively splitting the edge into parallel copies.

Codimension 2 singularities happen at the crossings of codimension 1 singularities (no problem), or at the image of a vertex (problem!).

But the inverse image of a neighborhood of the vertex in the plane is a ball: a neighborhood of the corresponding vertex in M. So we can cut out a neighborhood of the vertex, forget about it, and glue a ball in at the end.

Main point: Interesting singularities come from crossings of edges of the triangulation; this is quadratic in the number of tetrahedra. $\square$

Part 3: Going further

4-manifolds bounding 5-manifolds?

Why doesn't this proof work to show that 4-manifolds bound 5-manifolds? Start the same way: pick a generic map from your 4-manifold to $\mathbb{R}^3$...

There are some new codimension 3 singularities. For some of them, the inverse image of a neighborhood, filled in around the boundary, is not S^4, but rather $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$.

This does give a proof that every smooth 4-manifold is cobordant to a union of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$'s.

If you want to get bounds on the complexity, you also need to worry about generic PL maps which are not generic smooth maps in codimension 2.

The shadow world

A soap bubble locally looks like:

- A plane;
- A 3-page book; or
- A cone over the 1-skeleton of a tetrahedron.

A 2-complex Σ whose only singularities are in the list above is called a *simple polyhedron*.

If such a Σ is embedded in a 4-manifold W in a locally flat way, and W collapses onto Σ, then we call Σ a *shadow representation* of W and its boundary ∂W.

Contrast this with the standard spines for 3-manifolds, where the 3-manifold is generically an interval bundle over Σ, rather than a disk bundle.

To determine W from Σ, also need to specify some integers or half-integers (the *gleams*) on the 2-dimensional faces. If Σ is a closed surface, the gleam is the Euler class of the disk bundle. In general, it is a relative Euler class.

Shadow number and Gromov norm

Definition. The *shadow number* of a 3-manifold is the minimum number of vertices in a shadow representation of the manifold.

There is an improved version of the main theorem that deals with ideal triangulation or spun ideal triangulations:

Theorem. *A manifold with a (spun) ideal triangulation with n tetrahedra has a shadow diagram with $O(n^2)$ vertices.*

Theorem. *The 3-manifolds with shadow diagrams with no vertices are exactly the graph manifolds (manifolds which can be cut up into Seifert-fibered pieces).*

Theorem (W. Thurston)**.** *A hyperbolic manifold with volume V has a spun ideal triangulation with $O(V)$ tetrahedra.*

Corollary. *A manifold M with Gromov norm $\|M\|$ satisfying the Geometrization Conjecture has a shadow diagram has shadow number S satisfying*

$$C_1\|M\| \leq S \leq C_2\|M\|^2$$

for suitable constants C_1, C_2.

Open questions

- Lower bounds or better upper bounds for G_3, 3-manifolds bounding 4-manifolds.
- Lower bounds for G_{Pachner}, Pachner moves to make a triangulation of S^3 standard.
- Bounds for 3-manifolds to bound special 4-manifolds, like simply-connected or spin.
- Lower bounds with a coarser notion of the complexity of the 4-manifold (e.g., the order of the second homology).

One approach to a lower bound for G_3:

- Pick an invariant I of 3-manifolds that is defined from a 4-manifold bounded by the 3-manifold;
- Show that I is linearly bounded by the complexity of the 4-manifold;
- Find a family of 3-manifolds for which I grows quadratically in the complexity of the 3-manifold.