that he was almost certain to make mistakes in going over a piece of work for the first time, and that he could generally not find his errors until he would work over the problem a second time using different symbols. He frequently got very enthusiastic over the theory of groups remarking (not very seriously) that everything could be done by means of groups.

The students talked a great deal about his peculiarities but they had great respect for his attainments. In speaking about Klein and Lie, a Frenchman who had studied under both remarked, "Klein is a gentleman while Lie is just a good fellow." I presume many of those who studied under both would agree that this statement conveys a great deal of truth.

I feel encouraged to publish these few personal observations because I believe that Lie would have liked to have his faults go with his merits. I do not think he wanted to be regarded as one who thought that he had mastered every part of the extensive science of mathematics or as one who thought that all his habits were exemplary. He believed that he had contributed materially towards the advancement of the science of mathematics and he worked hard to accomplish this end. Future developments along the lines which have been emphasized or opened by him will have a great influence on his relative position among the mathematicians of this century.

Cornell University, June, 1899.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

Grassmann's *Ausdehnungslehre* is one of the few great works of mathematics of the 19th century. Appearing first in 1844 and rewritten in 1862, it is only within the last decade or two that it has received a tardy recognition. One reason for this is found in the difficulty of the subject itself, being unlike other mathematics ; and another, in the rigorous methods of presentation adopted by the author. In the *Ausdehnungslehre* of 1862, following some 150 pages of theory, the author for the first time gives the subject concrete form by applying his method to geometry. The theoretical part is naturally the more difficult, while the application to geometry is the more interesting. Hyde, in his *Directional Calculus*, purposing to present the *Ausdehnungslehre* to American readers, cut the knot of the difficulty by taking the results of the theoretical part for granted and giving only the application to geometry, and by limiting his treatment to two and three dimensional space.

An elementary exposition which will give the simpler portions of the theoretical part as well as the applications of the theory seems to be needed.

Such an exposition should serve the needs of two classes of readers : First, of those who would like to have a good general idea of the subject without going very deeply into its particulars ; and secondly, of those who, expecting to make a thorough study of the subject, wish first to read an introduction to it. To meet this want the following pages have been written. In some places for the purpose of making the subject clearer, changes have been made, but wherever they have been introduced, attention is always called to them.

CHAPTER I.

INTRODUCTION.

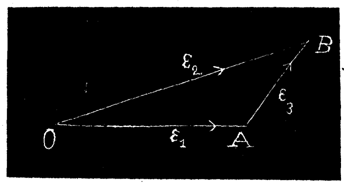
1. In elementary mathematics only one kind of unit is admitted, or at most two, viz., 1 and $\sqrt{-1}$. In the Theory of Extension besides the absolute unit of arithmetic and algebra, *extensive quantities* appear. The extensive quantities are different in nature from the absolute unit and different from each other. As a simple example of an extensive quantity we may name a *vector* (§3).

2. A *Scalar* quantity is a quantity of elementary mathematics, *i. e.*, a simple number, either positive or negative.

3. *A Vector* is a straight line whose length and direction are fixed but not its position. Thus any two parallel and equal straight lines may represent the same vector. A vector gives the relative position of one point with reference to another, viz., a certain distance in a certain direction.

4. Vectors can be *added* and *subtracted.*

Thus if ε_1 is the vector from O to A and ε_2 the vector from A to B, the sum of ε_1 and ε_2 is ε_3, because translation from O to B along the straight line OB is equivalent, or equal in the vector sense, to translation along OA and AB. Thus,



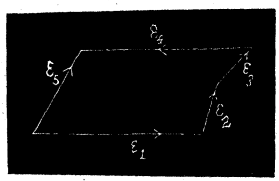
$$\varepsilon_1+\varepsilon_2=\varepsilon_3.$$

Transposing, we get

$$\varepsilon_1=\varepsilon_3-\varepsilon_2=\varepsilon_3+(-\varepsilon_2).$$

Interpreting this equation we see that translation along ε_3 followed by translation along ε_2 *in the negative direction* is equal to translation along ε_1.

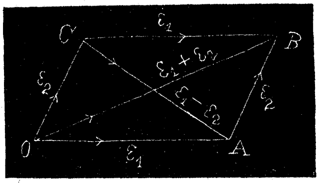
The sum of any number of vectors may evidently be found in the same way. Thus, in the figure



$$\varepsilon_5=\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4.$$

Stated generally, we have

The sum of any number of vectors is found by joining the beginning point of the second vector to the end point of the first, the beginning point of the third to the *end point of the second, and so on ; the vector from the beginning point of the first vector to the end point of the last is the sum required.*



The sum and difference of two vectors are the diagonals of the parallelogram whose adjacent sides are the given vectors.

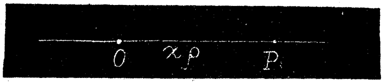
Or, more explicitly—

(1). *The sum of two vectors going out from an origin and forming two sides of a parellelogram is that diagonal of the parallelogram which passes through the origin.*

(2). *The difference of two such vectors is that diagonal which proceeds from the end of the subtrahend vector to the end of the minuend vector.*

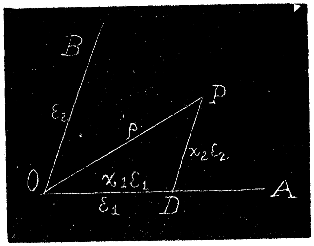
5. Vectors and line segments give us simple examples of extensive quantities. We proceed to show how lines and vectors can be used in a system of coördinates.

6. The simplest case of this is where a point is located on a given line. Let ρ denote any given line, and let O be an origin on it. Let further x be a scalar. Then by giving the proper value to x, $x\rho$ will locate any point P whatever on the line. Here ρ is to be regarded as an extensive quantity since it denotes not a number but the position and length of a line.



7. The next simplest case of coördinates is that in which a point is located in a plane by means of two vectors.

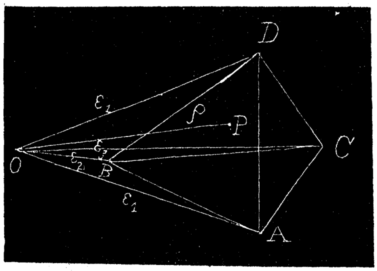
Let O, the origin, be a point in the given plane, and ε_1 and ε_2 two unit vectors in this plane. Then, by making



$$\rho = x_1\varepsilon_1 + x_2\varepsilon_2$$

$$\text{where } x_1 = \frac{r\sin BOP}{\sin BOA}, \quad x_2 = \frac{r\sin POA}{\sin BOA},$$

and r=length of ρ, the point P may be located at any point of the plane. Here ε_1, ε_2 and ρ are extensive quantities, and x_1 and x_2 are scalars.



8. In space we may have a similar system containing three vectors. Thus, if

$$\rho = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3$$

by assigning values to x_1, x_2, and x_3, P the extremity of ρ may be located at any point in space.

9. In the same way we can have a system including four vectors. Thus, if P is any point, $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ are four unit vectors, and

$$x_1 = \frac{\text{pyramid } P-BDC}{\text{pyramid } A-BDC}, \quad x_2 = \frac{\text{pyramid } P-ADC}{\text{pyramid } B-ADC},$$

etc., we have the equation

$$\rho = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4,$$

i. e. translation along ε_1 a distance equal to x_1, followed by translation along ε_2 a distance equal to x_2, and so on, is equivalent to translation from O to P direct. A geometrical proof of the truth of this can be given, but it is not thought necessary to insert it here.

REMARK. In the preceding the ε's in a certain sense denote dimensions. Then the *space* considered in this last article is of the *fourth* order.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF EXTENSIVE QUANTITIES.

10. DEFINITION. A quantity is said to be *independent* when it can not be expressed in terms of others. A quantity is said to be *dependent* when it can be numerically expressed in terms of others, *i. e.* as a sum formed out of numerical multiples of these quantities.

Thus if a_1, a_2, ... are extensive quantities, and α_1, α_2, ... are real numbers, positive or negative, and

$$a = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \ldots\ldots$$

a is said to be a dependent quantity and to be '*numerically expressed*' in terms of a_1, a_2, a_3, ...

11. A quantity a_1 is a *Unit* if it can serve along with other like units, a_2, a_3, ... to give a series of numerically derived quantities, a...... A unit is said to be *original* if it is not derived from other units. A set of quantities which are independent, *i. e.*, no one of which is numerically expressible in terms of one or more of the others is called a *System of Units*, provided any number of other quantities can be expressed in terms of them.

As an illustration of such a system we may take the set of units given in Art. 8. There ε_1, ε_2, ε_3 are a set of quantities which are independent because any sum formed out of multiples of ε_1 and ε_2 can never be a quantity like ε_3, since any sum formed from ε_1 and ε_2 would be a quantity in the plane of these two (7) while ε_3 is outside of this plane. Moreover, any number of other quantities, ρ's, can be derived from ε_1, ε_2, ε_3.

12. DEFINITION. An Extensive *Quantity* is a quantity numerically derived from a *system* of units. If an extensive quantity can be derived from the original units it is called an extensive quantity of the *first kind*.

13. DEFINITION. Quantities from the same system can be added (or subtracted) by adding the numerical coefficients of the same units.

Thus

$$\Sigma \alpha e + \Sigma \beta e = \Sigma(\alpha+\beta)e$$

where the α's and β's under the summation signs are numbers, and the e's are extensive units. We may add here that in the *Ausdehnungslehre* the distributive law is always assumed to hold.

REMARK. To remove ambiguity it will be understood that all indicated operations are performed as one comes to them from the left. Thus $a+b+c$ means $(a+b)+c$, and abc means $(ab)c$.

14. The following formulas underlie and justify all the operations involved in addition and subtraction in algebra. They follow directly from the definition in 13.

(1). $a+b=b+a$, a commutative law in addition and subtraction.

(2). $a+(b+c)=a+b+c$, associative law in addition and subtraction.

$\left.\begin{array}{l}(3).\ a+b-b=a \\ (4).\ a-b+b=a\end{array}\right\}$ *opposite* character of addition and subtraction.

Hence all the laws for addition and subtraction of algebraic numbers hold also for extensive quantities.

15. DEFINITION. When an extensive quantity is multiplied (or divided) by a *number* each of its coefficients is multiplied by that number.

Thus

$$\Sigma \alpha e.\beta = \Sigma(\alpha\beta)e.$$

REMARK. If a is an extensive quantity and α a number, then in αa or $a\alpha$ the numerical factor is the multiplier and the other factor is the multiplicand.

16. From Art. 15 we infer the following formulas :

(1). $a\alpha=\alpha a$,
(2). $a\beta\gamma=a(\beta\gamma)$,
(3). $(a+b)\gamma=a\gamma+b\gamma$,
(4). $a(\beta+\gamma)=a\beta+a\gamma$,

where, as heretofore, the Greek letters denote real numbers and the Roman letters, extensive quantities. From these formulas it follows—

That all the laws of multiplication and division of algebraic quantities hold also for extensive quantities multiplied or divided by numbers.

17. DEFINITION. The totality of quantities which are derivable from a series of extensive quantities, a_1, a_2, a_3, ... a_n is called the *Space* of those quantities. A space which can be formed out of not less than n such quantities each of the first kind (12) is called a space of the nth order.

18. DEFINITION. If every quantity of a space (A) is at the same time a quantity of another space (B), while the converse is not true, then the spaces are said to be *incident ;* the first is said to be *subordinate* to the second, and the second, to *include* the first.

19. *If n independent quantities a_1 ..... a_n can be numerically expressed in*

terms of n other quantities $b_1 \ldots\ldots b_n$, *then is the space of the first quantities identical with that of the last quantities. But if the n quantities* $a_1 \ldots\ldots a_n$ *can be expressed in terms of less than n quantities* $b_1 \ldots\ldots b_n$, *then* $a_1 \ldots\ldots a_n$ *are not independent, and some of them can be numerically expressed in terms of others.*

20. *Two quantities of a space of the nth order are equal to each other when and only when their numerical coefficients of the same units are equal.* This is analogous to the algebraic theorem which says that two complex numbers are equal only when their real parts are equal and also their imaginary parts.

21. *If the coefficients* $x_1 \ldots\ldots x_n$ *by which an extensive quantity* x *is expressed in terms of the units* $e_1 \ldots\ldots e_n$ *satisfy an equation of the mth degree* $f(x_1 \ldots\ldots x_n)=0$, *then the coefficients* $y_1 \ldots\ldots y_n$ *by which* x *is expressed in terms of* $a_1 \ldots\ldots a_n$ *of the same space also satisfy an equation of the mth degree, and if the first equation is homogeneous, the latter is also.*

PROOF. Let $a_1 = \Sigma\alpha_{1r}e_r, \ldots\ldots$ Then we have

$$x_1e_1+x_2e_2+\ldots\ldots+y_1\Sigma\alpha_{1r}e_r+y_2\Sigma\alpha_{2r}e_r+\ldots\ldots=\Sigma y_r\alpha_{r1}.e_1+\Sigma y_r\alpha_{r2}.e_2+\ldots\ldots$$

$$\therefore\ x_1=\Sigma y_r\alpha_{r1},\ x_2=\Sigma y_r\alpha_{r2},\ \ldots\ldots \quad \text{(Art. 20)}.$$

But if these values are substituted in $f(x_1, \ldots\ldots x_n)=0$, we get an equation of the mth degree in $y_1, y_2, \ldots\ldots$, and, indeed, homogeneous if the first equation is homogeneous.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

112. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$4.\frac{.297}{1.00\dot{2}}$. The selling price is $\$6.\frac{1000}{33337}$. What is the gain %?

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville Tenn., and the PROPOSER.

$297=\frac{297}{999}=\frac{11}{37}$; $1.003=1\frac{1}{300}=\frac{301}{300}$.

$\frac{11}{37}\div\frac{301}{300}=\frac{3300}{11137}$; $.\frac{3300}{11137}=\frac{330}{11137}$.