If we use only those from 1820 to 1880 inclusive we get:

$$p=7.29-0.28x+0.689x^2.$$

For the year 1894 the first of the formula gave 64.5 and the second 65.5; while the population reported in the World Almanac was 66.7.

The second formula gives for 1900 a population of 73.4, while the first formula gives a result somewhat lower. It appears then, that the population is increasing more rapidly than the parabolic curve indicates, and that if anything our forecast of 73.4 is somewhat low. It must be understood of course that the above formulae are strictly anti-expansion and make no allowance for our new possessions.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from the February Number.]

CHAPTER VI.

GEOMETRICAL ADDITION AND SUBTRACTION.

73. The general theory of the *Ausdehnungslehre* may be applied in such diverse sciences as geometry, mechanics, and logic. We proceed in this and the following chapter to apply it in geometry.

74. The concepts dealt with in geometry are the *point*, *line*, *surface*, and *solid*, which may or may not be fixed in position. For the sake of distinction a line whose length and direction are fixed but not its position is called a *vector*. (3). A portion of a plane whose direction and extent are fixed but not the position of the plane is called by analogy a *plane vector*.

75. As an introduction to the *Ausdehnungslehre* the addition and subtraction of vectors was treated in Chapter I. It is evident from what was given in that chapter that plane vectors may be added and subtracted in the same way as line vectors. One gets the parts whose sum is a given plane vector by projecting the given plane vector on the coördinate planes.

As we have already treated of the addition and subtraction of vectors, we proceed to apply the laws of addition and subtraction to points.

76. We will define a point as an infinitesimal portion of a line and denote it by p. When the point has position we will denote it by $p\rho$, in which p denotes the point at the extremity of the radius vector ρ from the origin, O. Evidently a line, or plane, or solid may be located by means of a radius vector in the same way. (See 168.)

Grassmann defines a point as that which has position and uses a single letter as A to denote it. In what follows if the ρ's be cut out of the formulas and the p's be given the subscripts of the ρ's, Grassmann's expressions will result.

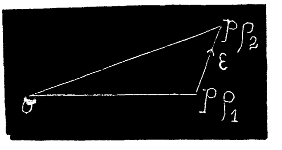
The reasons for using the complex symbol $p\rho$, instead of the simpler A, are : (1) Because the concept is complex and therefore for clearness should be represented by a complex symbol. (2) Since position is relative, for the proper representation of positions an origin is needed. (3) Because this notation shows plainly the relation which exists between point and vector analysis.

77. What we will call *unit* points all have the same (infinitesimal) unit length. This length as also that of the radius vector may be multiplied by any scalar, m. Thus $mp\rho$ denotes the point whose length or "weight," as it is called, is mp held in position by ρ, while $p(m\rho)$ denotes p held in position by $m\rho$, *i. e.*, m times the length of ρ.

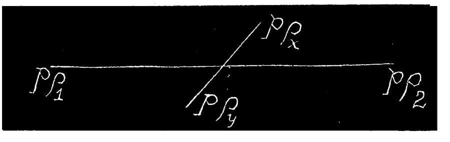
78. The difference between two unit points, since they can differ only in position, is a certain distance in a certain direction, *i. e.*, is a vector. (See 3.)

Thus $\quad p\rho_2-p\rho_1=\rho_2-\rho_1=\varepsilon.$ (4).

Similarly, $\quad mp\rho_2-mp\rho_1=m\varepsilon.$

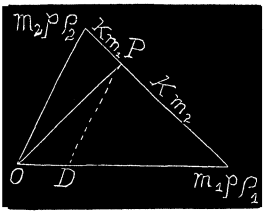


79. We next seek to find the sum of two or more points. What this sum is remains to be determined. Grassmann gives an investigation to show that the sum of two unit points is a point on the line joining them. We abridge this proof as follows : He begins by postulating (1) That whatever is true of one set of points is true of any congruent system wherever situated; (2) That the fundamental laws of addition and subtraction (14) hold. Then he assumes that the sum of two points is some point.



Let, in the figure, $p\rho_1$ and $p\rho_2$ be any two unit points whose sum is sought. Suppose $p\rho_1+p\rho_2=p\rho_x$. Then revolving the whole figure in the plane of the paper through 180°, $p\rho_1$ coincides with $p\rho_2$, $p\rho_x$ with $p\rho_y$, and $p\rho_2$ with $p\rho_1$. Thus we get $p\rho_2+p\rho_1=p\rho_y$. But by 14, $p\rho_2+p\rho_1=p\rho_1+p\rho_2$. Then $p\rho_x=p\rho_y$. This can only happen when they both coincide with the midpoint of the straight line joining the two given points.

80. Mechanics gives us a simpler and more general interpretation for the sum of two or more points. Let us regard the points as parallel (infinitesimal) forces whose magnitudes are represented by the weights of the points. The law for the addition of parallel forces gives a simple and consistent result. Thus



$$m_1p\rho_2+m_2p\rho_2=(m_1+m_2)p\left(\frac{m_1\rho_1+m_2\rho_2}{m_1+m_2}\right),$$

i. e., the sum of the two weighted points is a point on the line joining them whose *weight is the sum of the weights of the two points and the extremity of whose radius vector divides the line joining the two given points into segments inversely proportional to the weights of these points.*

That $\frac{m_1\rho_1+m_2\rho_2}{m_1+m_2}$ is the vector to the point described is evident from elementary geometry. Thus regarding the two radii vectores going out from O as axes, it is easy to show by similar triangles that

$$OD=\frac{m_1}{m_1+m_2}\rho_1, \text{ and } DP=\frac{m_2}{m_1+m_2}\rho_2.$$

NOTE.—The letter k which appears on the figure above is a factor chosen such that km_1 and km_2 equal the segments designated by them.

81. *Generalizing the result of the last article, we have*

$$\Sigma m_1 p\rho_1=\Sigma m_1 .p\left(\frac{\Sigma m_1\rho_1}{\Sigma m_1}\right).$$

82. When $\Sigma m_1=0$ in the preceding result, the weight of the sum point is zero and the radius vector is infinite in length. To interpret this, we get the sum of all the points except one and then add this partial sum to the remaining point. In this way we obtain an expression similar to that of 78 where the result is a *vector*.

Hence $\Sigma m_1 p\rho_1$ is a VECTOR when $\Sigma m_1=0$, and a POINT when Σm_1 is not equal to 0.

Thus a point at infinity (of zero weight) is equivalent to a vector.

83. Using the formula of 80 and putting

$$(m_1+m_1)=-m_3 \text{ and } \frac{m_1\rho_1+m_2\rho_2}{m_1+m_2}=\rho_3,$$

we see that $m_1 p\rho_1+m_2 p\rho_2+m_3 p\rho_3=0$, and $m_1+m_2+m_3=0$, *are the conditions that the three points $p\rho_1$, $p\rho_2$, $p\rho_3$ shall be collinear and the the vectors ρ_1, ρ_2, ρ_3, coplanar.*

84. For space of three dimensions we have (82)

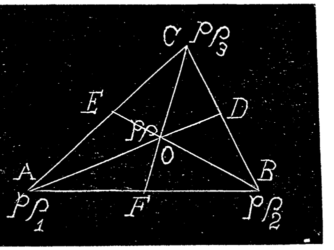
$$m_1 p\rho_1+m_2 p\rho_2+m_3 p\rho_3+m_4 p\rho_4=0, \text{ and } m_1+m_2+m_3+m_4=0,$$

for the conditions that the four points $p\rho_1$, $p\rho_2$, $p\rho_3$, $p\rho_4$ shall be coplanar.

85. *From the equations of* 80, 83, 84 *we see that two points are independent, three or more collinear points are dependent* (10), *three non-collinear points are independent, four or more coplaner points are dependent, four non-coplaner points are independent, and any five or more points are dependent in solid space.*

86. The calculus of this chapter is evidently adapted to dealing with theorems concerning the collinearity of points in geometry, and the center of parallel forces in mechanics.

87. We conclude this chapter with an example. *Required to find whether the three medians of a triangle meet in a point.*



Let the vertices of a triangle ABC be located by the unit points $p\rho_1$, $p\rho_2$, $p\rho_3$, and D, E, F be the mid-points of the sides. We have then

$$D=\frac{p\rho_2+p\rho_3}{2},\ E=\frac{p\rho_1+p\rho_3}{2}.$$

If $p\rho$ denote a unit point at O the intersection of AD and BE, and x, y, x', and y' arbitrary scalars, we may write

$$p\rho=xp\rho_1+y\frac{p\rho_2+p\rho_3}{2}=x'p\rho_2+y'\ \frac{p\rho_1+p\rho_3}{2}.$$

Then (20), $x=\frac{1}{2}y'$, $y=y'$; whence $x=\frac{1}{2}y$. But $x+y=1$ (77). Then $x=\frac{1}{3}$, $y=\frac{2}{3}$. Hence

$$p\rho=\tfrac{1}{3}p\rho_1+\tfrac{2}{3}\left(\frac{p\rho_2+p\rho_3}{2}\right)=\tfrac{1}{3}p\rho_1+\tfrac{1}{3}p\rho_2+\tfrac{1}{3}p\rho_3.$$

By symmetry we see that the intersection of AD and CF must be the same point. Or, supposing O to be the intersection of BE and CF, we may test A, O, and D for collinearity directly.

$$\begin{array}{lll} A & \quad O & \quad\quad D \\ \tfrac{1}{3}p\rho_1- & 1\,(\tfrac{1}{3}p\rho_1+\tfrac{1}{3}p\rho_2+\tfrac{1}{3}p\rho_3) & +\tfrac{2}{3}(\tfrac{1}{2}(p\rho_2+p\rho_3))\equiv 0. \quad (83). \end{array}$$

It is evident that the above equations can be interpreted as equations of ordinary vector analysis by dropping the p's. In this way is shown the relation existing between point and vector analysis.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

127. Proposed by **P. S. BERG, A. M.**, Principal of Schools, Larimore, N. D.

A man borrows \$1000 of a Building and Loan Association, and at the same time subscribes for 10 \$100-shares of stock. A membership fee of \$1 per share is charged. At the beginning of each month an installment of \$1 per share is paid, also 5% interest and 5% premium on the \$1000. The stock matures in 75 months and the debt is cancelled. What rate of interest does he pay per annum ?