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Convergence of quantum electrodynamics in a curved modification of Minkowski space

(ultraviolet divergences/nonlinear quantum field theory/fundamental length/Einstein universe/chronometric redshift theory)

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ABSTRACT **The interaction and total hamiltonians for quantum electrodynamics, in the interaction representation, are entirely regular self-adjoint operators in Hilbert space, in the universal covering manifold M of the conformal compactification of Minkowski space $\mathbf{M_o}$. (M is conformally equivalent to the Einstein universe E, in which $\mathbf{M_o}$ may be canonically imbedded.) In a fixed Lorentz frame this may be expressed as convergence in a spherical space with suitable periodic boundary conditions in time. The traditional relativistic theory is the formal limit of the present variant as the space curvature vanishes.**

The ultraviolet (UV) divergences of quantum electrodynamics (QED) have remained the fundamental challenge to the foundations of physical quantum field theory since its development in the late 1920s by Dirac (1–3), Heisenberg and Pauli (4, 5), and others. In work that is detailed elsewhere (6), we show that these divergences are absent when Minkowski space $\mathbf{M_o}$ is modified by the suitable introduction of a fundamental length. This is reminiscent of the classic suggestion by Heisenberg (7) that a model incorporating a small fundamental length might relieve the UV divergences. However, in contrast to this suggestion, the length R is large, and it is in the limit $R \to \infty$ that the present formalism converges to that of relativistic theory.

Indeed, R is naturally interpretable as the radius of the universe, as originally modeled by Einstein (8). The form of QED that we treat is based on the Maxwell–Dirac equations in the Einstein universe‡ E. The hamiltonian correspondingly represents the generator of temporal evolution in E, which differs from temporal evolution in the locally tangential Minkowski space. The relation between the Einstein and Minkowski hamiltonians is closely analogous to that between the Lorentz boosts in the Poincaré group and the corresponding generators of transformations to moving frames in the Galilean group; the limit $R \to \infty$ in the present context plays the role of the limit $c \to \infty$ in special relativity.§

Einstein temporal evolution is not equivalent (e.g., nonconjugate in the conformal group) to Minkowski temporal evolution, and the Einstein hamiltonians for free fields have discrete spectra that are strictly positive. In units in which $\hbar = c = 1$, the lowest Einstein photon energy takes the form $2/R$. The present fundamental length thus has a simple microscopic physical interpretation, as well as a cosmic one. It implies also a finite upper bound on the energy of a spinor particle of given mass.

The Einstein energy always exceeds the Minkowski (relativistic) energy, although microscopically it closely approximates it. Chronometric cosmology (11) proposes that the difference, as applied to a photon propagated freely during a long time period, represents the observed cosmic redshift. This cosmology appears to be valid (e.g., refs. 12 and 13), and if so R may be estimated from direct observations on apparently superluminal radio sources, as of the order of 5×10^8 light years (14). This implies a minimal photon Einstein energy of order 10^{-35} MeV. Applied to spinor particles as modeled by the Dirac equation, it implies that a particle of mass m_e has a cutoff on its energy of order 10^{37} GeV.

In formal terms we may state the

THEOREM. *Let* $\mathbf{K}_\gamma$ *denote the free photon Hilbert space of normalizable multiparticle photon states (in* $\mathbf{M_o}$ *or equivalently, by virtue of conformal invariance, in* **M**). *Let* $\mathbf{K_e}$ *denote the same for electrons of given mass satisfying the Dirac equation in* **E**. *Then the quantized interaction hamiltonian* $\mathrm{H_i} = \int_{S^3}:\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x):\, d_3x$ *exists as an essentially self-adjoint operator in the tensor product* $\mathbf{K} = \mathbf{K}_\gamma \times \mathbf{K_e}$, *on the domain of all linear combinations of tensor products of modes having exact Einstein quantum numbers. If* $\mathrm{H_o}$ *denotes the total free hamiltonian in* **K**, *then* $\mathrm{H_o} + \mathrm{H_i}$ *is essentially self-adjoint on the same domain, and for all sufficiently small constants* g, *the closure of* $\mathrm{H_o} + \mathrm{gH_i}$ *has a unique (within a constant factor) lowest eigenstate.*

The Einstein quantum numbers are defined and treated in refs. 15–17.

‡E represents the space $R^1 \times S^3$. All relevant free particle wave functions in E are suitably periodic in time and the underlying space-time may correspondingly be regarded as $S^1 \times S^3$, which is conformally equivalent to the fourfold cover of the conformal compactification of $\mathbf{M_o}$. In consequence, the interaction representation analysis of the coupled electron–photon fields in M may be confined to this fourfold cover. The interacting (Heisenberg) fields themselves, however, are not at all periodic in time.

§A conformal group generator distinct from the relativistic temporal evolution generator was proposed in place of the latter for fundamental microphysical purposes in ref. 9, on symmetry grounds similar to those of Minkowski's explication of special relativity (1908 address, *Space and Time*, see ref. 10). This generator is mathematically equivalent to that for temporal evolution in E.

This paper is dedicated to the memory of the late Stephen M. Paneitz, in recognition of his essential contributions.

1. Dirac, P. A. M. (1926) *Proc. R. Soc. London Ser. A* **112,** 661–677.
2. Dirac, P. A. M. (1927) *Proc. R. Soc. London Ser. A* **114,** 243–265.
3. Dirac, P. A. M. (1927) *Proc. R. Soc. London Ser. A* **114,** 710–728.
4. Heisenberg, W. & Pauli, W. (1929) *Z. Phys.* **56,** 1–61.
5. Heisenberg, W. & Pauli, W. (1930) *Z. Phys.* **59,** 168–190.
6. Segal, I. E. & Zhou, Z. (1993) *Ann. Phys.*, in press.
7. Heisenberg, W. (1946) *Z. Naturforsch.* **1,** 608–622.
8. Einstein, A. (1917) *Situngsber. Preuss. Akad. Wiss.*, 142–152.
9. Segal, I. E. (1951) *Duke Math. J.* **18,** 221–265.
10. Minkowski, H. (1952) *The Principle of Relativity* (Dover, New York), pp. 73–91.
11. Segal, I. E. (1972) *Astron. Astrophys.* **18,** 143–148.

12. Segal, I. E. (1993) *Proc. Natl. Acad. Sci. USA* **90,** 4798–4805.
13. Segal, I. E., Nicoll, J. F., Wu, P. & Zhou, Z. (1993) *Astrophys. J.* **411,** 465–483.
14. Segal, I. E. (1990) *Mon. Not. R. Astron. Soc.* **242,** 423–427.
15. Paneitz, S. M. & Segal, I. E. (1982) *J. Funct. Anal.* **47,** 78–142.
16. Paneitz, S. M. & Segal, I. E. (1982) *J. Funct. Anal.* **49,** 335–414.
17. Paneitz, S. M. (1983) *J. Funct. Anal.* **54,** 18–112.