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[^0]:    ${ }^{1}$ A NOTE TO THE READER: Some of the images in this document are referred as color ones. However, depending of the printed version, it is possible that they cannot be appreciated as originally intended. According to the case, we recommend to consult the electronic version of this work.

