Chapter 4
Four-Dimensional Orthogonal Polytopes

In section 1.6.1, it was presented one of the premises of this work: that it is essential, as a first step, to analyze the polytopes and the boundaries that compose them, to guarantee the validity of the visualization and analysis of the phenomena or data that they will represent (as we will see in the next chapters). Therefore, this chapter presents in first place (section 4.1) some definitions related with the 4D Orthogonal Pseudo-Polytopes (4D-OPP's), which are the polytopes that we will use subsequently for representing some multidimensional data and phenomena (as stated in section 1.6.4). Then, (in section 4.2) we will cover the analysis related to the configurations that can represent the 4D-OPP's. Moreover, the procedures for classifying edges and faces as manifold or non-manifold elements in 4D-OPP's will be described. For faces in 4D-OPP's the [Aguilera & Pérez, 02b]'s condition to classify them as manifold or non-manifold will be described (section 4.3). For the edges' analysis in 4D-OPP's (section 4.4) the two [Aguilera & Pérez, 03]'s approaches will be described: 1) The analogy between incident (manifold and non-manifold) edges to a vertex in 3D Orthogonal Pseudo-Polyhedra (3D-OPP's) with incident (manifold and non-manifold) faces to a edge in 4D-OPP's; and 2) The extension of the concept of "cones of faces" (which is applied for classifying a vertex in 3D-OPP's as manifold or non-manifold; and introduced in section 2.1.2) to "hypercones of volumes" for classifying an edge as manifold or non-manifold in 4D-OPP's (and introduced in section 2.1.4). The generalizations for classifying the n-3 and the n-2 dimensional boundary elements for n-dimensional Orthogonal Pseudo-Polytopes as manifold or non-manifold elements are also presented. Finally, (section 4.5) it will be considered the characterization of the 4D-OPP's

edges as Extreme or Non-Extreme. It will be described how this classification is the result of a 3D analysis over the possible configurations for the 4D-OPP's and, moreover, it is not only restricted to the (n-3)-dimensional elements, because it is present in the (n-1) and (n-2)-dimensional elements.

4.1 Definition

[Juan-Arinyo, 88] & [Preparata 85] define Orthogonal Polyhedra (3D-OP) as polyhedra with all their edges and faces oriented in three orthogonal directions. Orthogonal Pseudo-Polyhedra (3D-OPP) will refer to regular and orthogonal polyhedra with non-manifold boundary [Aguilera, 98].

Similarly, **4D Orthogonal Polytopes (4D-OP)** are defined as 4D polytopes with all their edges, faces and volumes oriented in four orthogonal directions and 4D Orthogonal Pseudo-Polytopes (4D-OPP) will refer to 4D regular and orthogonal polytopes with non-manifold boundary [Aguilera & Pérez, 02b].

Because the 4D-OPP's definition is an extension from the 3D-OPP's, it is easy to generalize the concept to define **n-dimensional Orthogonal Polytopes (nD-OP)** as n-dimensional polytopes with all their Π_{n-1}, Π_{n-2},..., Π_1 oriented in n orthogonal directions. Finally, **n-dimensional Orthogonal Pseudo-Polytopes (nD-OPP)** are defined as n-dimensional regular and orthogonal polytopes with non-manifold boundary [Aguilera & Pérez, 02b].

4.2 Adjacency Analysis For 2D, 3D And 4D-OPP's

4.2.1 Adjacency Analysis For 2D-OPP's

A set of quasi-disjoint rectangles determines a 2D-OPP whose vertices must coincide with some of the rectangles' vertices [Aguilera, 98]. Each of these rectangles' vertices can be considered as the origin of a 2D local coordinate system, and they may belong to up to four rectangles, one for each local quadrant. The two possible adjacency relations between the four possible rectangles can be of edge or vertex. There are $2^4 = 16$ possible combinations which, by applying symmetries and rotations, may be grouped into six equivalence classes, also called configurations [Srihari, 81]. Moreover, [Aguilera, 98] has identified that each possible combination has its complementary combination, and each configuration has its complementary configuration which is the class that contains the complementary combinations of all the combinations in the given class.

[Aguilera, 98] describes that these 16 possible combinations are distributed in the following way:

$$2^4 = \sum_{k=0}^{4} C\binom{4}{k} = 1 + 4 + 6 + 4 + 1 = 16$$

And using combinatorial analysis, there are:

- $C\binom{4}{0} = 1$ combination with zero surrounding rectangles (configuration a, **Table 4.1**).
- $C\binom{4}{1} = 4$ combinations with one surrounding rectangle (configuration b).

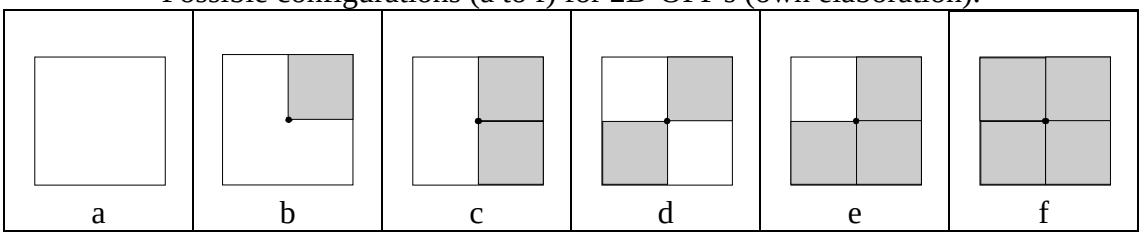
- $C\binom{4}{2} = 6$ combinations with two surrounding rectangles (configurations c and d).

- $C\binom{4}{3} = 4$ combinations with three surrounding rectangles (configuration e).

- $C\binom{4}{4} = 1$ combination with four surrounding rectangles (configuration f).

[Aguilera, 98] identifies that configurations a and f, as well as configurations b and e, are complementary to each other. Configurations c and d are self-complementary.

TABLE 4.1
Possible configurations (a to f) for 2D-OPP's (own elaboration).



4.2.2 Adjacency Analysis For 3D-OPP's

[Aguilera, 98] describes that a set of quasi-disjoint boxes determines a 3D-OPP whose vertices must coincide with some of the boxes' vertices. Similarly to the 2D case, each of these boxes' vertices can be considered as the origin of a 3D local coordinate system, and they may belong to up to eight boxes, one for each local octant. The three possible adjacency relations between the eight possible boxes can be of face, edge or vertex. There are $2^8 = 256$ possible combinations which, by applying symmetries and rotations, may be grouped into 22 equivalence classes [Lorensen, 87], also called

configurations [Srihari, 81]. As in the 2D case, each possible combination has its complementary combination, and each configuration has its complementary configuration which is the class that contains the complementary combinations of all the combinations in the given class [Aguilera, 98]. Grouping complementary configurations leads to the 14 major cases [Van Gelder, 94].

Similarly to the 2D case, [Aguilera, 98] describes that these 256 possible combinations are distributed in the following way:

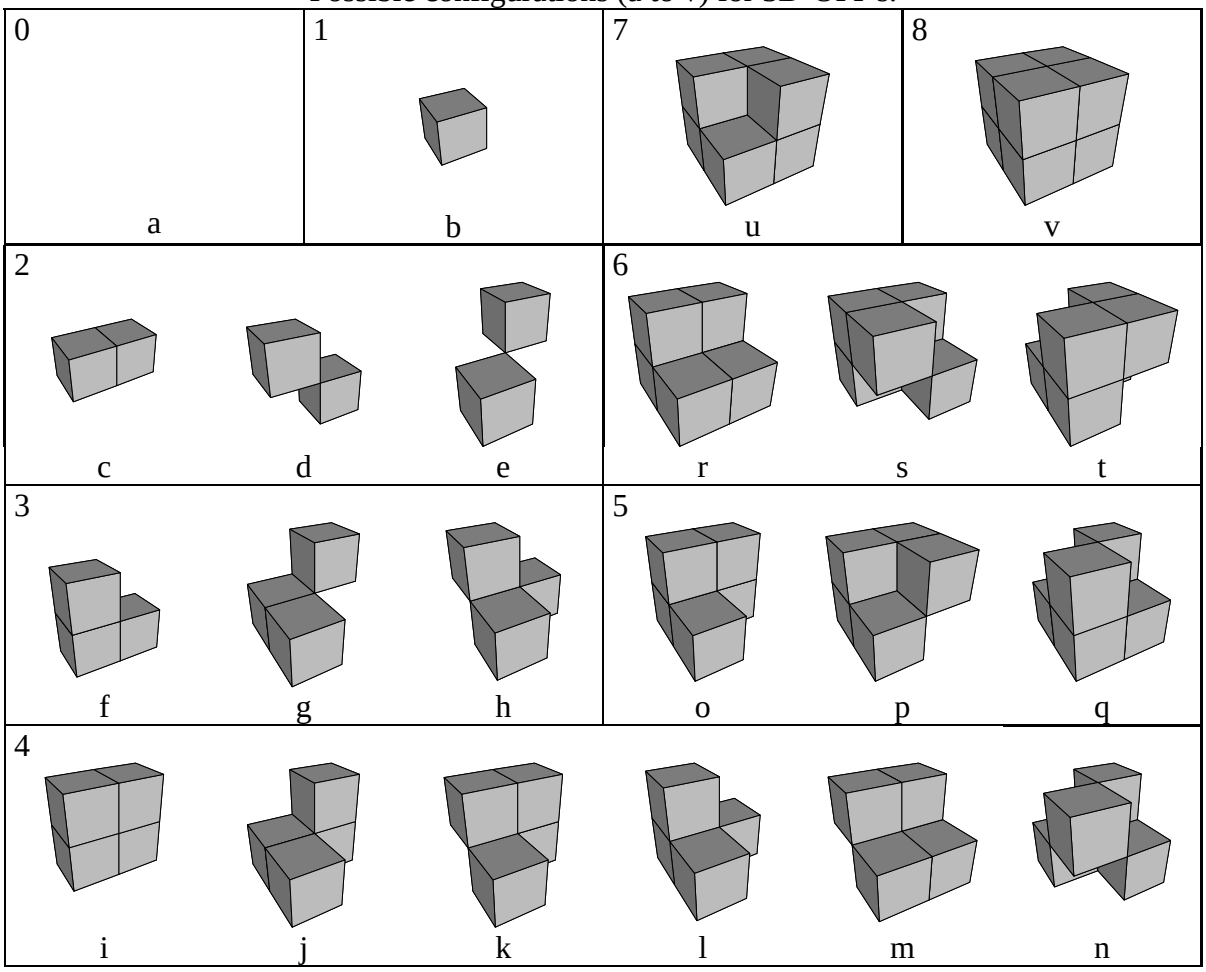
$$2^8 = \sum_{k=0}^{8} C\binom{8}{k} = 1+8+28+56+70+56+28+8+1 = 256$$

And using combinatorial analysis, there are:

- $C\binom{8}{0} = 1$ combination with zero surrounding boxes (configuration a, **Table 4.2**).
- $C\binom{8}{1} = 8$ combinations with one surrounding box (configuration b).
- $C\binom{8}{2} = 28$ combinations with two surrounding boxes (configurations c, d and e).
- $C\binom{8}{3} = 56$ combinations with three surrounding boxes (configurations f, g and h).
- $C\binom{8}{4} = 70$ combinations with 4 surrounding boxes (configurations i, j, k, l, m and n).

TABLE 4.2

Possible configurations (a to v) for 3D-OPP's.



The remaining combinations with 5, 6, 7 and 8 surrounding boxes are complementary, and thus analogous, to combinations with 3, 2, 1 and 0 surrounding boxes, respectively [Aguilera, 98]. Finally, each configuration, with four surrounding boxes is self-complementary.

4.2.3 Adjacency Analysis For 4D-OPP's

By analogy, we can assume that a set of quasi-disjoint *hyper-boxes* (hypercubes, for example) determines a 4D-OPP whose vertices must coincide with some of the *hyper-boxes'* vertices. We will consider the *hyper-boxes'* vertices as the origin of a 4D local coordinate system, and they may belong to up to 16 *hyper-boxes'*, one for each local *hyper-octant*. The 4D-OPP's vertices are determined according to the presence or absence of each of this 16 surrounding *hyper-boxes*. The four possible adjacency relations, extended by analogy, between the 16 possible *hyper-boxes* can be of volume, face, edge or vertex. There are 2^{16} = 65,536 possible combinations of vertices in 4D-OPP's. Through a computer program, we exhaustively analyzed those 65,536 combinations and found that all of them can be grouped, applying symmetries and rotations, into 253 equivalence classes, which we shall call *configurations*, as in 2D and 3D cases. Similarly, we found that each possible combination has its complementary combination, and each configuration (i.e., each class) has its complementary configuration which is the class that contains the complementary combinations of all the combinations in the given class. Grouping complementary configurations leads us to the 145 major cases.

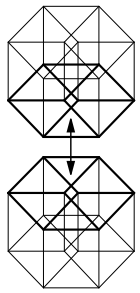
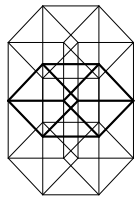
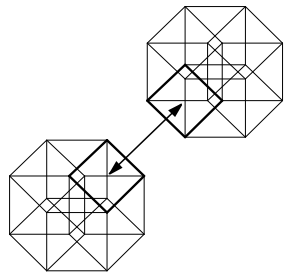
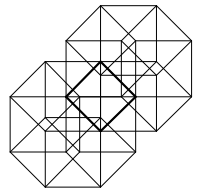
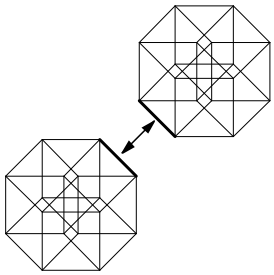
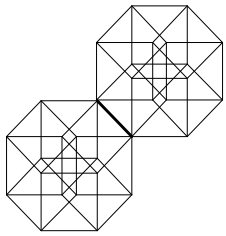
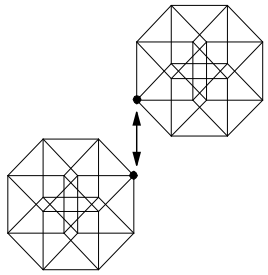
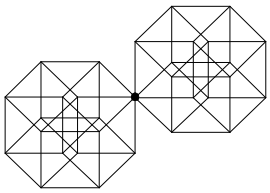
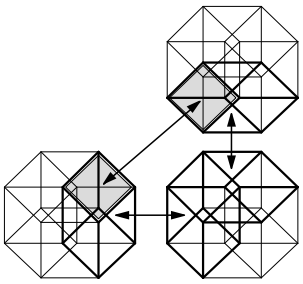
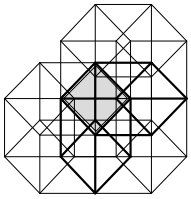
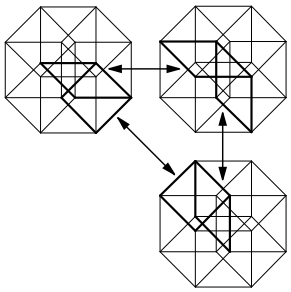
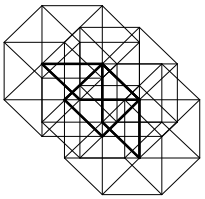
The 65,536 possible combinations are distributed in the following way:

$$2^{16} = \sum_{k=0}^{16} \binom{16}{k} = \left\{ \begin{matrix} 1+16+120+560+1{,}820+4{,}368+8{,}008+11{,}440+12{,}870 \\ 11{,}440+8{,}008+4{,}368+1{,}820+560+120+16+1 \end{matrix} \right\} = 65{,}536$$

And using combinatorial analysis, there are:

- $C\binom{16}{0} = 1$ combination with zero surrounding hyper-boxes (configuration 1).
- $C\binom{16}{1} = 16$ combinations with one surrounding hyper-box (configuration 2).
- $C\binom{16}{2} = 120$ combinations with two surrounding hyper-boxes (configurations 3, 4, 5 and 6, shown in **Table 4.3**).
- $C\binom{16}{3} = 560$ combinations with three surrounding hyper-boxes (configurations 7 to 12. Configuration 7 and 8 are shown in **Table 4.3**).
- $C\binom{16}{4} = 1{,}820$ combinations with 4 surrounding hyper-boxes (configurations 13 to 28).
- $C\binom{16}{5} = 4{,}368$ combination with 5 surrounding hyper-boxes (configurations 29 to 48).
- $C\binom{16}{6} = 8{,}008$ combinations with 6 surrounding hyper-boxes (configurations 49 to 78).
- $C\binom{16}{7} = 11{,}440$ combinations with 7 surrounding hyper-boxes (configurations 79 to 108).
- $C\binom{16}{8} = 12{,}870$ combinations with 8 surrounding hyper-boxes (configurations 109 to 145).

TABLE 4.3
Configurations 3 to 8 for 4D-OPP's
(see **Appendix A** for a description of the remaining configurations).

Adjacencies between hyper-boxes	Configuration	Adjacencies between hyper-boxes	Configuration
	 **3**		 **4**
	 **5**		 **6**
	 **7**		 **8**

If has been found, through exhaustive analysis by a computer system, that the remaining combinations with 9, 10, 11, 12, 13, 14, 15 and 16 surrounding hyper-boxes are complementary, and thus analogous, to combinations with 7, 6, 5, 4, 3, 2, 1 and 0 surrounding hyper-boxes, respectively. Finally, each configuration, with eight surrounding hyper-boxes is self-complementary. In **Appendix A** are presented all the 253 4D configurations.

4.2.4 Determining the Sum of Adjacencies for Configurations in nD-OPP's

In the **Table 4.4** is presented a summary of the adjacency analysis for 2D-OPP's. In the table are presented the number of vertex adjacencies and the number of edge adjacencies that are possible between the rectangles of each configuration.

TABLE 4.4
Counting the edge and vertex adjacencies in 2D-OPP's configurations (own elaboration).

Number of Rectangles	**Configuration**	**Number of edge adjacencies**	**Number of vertex adjacencies**	**Sum of adjacencies**
0	a	0	0	**0**
1	b	0	0	**0**
2	c	1	0	**1**
	d	0	1	**1**
3	e	2	1	**3**
4	f	4	2	**6**

In the **Table 4.5** is presented a summary of the adjacency analysis for 3D-OPP's. In the table are presented the number of vertex adjacencies, edges adjacencies and face adjacencies that are possible between the boxes of each configuration.

TABLE 4.5

Counting the face, edge and vertex adjacencies in 3D-OPP's configurations (own elaboration).

Number of boxes	Configuration	Number of face adjacencies	Number of edge adjacencies	Number of vertex adjacencies	Sum of adjacencies
0	a	0	0	0	**0**
1	b	0	0	0	**0**
2	c	1	0	0	**1**
	d	0	1	0	**1**
	e	0	0	1	**1**
3	f	2	1	0	**3**
	g	1	1	1	**3**
	h	0	3	0	**3**
4	i	4	2	0	**6**
	j	3	2	1	**6**
	k	2	3	1	**6**
	l	3	3	0	**6**
	m	2	2	2	**6**
	n	0	6	0	**6**
5	o	5	4	1	**10**
	p	4	4	2	**10**
	q	3	6	1	**10**
6	r	7	6	2	**15**
	s	6	7	2	**15**
	t	6	6	3	**15**
7	u	9	9	3	**21**
8	v	12	12	4	**28**

From **Tables 4.4 and 4.5** it can be observed that the sums of adjacencies from 0 to 4 rectangles/boxes are the same. If the sums presented in **Table 4.5** are compared with the 4D-OPP's configurations presented in **Appendix A**, the same property is present: the sums of adjacencies from 0 to 8 boxes/*hyper-boxes* are the same. Moreover, the sums of adjacencies for configurations with the same number of boxes are equal for all those configurations, regardless they correspond to 2D-OPP's, 3D-OPP's or 4D-OPP's (see the configurations' case, for example, with 4 "boxes" in **Tables 4.4**, **4.5** and **Appendix A**). Based in these properties, we can assume that the sum of adjacencies for any configuration

with x "*boxes*" (rectangles, boxes, *hyper-boxes*, etc.) independently of the number of dimensions (in an Euclidean Space), is:

$$\frac{x(x-1)}{2}$$

See in Chapter 5 its corresponding Theorem and proof.

4.3 The Π_{n-2} Analysis for 2D, 3D and 4D-OPP's

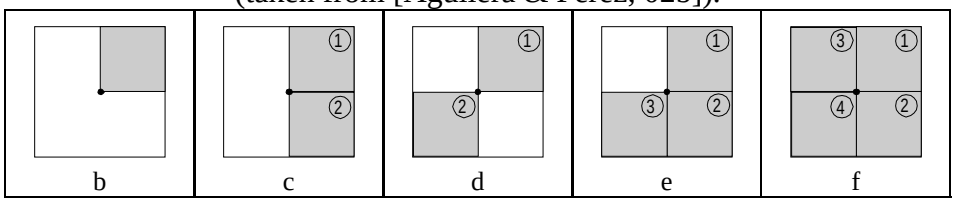
In the following sections (4.3.1, 4.3.2 and 4.3.3) we will classify, as manifold or non-manifold, to vertices (Π_0's) in the 2D-OPP's, edges (Π_1's) in the 3D-OPP's and faces (Π_2's) in the 4D-OPP's in terms of the number of their incident edges, faces and volumes respectively. Finally, in section 4.3.4, we consider the general condition, presented originally in [Aguilera & Pérez, 02b], to classify the Π_{n-2}'s in the nD Orthogonal Pseudo-Polytopes.

4.3.1 The Π_0 (Vertex) Analysis for 2D-OPP's

Because we are interested in the vertex analysis, we will consider only those configurations where all their rectangles are incident to a vertex. According to the configurations' nomenclature presented in [Aguilera, 98], the studied configurations are b, c, d, e and f (**Table 4.6**).

TABLE 4.6

2D configurations where all the rectangles are incident to a vertex (taken from [Aguilera & Pérez, 02b]).



We will classify a vertex in terms of the incident edges to it. Those edges that are shared by two rectangles (edge adjacency) will not be considered because they are not valid edges since they do not belong to the final OPP. From this analysis it results that there are only two types of vertices in a 2D-OPP: the manifold vertex with two incident edges (configurations b and e), and the non-manifold vertex with four incident edges (configuration d) [Aguilera, 98]. The remaining configurations represent no vertex because in configuration c there are only two incident and collinear edges, and in configuration d there are no incident edges.

From the adjacency analysis we can observe that in the configurations all possible types of adjacency for rectangles are present: vertex and edge adjacency. **Table 4.7** summarizes the rectangles' adjacency analysis and the vertex classification.

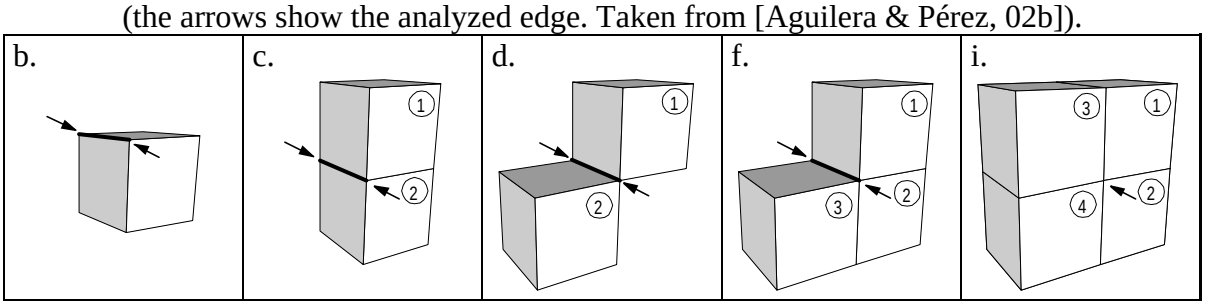
TABLE 4.7
Resume of adjacency analysis for each configuration and vertex classification
(refer to **Table 4.6** to rectangles' numbering).

Configuration	Number of involved rectangles	Adjacencies between rectangles	Number of valid incident edges to vertex	Vertex classification
b	1	-	2	Manifold
c	2	1-2: Edge	-	No vertex
d	2	1-2: Vertex	4	Non manifold
e	3	1-2: Edge 1-3: Vertex 2-3: Edge	2	Manifold
f	4	1-2: Edge 1-3: Edge 1-4: Vertex 2-3: Vertex 2-4: Edge 3-4: Edge	-	No vertex

4.3.2 The Π_1 (Edge) Analysis for 3D-OPP's

Because we are interested in the edge analysis, we will consider only those configurations where all their boxes are incident to just one edge. According to the configurations' associated nomenclature, presented in [Aguilera, 98], the studied configurations are b, c, d, f and i (**Table 4.8**).

TABLE 4.8
3D configurations where all the boxes are incident to an edge
(the arrows show the analyzed edge. Taken from [Aguilera & Pérez, 02b]).



We will classify an edge in terms of the incident faces to it. Those faces that are shared by two boxes (face adjacency) will not be considered because they are not valid faces since they do not belong to the final OPP. From this analysis it results that there are only two types of edges in a 3D-OPP [Aguilera, 98]:

- **The manifold edge with two incident faces.** This type of edges is found in configurations b and f. The edge's two incident faces in configuration b belong to one cube's boundary and they are perpendicular to each other. The edge's two incident faces in configuration f belong to two different cubes with edge adjacency and they result perpendicular to each other.
- **The non-manifold edge with four incident faces.** This type of edge is found in configuration d, where two of its faces belongs to a cube and the remaining two belong to a second cube with edge adjacency.

The remaining configurations represent no edge because in configuration c there are only two incident and coplanar faces, and in configuration i there are no incident faces. **Table 4.9** resumes the boxes' adjacency analysis and the edge classification.

TABLE 4.9
Resume of adjacency analysis for each configuration and edge classification (refer to **Table 4.8** to boxes' numbering).

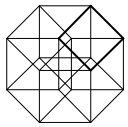
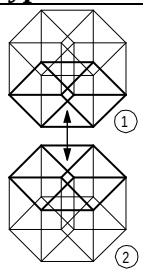
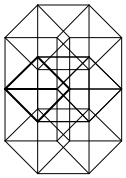
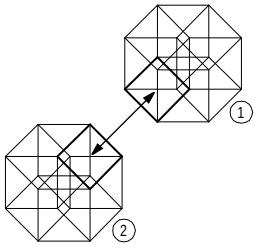
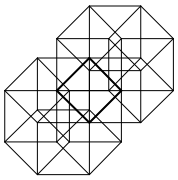
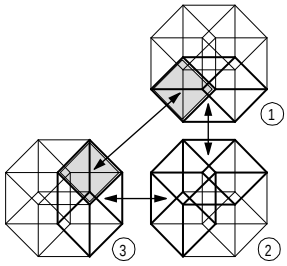
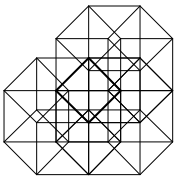
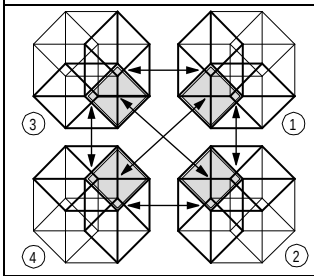
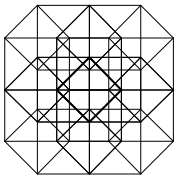
Configuration	**Number of involved boxes**	**Adjacencies between boxes**	**Number of incident valid faces to edge**	**Edge classification**
b	1	-	2	Manifold
c	2	1-2: Face	-	No edge
d	2	1-2: Edge	4	Non manifold
f	3	1-2: Face 1-3: Edge 2-3: Face	2	Manifold
i	4	1-2: Face 1-3: Face 1-4: Edge 2-3: Edge 2-4: Face 3-4: Face	-	No edge

4.3.3 The Π_2 (Face) Analysis for 4D-OPP's

Because we are interested only in the face analysis, we will consider only those configurations where all their *hyper-boxes* are incident to just one face. It has been found that there are only 5 of such configurations. These configurations are 2, 3, 4, 7 and 13. See **Table 4.10**.

TABLE 4.10

4D configurations where all the *hyper-boxes* are incident to a face (the analyzed face is indicated in the 2nd column. Taken from [Pérez & Aguilera, 03]).

Adjacencies between *hyper-boxes*	**Configuration**	**Adjacencies between *hyper-boxes***	**Configuration**
	 2.		 3.
	 4.		 7.
	 13.		

We will classify a face in terms of the incident volumes to it. Those volumes that are shared by two *hyper-boxes* (volume adjacency) will not be considered because they are not valid volumes since they do not belong to the final OPP. From this analysis, and analogous to the 3D case (see section 4.3.2), it results that there are only two types of faces in a 4D-OPP [Aguilera & Pérez, 02b]:

- **Faces with two incident volumes.** This type of faces is found in configurations 2 and 7. The face's two incident volumes in configuration 2 belong to the boundary of only one hypercube and they are perpendicular to each other. While in configuration 7, the face's two incident volumes belong to two different hypercubes with face adjacency and they result perpendicular to each other.

- **Faces with four incident volumes.** This type of faces is found in configuration 4, where two of its incident volumes belong to a hypercube and the remaining two belong to a second hypercube with face adjacency.

The remaining configurations represent no face because in configuration 3 there are only two incident and *co-hyperplanar* volumes, and in configuration 13 there are no incident volumes (analogous to 3D configurations c and i in **Table 4.8**).

From the adjacency analysis we can observe that in the studied configurations the possible types of adjacency for *hyper-boxes* are face and volume adjacency. The vertex and edge adjacencies are not present. Also, we can observe that the adjacency analysis and the number of incident volumes are analogous with the vertex analysis for 2D-OPP's and the edge analysis for 3D-OPP's.

[Coxeter, 63] defines that a Π_{n-2} belongs to just two of the Π_{n-1}'s in any n-dimensional polytope. In section 2.1.5 were described the properties, described by [Hansen,93], for the representation of a nD Polytope's boundary as a closed set of n-manifolds:

1. A 0-manifold is a point, and it has no boundary.
2. All boundary elements of an n-manifold are (n-1)-manifold elements.
3. All (n-1)-dimensional elements belong to exactly two n-manifold elements (or twice to the same element).
4. Manifold elements may not intersect each other except at common boundary elements.

For example, [Hansen, 93] shows the use of their rules in the cube (a 3-dimensional solid, bounded by a closed set of 2-manifolds, the faces). For this case, n = 2, property 2 says, that every boundary element of each 2-manifold (face) is a 1-manifold (edge), and property 3 says, that every 1-manifold (edge) belongs to two 2-manifold (faces). If we apply the same rules to the 4D hypercube (a 4-dimensional polytope, bounded by a closed set of 3-manifolds, the volumes), then for n = 3, property 2 says that every boundary element of each 3-manifold (volume) is a 2-manifold (face), and property 3 says, that every 2-manifold (face) belongs to two 3-manifolds (volumes).

According to [Coxeter, 63] and [Hansen, 93], if the 4D Orthogonal Pseudo-Polytope's two perpendicular volumes (Π_3) are incident to a face (Π_2), then it is a manifold face. Otherwise, if the 4D Orthogonal Pseudo-Polytope's four volumes are incident to a face, [Aguilera & Pérez, 02b] suggest, by analogy, that it is a non-manifold face. These properties are present in the face analysis (see **Table 4.11**).

TABLE 4.11
Resume of adjacency analysis for each configuration
(refer to **Table 4.10** to *hyper-boxes*' numbering).

Configuration	**Number of involved hyper-boxes**	**Adjacencies between hyper-boxes**	**Number of incident valid volumes to face**	**Face classification**
1	1	-	2	Manifold
2	2	1-2: Volume	-	No face
3	2	1-2: Face	4	Non manifold
4	3	1-2: Volume 1-3: Face 2-3: Volume	2	Manifold
5	4	1-2: Volume 1-3: Volume 1-4: Face 2-3: Face 2-4: Volume 3-4: Volume	-	No face

4.3.4 Classifying the Π_{n-2}'s in nD-OPP's

Finally, the generalized conditions to classify a Π_{n-2} as manifold or non-manifold in a nD-OPP are [Aguilera & Pérez, 02b]:

- If two perpendicular Π_{n-1}'s are incident to a Π_{n-2} then it must be classified as **manifold**.
- If four Π_{n-1}'s are incident to a Π_{n-2} then it must be classified as **non-manifold**.

4.4 The Π_{n-3} Analysis for 3D and 4D-OPP's

In the following sections we will analyze and classify, as manifold or non-manifold the different types of vertices (Π_0's) in the 3D-OPP's and edges (Π_1's) in the 4D-OPP's. We will consider two approaches:

a) Classification of these elements by means of their incident manifold or non-manifold edges (3D-OPP's) and faces (4D-OPP's, sections 4.4.1 and 4.4.2 respectively).

b) Application of the concepts of "cones of faces" (section 4.4.3) and "hypercones of volumes" (4.4.4) to classify vertices in the 3D-OPP's and edges in the 4D-OPP's respectively.

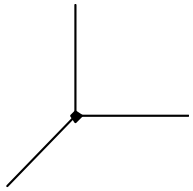
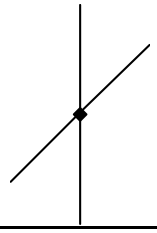
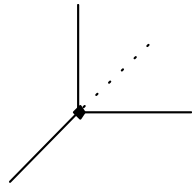
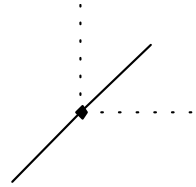
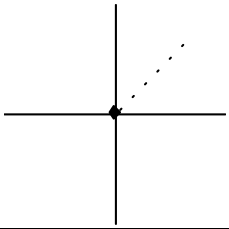
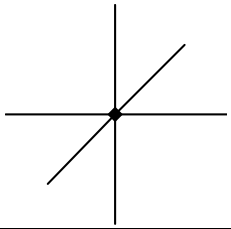
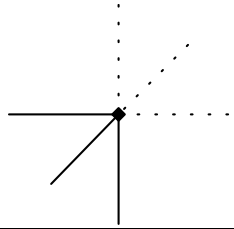
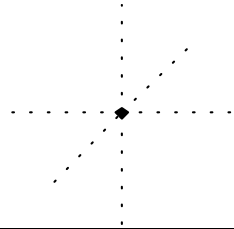
In sections 4.4.5 and 4.4.6 are presented the generalizations of the two approaches for classifying the Π_{n-3}'s in the nD Orthogonal Pseudo-Polytopes.

4.4.1 The Π_0 (Vertex) Analysis for 3D-OPP's

[Aguilera, 98] identifies eight types of vertices (also two non valid vertices are identified) for 3D-OPP's. These vertices can be classified depending on the number of two-manifold and non-manifold edges incident to them and they are referred as V3, V4, V4N1, V4N2, V5N, V6, V6N1 and V6N2 [Aguilera,98] (**Table 4.12**). In this nomenclature "V" means vertex, the first digit shows the number of incident edges, the "N" is present if at least one non-manifold edge is incident to the vertex and the second digit is included to distinguish between two different types that otherwise could receive the same name.

TABLE 4.12

Vertices present in 3D-OPP's (dotted lines indicate non-manifold edges and continuos lines indicate manifold edges. Taken from [Pérez & Aguilera, 03b]).

V3 	V4 	V4N1 	V4N2 
V5N 	V6 	V6N1 	V6N2 
Non valid vertex 1	Non valid vertex 2		

Each vertex has the following properties [Aguilera, 98]:

- V3: all three incident edges are two-manifold and perpendicular. It is present in configurations b, f, o and u.
- V4: all four incident edges are two-manifold, they lie on a plane, and can be grouped in two couples of collinear edges. It is present in configuration j.
- V4N1: three of its four incident edges are perpendicular and also two-manifold ones, while the fourth is non-manifold and collinear to one of the other three. It is present in configurations g and p.
- V4N2: two of its four incident edges are two-manifold and collinear, while each of its other two is non-manifold and perpendicular to the other three. It is present in configuration k.

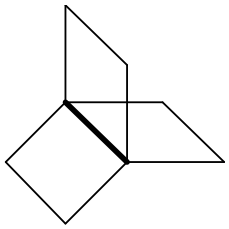
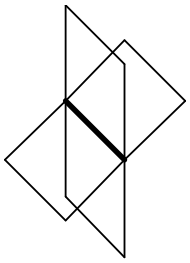
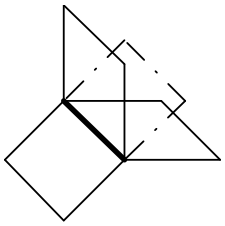
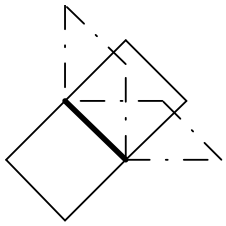
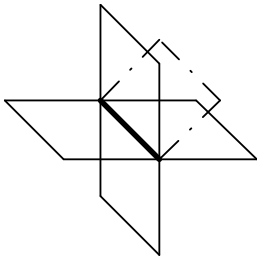
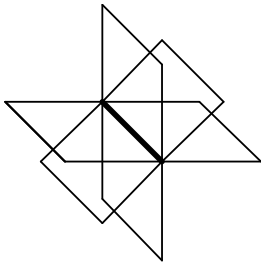
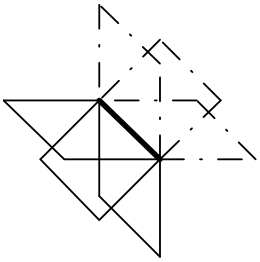
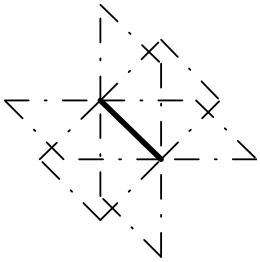
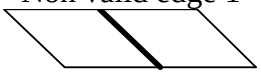
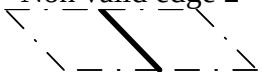
- V5N: four of its five incident edges are two-manifold and lie in a plane, while the fifth is non-manifold and perpendicular to the rest of them. It is present in configurations d and s.
- V6: all six incident edges are two-manifold. It is present in configurations e, l and t.
- V6N1: three of its six incident edges are perpendicular and also two-manifold ones, while each of its remaining three edges is non-manifold and collinear to one of the first three. It is present in configurations h and q.
- V6N2: all of its six incident edges are non-manifold. It is present in configuration n.
- Non valid vertex 1: its two manifold edges are collinear . It is present in configurations c and r.
- Non valid vertex 2: its two non-manifold edges are collinear. It is present in configuration m.

4.4.2 The Π_1 (Edge) Analysis For 4D-OPP's

[Aguilera, 98] defines vertex types in terms of the manifold or non-manifold edges that are incident to these vertices in 3D-OPP's. [Pérez & Aguilera, 03c] extend the same process to describe edge types in terms of the manifold or non-manifold faces that are incident to those edges in 4D-OPP's. In this way, we have identified eight types of edges and two non valid edges. We will also extend the nomenclature used by [Aguilera, 98] to describe them. Such edges will be referred as E3, E4, E4N1, E4N2, E5N, E6, E6N1 and E6N2 (**Table 4.13**). The only difference with the nomenclature used by [Aguilera, 98] is that "E" means edge instead of "V" that means vertex.

TABLE 4.13

Edges present in 4D-OPP's (dotted lines indicate non-manifold faces and continuos lines indicate manifold faces. Taken from [Pérez & Aguilera, 03b]).

E3 	E4 	E4N1 	E4N2 
E5N 	E6 	E6N1 	E6N2 
Non valid edge 1 	Non valid edge 2 		

Each edge, identified by [Pérez & Aguilera, 03], has the following properties:

- E3: all three incident faces are manifold and perpendicular.
- E4: all four incident faces are manifold, they lie on a hyperplane, and can be grouped in two couples of coplanar faces.
- E4N1: three of its four incident faces are perpendicular and also manifold ones, while the fourth is non-manifold and coplanar to one of the other three.
- E4N2: two of its four incident faces are manifold and coplanar, while each of its other two is non-manifold and perpendicular to the other three.
- E5N: four of its five incident faces are manifold and lie in a hyperplane, while the fifth is non-manifold and perpendicular to the rest of them.

- E6: all six incident faces are manifold.
- E6N1: three of its six incident faces are perpendicular and also manifold ones, while each of its remaining three faces is non-manifold and coplanar to one of the first three.
- E6N2: all of its six incident faces are non-manifold.
- Non valid edge 1: its two manifold faces are coplanar.
- Non valid edge 2: its two non-manifold faces are coplanar.

It results interesting that the number, classifications and positions of the incident faces to an edge in 4D-OPP's are analogous to the way that a set of edges are incident to a vertex in 3D-OPP's.

4.4.3 Classifying the Π_0 in Polyhedra Through its Cones of Faces

As commented in section 2.1, edges and vertices, as boundary elements for polyhedra, may be either two-manifold (or just manifold) or non-manifold elements. In the case of edges, they are (non) manifold elements when every points of it is also a (non) manifold point, except that either or both of its ending vertices might be a point of the opposite type [Aguilera, 98]. A manifold edge is adjacent to exactly two faces, and a manifold vertex is the apex (i.e., the common vertex) of only one **cone of faces**. Conversely, a non-manifold edge is adjacent to more than two faces, and a non-manifold vertex is the apex (i.e., the common vertex) of more than one **cone of faces** [Rossignac,91].

Using the concept of cones of faces, it is easy to construct an algorithm to determine the classification of a vertex as manifold or non-manifold in any polyhedron or pseudo-polyhedron. The algorithm will be defined with the following steps (1 to 6):

1 Get the set of the polyhedron's 2D faces that are incident to 0D vertex *A*.

2 From the set of faces select one of them.

3 The selected face has two 1D edges that are incident to *A*, get one of them and label it as *START* and *ANOTHER*

4 Repeat

4.1 If the number of incident faces to *ANOTHER* is more than one, then *A* is a non-manifold vertex.

4.2 The *ANOTHER* edge is common to another face, find it.

4.3 The face has another edge that is common to *A*, find it and label it as *ANOTHER*.

4.4 Until *START* = *ANOTHER* (it has been found a cone of faces).

5 If there are more faces to analyze then A is a non-manifold vertex (there are more cones of faces).

6 Otherwise, *A* is a manifold vertex (*A* is the apex of only one cone of faces).

The following is an implementation of the algorithm in the high level language Java (see in [Gosling, 00] the language's specifications). For this code, a vertex "v" is evaluated to classify it as manifold or non-manifold. If the vertex is manifold (and for instance, the apex of only one cone of faces), then the method returns true, otherwise, the vertex is non-manifold (it is the apex of more than one cone of faces) and it returns false.

```
        boolean isManifoldVertex(Polygon p, Vertex v)
        {
1               Face faces[ ] = getFacesIncidentToVertex(p, v);
2               Face firstFace = selectAndRemoveFace(faces);
3               Edge e1 = getIncidentEdgeToVertex(firstFace, v);
                Edge start = e1;
                Edge another = e1;
4               do
                {
4.1                     if(getNumberOfIncidentFacesToEdge(faces, another) > 1)
                        {
                                return false;
                        }
4.2                     Face f = removeFaceIncidentToEdge(faces, another);
4.3                     Edge e2 = getIncidentEdgeToVertex(f, another, v);
                        another = e2;
                }
4.4             while(another != start);
5               if(faces.length > 0)
                {
                        return false;
                }
6               return true;
        }
```

Using this algorithm over the possible vertices in 3D-OPP's (section 4.4.1) we have the results presented in **Table 4.14** which coincide with those presented in [Aguilera, 98].

TABLE 4.14

3D-OPP's vertices classification.

3D vertex	**Configuration(s)**	**Classification**
V3	b, f, o, u	Manifold
V4	j	Manifold
V4N1	g, p	Non-manifold
V4N2	k	Non-manifold
V5N	d, s	Non-manifold
V6	e, l, t	Non-manifold for configurations e and t. Manifold for configuration l.
V6N1	h, q	Non-manifold
V6N2	n	Non-manifold

4.4.4 Classifying the Π_1 in 4D Polytopes Through its Hyper-Cones of Volumes

Due to the analogy between 3D-OPP's vertices described in terms of their incident manifold or non-manifold edges, and 4D-OPP's edges described in terms of their incident manifold or non-manifold faces, [Pérez & Aguilera, 03] consider that the next logical step is to extend the concept of cones of faces presented in section 4.4.3 to classify 4D polytopes' edges as manifold or non-manifold.

As introduced in section 2.1.4, the faces, edges and vertices, as boundary elements for 4D polytopes and pseudo-polytopes, may be either manifold or non-manifold elements. We have stated that a manifold face is adjacent to exactly two volumes [Hansen, 93] while [Pérez & Aguilera, 03] suggest that a manifold edge is the common edge of only one **hyper-cone of volumes**. Conversely, we have suggested that a non-manifold face is adjacent to more than two volumes, and now we suggest that a non-manifold edge is the common edge of more than one **hyper-cone of volumes**.

Using the concept of hyper-cones of volumes, it is easy to extend the algorithm presented in section 4.4.3 to allow us classifying an edge, as manifold or non-manifold, in any 4D polytope or 4D pseudo-polytope [Pérez & Aguilera, 03]. The algorithm defined by [Pérez & Aguilera, 03] with the following steps (1 to 6) that are an extension of those presented in section 4.4.3:

1 Get the set of 3D Π_3's that are incident to edge *A* (a Π_1).

2 From the set of Π_3's select one of them.

3 The selected Π_3 has two Π_2's that are incident to *A*, get one of them and label it as *START* and *ANOTHER*.

4 Repeat

4.1 If the number of incident Π_3's to *ANOTHER* is more than one, then *A* is a non-manifold Π_1.

4.2 The *ANOTHER* Π_2 is common to another Π_3, find it.

4.3 The Π_3 has another Π_2 that is common to *A*, find it and label it as *ANOTHER*.

4.4 Until *START* = *ANOTHER* (a hyper-cone of volumes has been found).

5 If there are more Π_3's to analyze then *A* is non-manifold (there are more hyper-cones of volumes).

6 Otherwise, *A* is manifold (*A* is the common edge of only one hyper-cone of volumes).

The following is an implementation of the algorithm in the high level language Java (see in [Gosling, 00] the language's specifications). For this code, an edge "e" is evaluated to classify it as manifold or non-manifold. If the edge is manifold (and for instance, the common edge of only one hyper-cone of volumes), then the method returns true, otherwise, the edge is non-manifold (it is the common edge of more than one hyper-cone of volumes) and it returns false.

```
        boolean isManifoldEdge(Polytope p, Edge e)
        {
1               Volume volumes[ ] = getVolumesIncidentToEdge(p, e);
2               Volume firstVolume = selectAndRemoveVolume(volumes);
3               Face f1 = getIncidentFaceToEdge(firstVolume, e);
                Face start = f1;
                Face another = f1;
4               do
                {
4.1                     if(getNumberOfIncidentVolumesToFace(volumes, another) > 1)
                                return false;
4.2                     Volume v = removeVolumeIncidentToFace(volumes, another);
4.3                     Face f2 = getIncidentFaceToEdge(v, another, e);
                        another = f2;
                }
4.4             while(another != start);
5               if(volumes.length > 0)
                {
                        return false;
                }
6               return true;
        }
```

Using this algorithm over the possible edges in 4D-OPP's (section 4.4.2) we have that the edges' classifications are analogous to the 3D-OPP's vertices' classifications. **Table 4.15** shows the edges' classifications given by the extended algorithm and their analogous 3D results.

TABLE 4.15
4D-OPP's edges classifications and their analogy with 3D-OPP's vertices
(taken from [Pérez & Aguilera, 03c]).

4D edge	**Classification through hyper-cones of volumes**	**3D vertex**	**Classification through cones of faces**
E3	Manifold	V3	Manifold
E4	Manifold	V4	Manifold
E4N1	Non-manifold	V4N1	Non-manifold
E4N2	Non-manifold	V4N2	Non-manifold
E5N	Non-manifold	V5N	Non-manifold
E6	*Non-manifold* when 2 or 6 hypervolumes are incident to it. *Manifold* when 4 hypervolumes are incident to it.	V6	*Non-manifold* for configurations e and t. *Manifold* for configuration l.
E6N1	Non-manifold	V6N1	Non-manifold
E6N2	Non-manifold	V6N2	Non-manifold

4.4.5 Classifying the Π_{n-3} in nD Polytopes Through its nD Hyper-Cones of Π_{n-1}'s

Due to the analogy found between 3D vertices and 4D edges with the extension of the concept of cones of faces, it is feasible to generalize the algorithms presented in sections 4.4.3 and 4.4.4 to classify the Π_{n-3} as manifold or non-manifold in nD polytopes through their nD hyper-cones of Π_{n-1}'s. The general algorithm proposed by [Pérez & Aguilera, 03] is the following:

1 Get the set of Π_{n-1}'s that are incident to Π_{n-3} *A*.

2 From the set of Π_{n-1}'s select one of them.

3 The selected Π_{n-1} has two Π_{n-2}'s that are incident to Π_{n-3} A, get one of them and label it as *START* and *ANOTHER*.

4 Repeat

4.1 If the number of incident Π_{n-1}'s to *ANOTHER* is more than one, then *A* is a non-manifold Π_{n-3}.

4.2 The *ANOTHER* Π_{n-2} is common to another Π_{n-1}, find it.

4.3 The Π_{n-1} has another Π_{n-2} that is common to *A*, find it and label it as *ANOTHER*.

4.4 Until *START* = *ANOTHER* (it has been found a nD hyper-cone of Π_{n-1}'s).

5 If there are more Π_{n-1}'s to analyze then Π_{n-3} *A* is non-manifold (there are more nD hyper-cones of Π_{n-1}'s).

6 Otherwise, Π_{n-3} *A* is manifold (*A* is the apex of only one nD hyper-cone of Π_{n-1}'s).

4.4.6 The Eight Types of Π_{n-3}'s in n-Dimensional Orthogonal Pseudo-Polytopes

Due to the analogy between vertices in 3D-OPP's and edges in 4D-OPP's (see **Table 4.15**), [Pérez & Aguilera, 03] extend their properties to propose the eight types of Π_{n-3}'s in nD Orthogonal Pseudo-Polytopes. Such Π_{n-3}'s will be referred as $\Pi_{n-3}3$, $\Pi_{n-3}4$, $\Pi_{n-3}4N1$, $\Pi_{n-3}4N2$, $\Pi_{n-3}5N$, $\Pi_{n-3}6$, $\Pi_{n-3}6N1$ and $\Pi_{n-3}6N2$. In this nomenclature (just as the used in sections 4.4.1 and 4.4.2) "Π_{n-3}" indicates the (n-3)-dimensional element (i.e. vertices in 3D-OPP's and edges in 4D-OPP's), the first digit shows the number of incident Π_{n-2} (i.e. edges in 3D-OPP's and faces in 4D-OPP's), the "N" is present if at least one non-manifold Π_{n-2} is incident to the Π_{n-3} and the second digit is included to distinguish between two different types of Π_{n-3}'s that otherwise could receive the same name.

[Pérez & Aguilera, 03c] describe the following properties for each Π_{n-3}:

- $\Pi_{n-3}3$: all three incident Π_{n-2}'s are manifold and perpendicular to each other.
- $\Pi_{n-3}4$: all four incident Π_{n-2}'s are manifold, they lie on a hyperplane, and can be grouped in two couples of *co-hyperplanar* Π_{n-2}'s.
- $\Pi_{n-3}4N1$: three of its four incident Π_{n-2}'s are perpendicular to each other and also manifold ones, while the fourth is non-manifold and *co-hyperplanar* to one of the other three.
- $\Pi_{n-3}4N2$: two of its four incident Π_{n-2}'s are manifold and *co-hyperplanar*, while each of its other two is non-manifold and perpendicular to the other three.
- $\Pi_{n-3}5N$: four of its five incident Π_{n-2}'s are manifold and lie in a hyperplane, while the fifth is non-manifold and perpendicular to the rest of them.

- $\Pi_{n-3}6$: all six incident Π_{n-2}'s are manifold.
- $\Pi_{n-3}6N1$: three of its six incident Π_{n-2}'s are perpendicular to each other and also manifold ones, while each of its remaining three Π_{n-2}'s is non-manifold and *co-hyperplanar* to one of the first three.
- $\Pi_{n-3}6N2$: all of its six incident Π_{n-2}'s are non-manifold.

4.5 Extreme Edges in the 4D-OPP's

In this section we will introduce the Aguilera & Ayala's concept of Extreme Vertex and how it is possible to proceed, as we have seen in the previous analogies between vertices in the 3D-OPP's and edges in the 4D-OPP's, to define its four-dimensional space's analogue: the Extreme Edges. In the last section, we will show that the (n-1)-dimensional elements can be classified as extreme while the (n-2)-dimensional elements can be classified as extreme or non-extreme.

4.5.1 Extreme Vertices in the 3D-OPP's

[Aguilera, 98] defines a brink or extended-edge as the maximal uninterrupted segment, built out of a sequence of collinear and contiguous two-manifold edges of a 3D-OPP with the following properties:

- Non-manifold edges do not belong to brinks.
- Every two-manifold edge belongs to a brink, whereas every brink consists of m edges ($m \geq 1$), and contains $m+1$ vertices.

- Two of the vertices of type V3, V4N1 or V6N1 (section 4.4.1) are at either extreme of the brink (***Extreme Vertices***). These vertices have in common that they are the only ones that have exactly three incident two-manifold and perpendicular edges, regardless of the number of incident non-manifold edges, therefore those vertices mark the end of brinks in all three orthogonal directions.
- The $m-1$ vertices of type V4, V4N2, V5N or V6 are the only common point of two collinear edges of a same brink (interior vertices).
- Due to all six incident edges of a V6N2 vertex are non-manifold edges, none of them belongs to a brink, thus this vertex does not belong to any brink.

See **Figure 4.1.a** for an example of a 3D-OPP's wireframe model. Also in **Figure 4.1.b** are shown the OPP's brinks parallel to X_1 axis. The continuous lines indicate manifold edges and the dotted one a non-manifold edge (it does not belong to a brink). The points at both extremes of the brinks are Extreme Vertices.

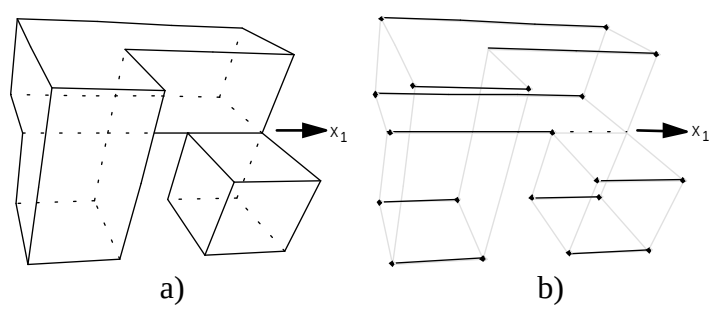


FIGURE 4.1
Example of a 3D-OPP. a) Wireframe model. b) Their brinks parallel to X_1 axis (see text for details. Taken from [Pérez & Aguilera, 03b]).

Based in the previous analysis for brinks, [Aguilera, 98] presents the following properties for the **Extreme Vertices** in the 3D-OPP's:

- Every Extreme Vertex of a 3D-OPP has exactly 3 incident manifold edges perpendicular to each other. This number is even for every non-extreme vertex.
- Every Extreme Vertex has an odd number of incident faces, and every non-extreme vertex has an even number of incident faces.
- Any Extreme Vertex of a 3D-OPP, when is locally described by a set of surrounding boxes, is surrounded by an odd number of such boxes. An even number of surrounding boxes either defines a non-extreme vertex, or does not define any vertex at all (i.e., a non-valid vertex).

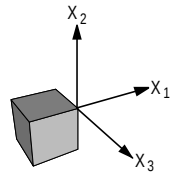
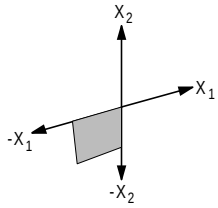
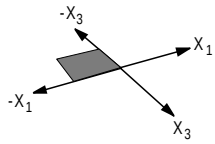
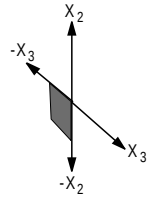
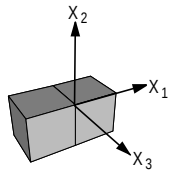
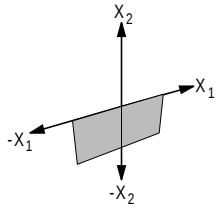
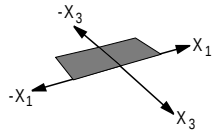
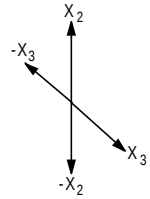
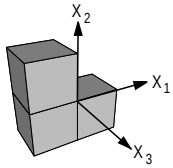
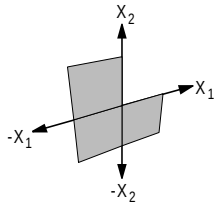
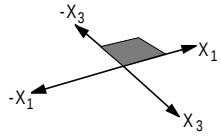
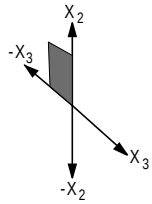
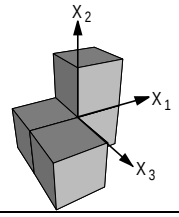
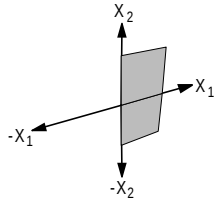
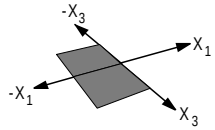
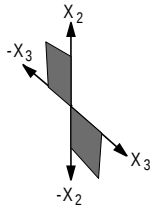
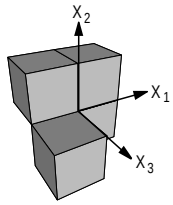
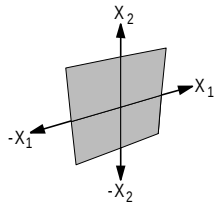
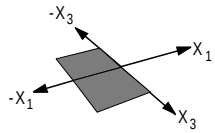
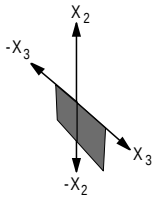
4.5.2 The 2D Analysis for Vertices in 3D-OPP's

In section 4.2.2 were presented the 22 configurations, identified by [Aguilera, 98], which determine a 3D-OPP through a set of quasi-disjoint boxes (cubes). Each of these boxes' vertices can be considered as the origin of a 3D local coordinate system. In such 3D local coordinate system can be identified three main planes: X_1X_2, X_1X_3 and X_2X_3. If the faces that are coplanar to such main planes are grouped, ignoring those faces that are shared by two cubes (face adjacency), they compose three 2D configurations (one for each main plane). For these 2D configurations the vertex can be classified as manifold or non-manifold (section 4.3.1). See **Table 4.16** for examples for 3D configurations b to k.

By applying this analysis over the 22 configurations for the 3D-OPP's, it results that for those configurations whose vertex is extreme (V3, V4N1 or V6N1) and their number of boxes is odd, the three vertex analysis for their 2D configurations classify the 2D vertex as manifold (in **Table 4.16**, configurations b and f, for example). From this pattern, we can infer if a vertex is extreme or non-extreme.

TABLE 4.16

Vertex analysis for 2D configurations on the main planes in 3D configurations b to k (taken from [Pérez & Aguilera, 03b]).

3D configuration	**2D configuration on X_1X_2 Plane**	**2D configuration on X_1X_3 Plane**	**2D configuration on X_2X_3 Plane**	**Analysis for 2D vertex**
b 	b 	b 	b 	***X_1X_2: Manifold*** ***X_1X_3: Manifold*** ***X_2X_3: Manifold***
c 	c 	c 	a 	X_1X_2: Non vertex X_1X_3: Non vertex X_2X_3: Non vertex
f 	e 	b 	b 	***X_1X_2: Manifold*** ***X_1X_3: Manifold*** ***X_2X_3: Manifold***
j 	c 	c 	d 	X_1X_2: Non vertex X_1X_3: Non vertex X_2X_3: Non manifold
k 	f 	c 	c 	X_1X_2: Non vertex X_1X_3: Non vertex X_2X_3: Non vertex

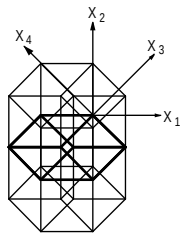
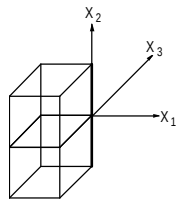
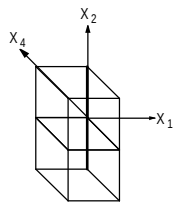
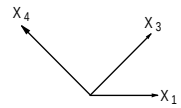
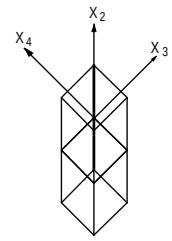
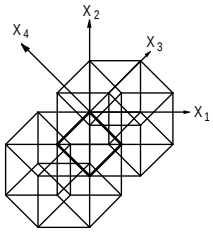
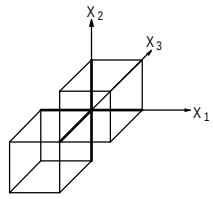
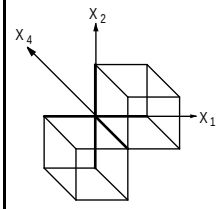
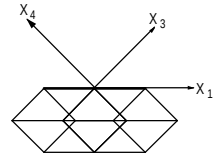
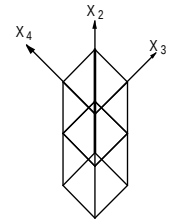
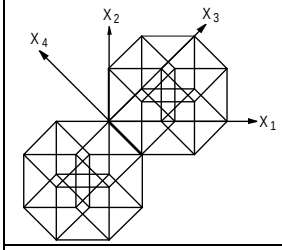
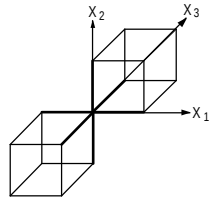
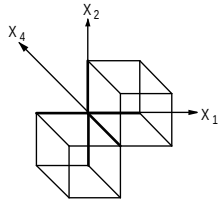
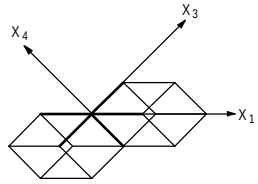
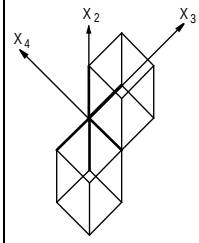
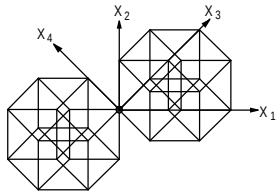
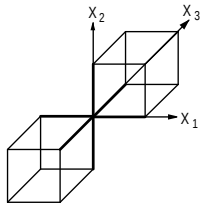
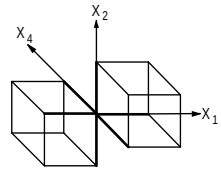
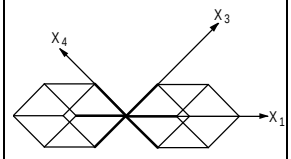
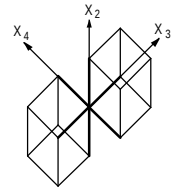
4.5.3 The 3D Analysis for Edges in 4D-OPP's

The vertex analysis for 2D configurations embedded in the main planes of a 3D configuration (previous section) classify the 2D vertex as manifold or non-manifold, and through these three 2D analysis we can infer if the 3D vertex is extreme or non-extreme. For consequence, in analogous way, [Pérez & Aguilera, 03b] propose that we can assume that the edges analysis for 3D configurations embedded in the main hyperplanes of a 4D configuration will classify to 3D edges as manifold or non-manifold, and through these 3D analysis we can infer, due to the analogy with 3D vertex, if the 4D edges are "**Extreme**" or "**Non-Extreme**".

In section 4.2.3 and **Appendix A** are presented the 253 configurations which determine a 4D-OPP through a set of quasi-disjoint hyper-boxes (hypercubes). Each of these hyper-boxes' vertices can be considered as the origin of a 4D local coordinate system. In such 4D local coordinate system can be identified four main hyperplanes: $X_1X_2X_3$, $X_1X_2X_4$, $X_1X_3X_4$ and $X_2X_3X_4$. If the volumes that are *co-hyperplanar* to such main hyperplanes are grouped, ignoring those volumes that are shared by two hypercubes (volume adjacency), they will compose four 3D configurations (one for each main hyperplane). **Table 4.17** presents the four 3D configurations that are present in 4D configurations 3 to 6.

TABLE 4.17

Determining the 3D configurations on the main hyperplanes in 4D configurations 3 to 6 (taken from [Pérez & Aguilera, 03b]).

4D configuration	**3D configuration on $X_1X_2X_3$ hyperplane**	**3D configuration on $X_1X_2X_4$ hyperplane**	**3D configuration on $X_1X_3X_4$ hyperplane**	**3D configuration on $X_2X_3X_4$ hyperplane**
3 	b 	b 	a 	b 
4 	d 	d 	b 	b 
5 	e 	d 	d 	d 
6 	e 	e 	e 	e 

For the 3D configurations that are embedded in the main hyperplanes it is possible to analyze their edges and classify them as manifold or non-manifold (section 4.3.2). **Table 4.18** shows the edges analysis for the 3D configurations that are present in 4D configurations 3 to 6.

TABLE 4.18

Edges analysis for 3D configurations on the main hyperplanes in 4D configurations 3 to 6 (taken from [Pérez & Aguilera, 03b]).

	3D Edges Analysis			
4D Configuration	**Configuration on $X_1X_2X_3$ hyperplane**	**Configuration on $X_1X_2X_4$ hyperplane**	**Configuration on $X_1X_3X_4$ hyperplane**	**Configuration on $X_2X_3X_4$ hyperplane**
3	X_1: Non edge $-X_1$: Non edge ***X_2: Manifold*** ***$-X_2$: Manifold*** X_3: Non edge $-X_3$: Non edge	X_1: Non edge $-X_1$: Non edge ***X_2: Manifold*** ***$-X_2$: Manifold*** X_4: Non edge $-X_4$: Non edge	X_1: Non edge $-X_1$: Non edge X_3: Non edge $-X_3$: Non edge X_4: Non edge $-X_4$: Non edge	***X_2: Manifold*** ***$-X_2$: Manifold*** X_3: Non edge $-X_3$: Non edge X_4: Non edge $-X_4$: Non edge
4	***X_1: Manifold*** ***$-X_1$: Manifold*** ***X_2: Manifold*** ***$-X_2$: Manifold*** X_3: Non edge $-X_3$: Manifold	***X_1: Manifold*** ***$-X_1$: Manifold*** ***X_2: Manifold*** ***$-X_2$: Manifold*** X_4: Non manifold $-X_4$: Non edge	***X_1: Manifold*** ***$-X_1$: Manifold*** X_3: Non edge $-X_3$: Non edge X_4: Non edge $-X_4$: Non edge	***X_2: Manifold*** ***$-X_2$: Manifold*** X_3: Non edge $-X_3$: Non edge X_4: Non edge $-X_4$: Non edge
5	***X_1: Manifold*** ***$-X_1$: Manifold*** ***X_2: Manifold*** ***$-X_2$: Manifold*** ***X_3: Manifold*** ***$-X_3$: Manifold***	***X_1: Manifold*** ***$-X_1$: Manifold*** ***X_2: Manifold*** ***$-X_2$: Manifold*** X_4: Non edge $-X_4$: Non manifold	***X_1: Manifold*** ***$-X_1$: Manifold*** ***X_3: Manifold*** ***$-X_3$: Manifold*** X_4: Non edge $-X_4$: Non manifold	***X_2: Manifold*** ***$-X_2$: Manifold*** ***X_3: Manifold*** ***$-X_3$: Manifold*** X_4: Non edge $-X_4$: Non manifold
6	***X_1: Manifold*** ***$-X_1$: Manifold*** ***X_2: Manifold*** ***$-X_2$: Manifold*** ***X_3: Manifold*** ***$-X_3$: Manifold***	***X_1: Manifold*** ***$-X_1$: Manifold*** ***X_2: Manifold*** ***$-X_2$: Manifold*** ***X_4: Manifold*** ***$-X_4$: Manifold***	***X_1: Manifold*** ***$-X_1$: Manifold*** ***X_3: Manifold*** ***$-X_3$: Manifold*** ***X_4: Manifold*** ***$-X_4$: Manifold***	***X_2: Manifold*** ***$-X_2$: Manifold*** ***X_3: Manifold*** ***$-X_3$: Manifold*** ***X_4: Manifold*** ***$-X_4$: Manifold***

Through a computer program, the edges analysis for the 3D configurations embedded in the main hyperplanes of a 4D configuration, was applied over the 253 configurations for the 4D-OPP's and the obtained results are [Pérez & Aguilera, 03b]:

- An edge in a 4D-OPP can be classified by three 3D analysis (a 4D edge can only be present in three of the four main hyperplanes) as:
    - 3 times as manifold and 0 times as non-manifold, or
    - 0 times as manifold and once as non-manifold, or
    - 0 times as manifold and 3 times as non-manifold, or
    - 0 times as manifold and 0 times as non-manifold.

- The above patterns can be found in any 4D configuration because it can have from 0 to 8 incident edges to the origin.
- Following the analogy with the vertex analysis for 2D configurations embedded in the main planes of a 3D configuration (previous section), it can be proposed that if a edge in a 4D-OPP has been classified in the 3D analysis three times as manifold, then it can be considered as an **Extreme Edge**, and any other result will classify it as a **Non-Extreme Edge**.
- The manifold or non-manifold classification for a edge in a 4D-OPP is independent of its classification as extreme or non-extreme. Is the same situation for a vertex in a 3D-OPP, where its classification as extreme or non-extreme is independent of its classification as manifold or non-manifold.
- If we analyze the incident manifold or non-manifold faces that are incident to an extreme or non-extreme edge in 4D-OPP's, we can observe that the analogy with the description of extreme or non-extreme vertices in terms of the incident manifold or non-manifold edges that are incident to those vertices is preserved, as shown in **Table 4.19**.

TABLE 4.19
The 4D-OPP's edges classifications and their analogy with 3D-OPP's vertices (taken from [Pérez & Aguilera, 03b]).

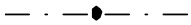
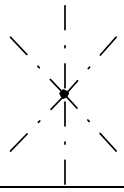
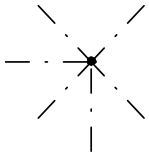
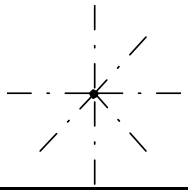
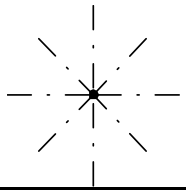
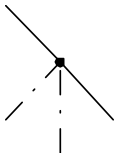
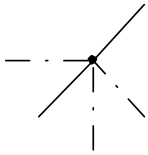
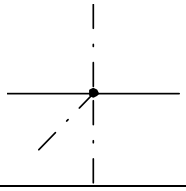
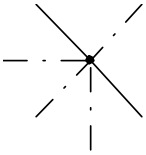
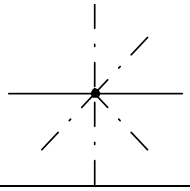
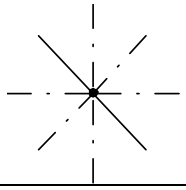
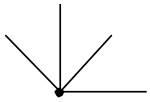
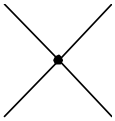
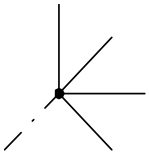
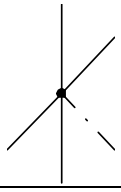
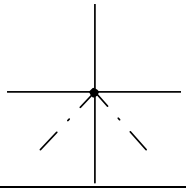
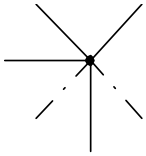
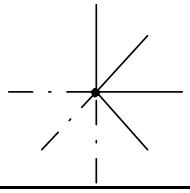
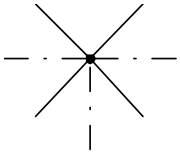
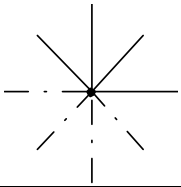
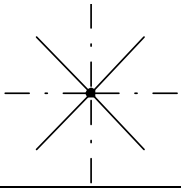
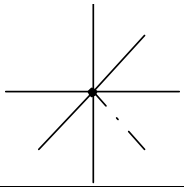
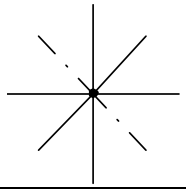
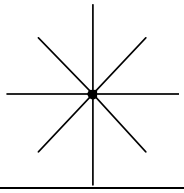
4D edge	Classification (manifold or non-manifold)	Classification (extreme or non-extreme)	3D vertex	Classification (manifold or non-manifold)	Classification (extreme or non-extreme)
E3	Manifold	***Extreme***	V3	Manifold	***Extreme***
E4	Manifold	Non extreme	V4	Manifold	Non extreme
E4N1	Non-manifold	***Extreme***	V4N1	Non-manifold	***Extreme***
E4N2	Non-manifold	Non extreme	V4N2	Non-manifold	Non extreme
E5N	Non-manifold	Non extreme	V5N	Non-manifold	Non extreme
E6	*Non-manifold* *Manifold*	Non extreme Non extreme	V6	*Non-manifold* *Manifold*	Non extreme Non extreme
E6N1	Non-manifold	***Extreme***	V6N1	Non-manifold	***Extreme***
E6N2	Non-manifold	Non extreme	V6N2	Non-manifold	Non extreme

4.5.4 The Vertices in 4D-OPP's Described in Terms of Extreme and Non-Extreme Edges

We will describe and classify vertices in 4D-OPP's depending on the number of extreme and non-extreme edges incident to them. In the nomenclature to use “V” means vertex, the "X" indicates that the vertex is described in terms of extreme and non-extreme edges, the first digit shows the number of incident extreme edges, the “N” followed by a digit is present if there are incident non-extreme edges to the vertex and the digit indicates the number of such edges, and a third digit is included to distinguish between two different types that otherwise could receive the same name. In **Table 4.20** are shown the 26 identified 4D vertices.

TABLE 4.20

Vertices present in 4D-OPP's described in terms of their incident extreme and non-extreme edges (dotted lines indicate non-extreme edges and continuous lines indicate extreme edges; own elaboration).

VX0 	VX0N2 	VX0N6 	VX0N6-2 
VX0N7 	VX0N8 	VX2 	VX2N2 
VX2N3 	VX2N3-2 	VX2N4 	VX2N5 
VX2N6 	VX4 	VX4-2 	VX4N1 
VX4N1-2 	VX4N2 	VX4N2-2 	VX4N3 
VX4N3-2 	VX4N4 	VX4N4-2 	VX6N1 
VX6N2 	VX8 		

Each vertex has the following properties [Pérez, 01]:

- VX0: Non valid vertex. It doesn't have incident extreme or non-extreme edges.
- VX0N2: Non valid vertex. Its two non-extreme edges are collinear.
- VX0N6: All six incident edges are non-extreme grouped in three pairs of collinear edges.
- VX0N6-2: All six incident edges are non-extreme, four of them lie on a plane, and can be grouped in two couples of collinear edges, while the remaining two are lineally independent.
- VX0N7: All seven incident edges are non-extreme, six of them lie in a hyperplane, while the seventh is perpendicular to the rest of them.
- VX0N8: All of its eight incident edges are non-extreme.
- VX2: Non valid vertex. Its two extreme edges are collinear.
- VX2N2: Two of its four incident edges are extreme and collinear, while each of its other two is non-extreme and perpendicular to the other three.
- VX2N3: Two of its five incident edges are extreme and collinear, while each of its other three is non-extreme and perpendicular to the other four.
- VX2N3-2: Two of its five incident edges are extreme and collinear, other two are non-extreme and collinear, both pairs lie in a plane, while the fifth is non-extreme and perpendicular to the other four.
- VX2N4: Two of its six incident edges are extreme and collinear, other two are non-extreme and collinear, both pairs lie in a plane, while each of its other two is non-extreme and perpendicular to the other five.

- VX2N5: Four of its seven incident edges are non-extreme and lie in a plane, while other two are extreme and collinear and the remaining is non-extreme and perpendicular to the other six.
- VX2N6: Six of its eight incident edges are non-extreme and lie in a hyperplane, while other two are extreme and collinear.
- VX4: All four incident edges are extreme and perpendicular.
- VX4-2: All four incident edges are extreme and lie in a plane.
- VX4N1: Four of its five incident edges are perpendicular and also extreme ones, while the fifth is non-extreme and collinear to one of the other four.
- VX4N1-2: Four of its five incident edges are extreme and lie in a plane, while the fifth is non-extreme and perpendicular to the rest of them.
- VX4N2: Four of its six incident edges are extreme and lie in a plane, while each of its other two is non-extreme and perpendicular to the other five.
- VX4N2-2: Four of its six incident edges are perpendicular and also extreme ones, while each of its remaining two edges is non-extreme and collinear to one of the first four.
- VX4N3: Four of its seven incident edges are perpendicular and also extreme ones, while each of its remaining three edges is non-extreme and collinear to one of the first four.
- VX4N3-2: Four of its seven incident edges are extreme and lie in a plane, while other two are non-extreme and collinear and the remaining edge is non-extreme and perpendicular to the other six.
- VX4N4: Four of its eight incident edges are perpendicular and also extreme ones, while each of its remaining four edges is non-extreme and collinear to one of the first four.

- VX4N4-2: Four of its eight incident edges are extreme and lie in a plane, while the remaining four are non-extreme and lie in a plane.
- VX6N1: Six of its seven incident edges are extreme and lie in a hyperplane, while the seventh is non-extreme and perpendicular to the rest of them.
- VX6N2: Six of its eight incident edges are extreme and lie in a hyperplane, while the remaining two are collinear and non-extreme ones.
- VX8: All eight incident edges are extreme.

4.5.5 The Extreme and Non-Extreme (n-1), (n-2) and (n-3)-Dimensional Elements

Although the previous properties, presented in the section 4.5.1, define brinks in the 3D-OPP's, [Aguilera, 98] also defines the properties for brinks in the 1D-OPP's and 2D-OPP's as follows:

- In the 1D-OPP's the only elements which exist are vertices and edges. If a vertex has only one incident edge, then it is an Extreme Vertex. Then, edges and brinks are in this case equivalent.
- In the 2D-OPP's there are only two types of vertices (section 4.3.1): the vertex with two incident Manifold edges (V2) and the vertex with four incident Manifold edges (V4N). In a 2D-OPP's brink, vertices of type V2 are Extreme Vertices because each one of these vertices has two incident Manifold and perpendicular edges, while vertices of type V4N are interior vertices because each one is the common point of two edges in a brink, therefore they cannot be the brink's ending vertices.

Based in the previous analysis, [Aguilera, 98] presents the following properties for Extreme Vertices in the 2D-OPP's and 1D-OPP's:

- An Extreme Vertex in the 1D space has only one incident manifold edge. Any non extreme vertex will be the common point of two edges.
- Each Extreme Vertex in the 2D space has exactly two incident manifold and perpendicular edges.
- Any 2D-OPP's Extreme Vertex when is locally described by a set of surrounding rectangles (see section 4.2.1), it is surrounded by a odd number of those rectangles. An even number of surrounding rectangles defines either a non extreme vertex or a non valid vertex.

From these properties it is found that vertices, the (n-1)-dimensional elements of a segment (1D-OPP), are either extreme and 0-manifold (or just manifold, see the [Hansen,93]'s rules in section 2.1.5) or non extreme and non-manifold (because they are the common point of two segments with vertex adjacency).

In section 4.3.1 it was presented that vertices, the (n-2)-dimensional elements in a 2D-OPP's, can have two possible characterizations: manifold or non-manifold. Due to the previous properties, it is known that a vertex is extreme when it has two incident manifold and perpendicular edges, otherwise it will be a non extreme vertex. By associating the manifold vertex's definition (section 4.3.1) with the extreme vertex's definition, we have that a vertex in a 2D-OPP is:

- Manifold and Extreme: when it has two incident and perpendicular edges.
- Non-Manifold and Non-Extreme: when it has four incident edges.

The extreme vertex and manifold vertex's concepts are equivalent in the 1D-OPP's, therefore, and due to its analogy with the (n-1)-dimensional elements (in other words, the cells Π_{n-1} described in section 2.1.5), it is possible to generalize such equivalence to propose:

A cell Π_{n-1} in a nD-OPP is Manifold and Extreme when it has just one incident cell Π_n.

For example, we can expect the following characterizations:

- 1D-OPP's: Manifold/Extreme Vertices.
- 2D-OPP's: Manifold/Extreme Edges.
- 3D-OPP's: Manifold/Extreme Faces.
- 4D-OPP's: Manifold/Extreme Volumes.
- 5D-OPP's: Manifold/Extreme Hypervolumes.

In the 2D-OPP's, the extreme vertex and manifold vertex's concepts are equivalent. In the same way the non-extreme vertex and non-manifold vertex's concepts are equivalent. Due to their analogy with the (n-2)-dimensional elements, it is possible to generalize such equivalences to propose:

A cell Π_{n-2} in a nD-OPP can be:

1. Manifold/Extreme when it has two incident and perpendicular cells Π_{n-1}.
2. Non-Manifold/Non-Extreme when it has four incident cells Π_{n-1}.

Therefore, we can expect the following characterizations:

- 2D-OPP's: Manifold/Extreme Vertices and Non-Manifold/Non-Extreme Vertices.
- 3D-OPP's: Manifold/Extreme Edges and Non-Manifold/Non-Extreme Edges.
- 4D-OPP's: Manifold/Extreme Faces and Non-Manifold/Non-Extreme Faces.
- 5D-OPP's: Manifold/Extreme Volumes and Non-Manifold/Non-Extreme Volumes.

In the **Table 4.15** was presented the analogy between 3D-OPP's vertices and 4D-OPP's edges, the (n-3)-dimensional elements. Their characterizations are product of the methodologies described in sections 4.4.2 and 4.4.4 (Manifold or Non-Manifold vertex-edge). As observed in the mentioned sections, their classifications are the same, which led to [Pérez & Aguilera, 03] to generalize the eight possible Π_{n-3}'s in the nD-OPP's (section 4.4.6). As appreciated in section 4.5.3 (**Table 4.19**), the vertices and edges' classifications as extreme or non-extreme are consistent with the previously identified analogies between those elements, leading us to four possible characterizations:

- Manifold and Extreme Vertex (V3) in the 3D-OPP's or Edge (E3) in the 4D-OPP's.
- Manifold and Non-Extreme Vertex (V4, V6) in the 3D-OPP's or Edge (E4, E6) in the 4D-OPP's.
- Non-Manifold and Extreme Vertex (V4N1, V6N1) in the 3D-OPP's or Edge (E4N1, E6N1) in the 4D-OPP's.
- Non-Manifold and Non-Extreme Vertex (V4N2, V5N, V6, V6N2) in the 3D-OPP's or Edge (E4N2, E5N, E6, E6N2) in the 4D-OPP's.

Finally, for the eight Π_{n-3}'s in the nD-OPP's, described by [Pérez & Aguilera, 03c], it is possible to annex their corresponding characterization as Extreme or Non-Extreme (see their characteristics in section 4.4.6):

- Manifold/Extreme element $\Pi_{n-3}3$.
- Manifold/Non-Extreme element $\Pi_{n-3}4$.
- Non-Manifold/Extreme element $\Pi_{n-3}4N1$.
- Non-Manifold/Non-Extreme element $\Pi_{n-3}4N2$.
- Non-Manifold/Non-Extreme element $\Pi_{n-3}5N$.
- Manifold/Non-Extreme or Non-Manifold/Non-Extreme element $\Pi_{n-3}6$.
- Non-Manifold/Extreme element $\Pi_{n-3}6N1$.
- Non-Manifold/Non-Extreme element $\Pi_{n-3}6N2$.