Chapter 3
Techniques for Visualizing the Four-Dimensional Space

Recent interest has been growing in studying multidimensional polytopes (4D and beyond) for representing multidimensional phenomena in the Euclidean n-Dimensional space. Some of these phenomena's features rely on the polytope's geometric and topological relations (as we will see in the next chapters). However, [Banchoff, 96] motivates us to think about two important questions: Is it possible to visualize a polytope to know how it looks like? And if we can't see it, how can we be sure about the proper understanding of its relations and properties? The answer is that the task of visualizing polytopes in the fourth and higher dimensional spaces belongs to the computer graphics field [Banchoff, 96]. Visualizing these new dimensions leads us to learn and to understand the events, relationships and properties for these phenomena. In this chapter we will deal with three, yet introduced in chapter 1, methods for visualizing 4D and beyond Polytopes: (section 3.1) through their projections, (section 3.2) through their unravelings; and (section 3.3) through their slicings with three-dimensional space. Moreover, we will consider the visualization of two interesting events: a polytope's rotation around a plane and the 4D hypercube and simplex' unraveling processes.

3.1 Polytopes' Projection

3.1.1 The 3D-2D Projection

We can define a 3D-2D projection as the transformation of 3D scenes onto 2D viewing planes (a computer screen for example). A projection imitates the process by which the eye maps world scenes into images onto the retina. In general terms, a projection transform points in a nD space to points onto a lower dimensional space [Foley, 96].

The projection of 3D objects is defined by projection straight rays, which emanate from a center of projection to pass by each point of the object and to finally, intersect a plane and create the projection [Foley, 96] (**Figure 3.1**).

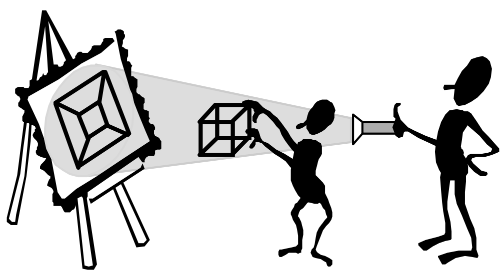


FIGURE 3.1
Projecting a cube onto a plane (taken from [Aguilera & Pérez, 02]).

When the center of projection is at the infinite then the projection rays are parallel between them. This projection is defined as ***3D-2D parallel projection***, which informally is just to remove the X_3 coordinate from the object's points if the projection plane is $X_3 = 0$ (which is the most popularly used in the Computer Graphics field). Then we have the matrix representation:

$$\begin{bmatrix} x_1 & x_2 & x_3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & 0 & 1 \end{bmatrix}$$

If the projection plane is $X_1 = 0$ or $X_2 = 0$ then it will be enough to remove (or replace with zero) the appropriate coordinate from the object’s points. Then the corresponding matrix representations are then:

$$[x_1 \quad x_2 \quad x_3 \quad 1]\cdot\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = [0 \quad x_2 \quad x_3 \quad 1]$$

$$[x_1 \quad x_2 \quad x_3 \quad 1]\cdot\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = [x_1 \quad 0 \quad x_3 \quad 1]$$

When the center of projection is on X_3 axis at a distance *pz* from the origin, and the projection plane is $X_3 = 0$, then we have a ***3D-2D perspective projection*** with the matrix representation:

$$[x_1 \quad x_2 \quad x_3 \quad 1]\cdot\begin{bmatrix}pz & 0 & 0 & 0\\ 0 & pz & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & pz\end{bmatrix} = [x_1 \cdot pz \quad x_2 \cdot pz \quad 0 \quad pz - x_3] = \begin{bmatrix}\dfrac{x_1 \cdot pz}{pz - x_3} & \dfrac{x_2 \cdot pz}{pz - x_3} & 0 & 1\end{bmatrix}$$

When the center of projection is on X_1 axis at a distance *px* from the origin, and the projection plane is $X_1 = 0$, then the matrix representation for this 3D-2D Perspective Projection is:

$$[x_1 \quad x_2 \quad x_3 \quad 1]\cdot\begin{bmatrix}0 & 0 & 0 & -1\\ 0 & px & 0 & 0\\ 0 & 0 & px & 0\\ 0 & 0 & 0 & px\end{bmatrix} = [0 \quad x_2 \cdot px \quad x_3 \cdot px \quad px - x_1] = \begin{bmatrix}0 & \dfrac{x_2 \cdot px}{px - x_1} & \dfrac{x_3 \cdot px}{px - x_1} & 1\end{bmatrix}$$

And the case when the center of projection is on X_2 axis at a distance *py* from the origin, and the projection plane is $X_2 = 0$, has the matrix representation:

$$[x_1 \quad x_2 \quad x_3 \quad 1]\cdot\begin{bmatrix}py & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & py & 0\\ 0 & 0 & 0 & py\end{bmatrix} = [x_1 \cdot py \quad 0 \quad x_3 \cdot py \quad py - x_2] = \begin{bmatrix}\dfrac{x_1 \cdot py}{py - x_2} & 0 & \dfrac{x_3 \cdot py}{py - x_2} & 1\end{bmatrix}$$

3.1.2 The 4D-3D Projection

[Banks, 92] establishes that the same techniques used to project 3D objects onto 2D planes can be applied to project 4D polytopes onto 3D hyperplanes (our 3D space for example). Then we have that a ***4D-3D parallel projection***, which informally is the X_1, X_2, X_3 or X_4 coordinate's removal from the polytope's points. It has the matrix representation (for the typically removed X_4 coordinate):

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & 1 \end{bmatrix}$$

And a ***4D-3D perspective projection*** is defined when the center of projection is on X_4 axis at a distance pw from the origin. If the projection hyperplane is $X_4 = 0$ then we have the matrix representation:

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & 1 \end{bmatrix} \cdot \begin{bmatrix} pw & 0 & 0 & 0 & 0 \\ 0 & pw & 0 & 0 & 0 \\ 0 & 0 & pw & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & pw \end{bmatrix} = \begin{bmatrix} x_1 \cdot pw & x_2 \cdot pw & x_3 \cdot pw & 0 & pw - x_4 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{x_1 \cdot pw}{pw - x_4} & \dfrac{x_2 \cdot pw}{pw - x_4} & \dfrac{x_3 \cdot pw}{pw - x_4} & 0 & 1 \end{bmatrix}$$

Because a 4D-3D projection will produce a volume as the "shadow" of a 4D polytope, [Hollasch, 91] considers valid to process this volume with some of the 3D-2D projections (parallel or perspective) to be projected finally onto a computer screen. Then we have four possible 4D-3D-2D projections:

4D-3D Perspective Projection - 3D-2D Perspective Projection

4D-3D Perspective Projection - 3D-2D Parallel Projection

4D-3D Parallel Projection - 3D-2D Perspective Projection

4D-3D Parallel Projection - 3D-2D Parallel Projection

In chapter 1, it was introduced the 4D hypercube's projection as a cube inside another cube, or in other words, its central projection (**Figure 1.4**). This visualization is commonly the result of applying the combination of 4D-3D perspective and 3D-2D perspective projections.

By applying the parallel and perspective projections, it is possible to visualize some interesting aspects related to some events in the 4D space. One of these events is the rotation of a hypercube. In **Figure 3.2** are presented some snapshots of the 4D hypercube's rotation around the X_1X_4 plane. The hypercube has its center at the origin and the rotation's angle is 180°. There were applied on the sequence the 4D-3D perspective and 3D-2D perspective projections.

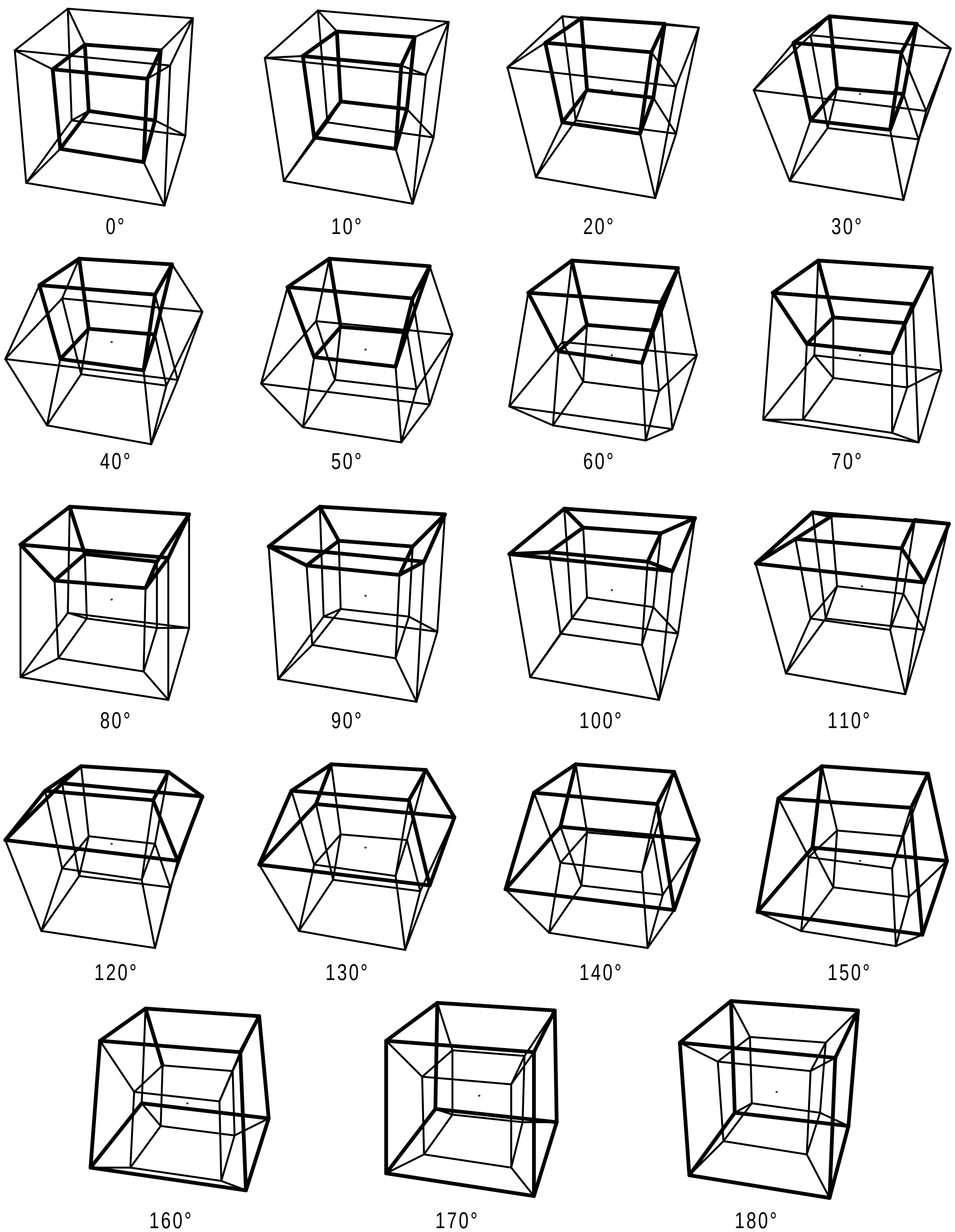


FIGURE 3.2
4D Hypercube's rotation around the X_1X_4 plane (own elaboration).

3.1.3 The nD – (n-1)D Projection

The projection's matrices used in 3D and 4D spaces can be generalized for any number of dimensions such that a n-dimensional polytope is projected onto a (n-1)-dimensional space, therefore, we have a nD – (n-1)D projection. For visualizing a nD polytope on a computer screen, for example, the projections must be repetitively applied, in other words, to consider the projections (n-1)D – (n-2)D, (n-2)D – (n-3)D. Finally, a three-dimensional object will be obtained, which represents the successive projections of the nD polytope [Noll, 67].

The Parallel Projection of a nD polytope onto a (n-1)D hyperplane, or in other words, the ***nD – (n-1)D Parallel Projection*** consists on just removing the n-th coordinate, whose corresponding axis is X_n, from the nD polytope's points. Then, the projection will be defined by the following operations:

$$\underbrace{[x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{n-1} \quad x_n \quad 1]}_{n+1} \cdot \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(n+1)\times(n+1)}$$

$$= \underbrace{[x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{n-1} \quad 0 \quad 1]}_{n+1}$$

The projection's matrix will have n+1 columns and n+1 rows because it is considered that the points have the homogeneous representation. All the elements in the matrix's main diagonal will be 1's except the position (n,n) that is zero (for eliminating the X_n-axis); the matrix's remaining elements will be all zero.

The ***Perspective Projection nD – (n-1)D*** is defined when the projection's center is on the X_n-axis (which corresponds to the n-th coordinate) to a distance *pn* from the origin. If the projection's (n-1)D hyperplane is $X_n = 0$, then we will have the matrix representation:

$$
\underbrace{[x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{n-1} \quad x_n \quad 1]}_{n+1} \cdot \begin{bmatrix} pn & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & pn & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & pn & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & pn & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & pn \end{bmatrix}_{(n+1)\times(n+1)}
$$

$$
= [\underbrace{x_1 \cdot pn \quad x_2 \cdot pn \quad x_3 \cdot pn \quad \cdots \quad x_{n-1} \cdot pn \quad 0 \quad pn - x_n}_{n+1}]
$$

$$
= \left[\underbrace{\frac{x_1 \cdot pn}{pn - x_n} \quad \frac{x_2 \cdot pn}{pn - x_n} \quad \frac{x_3 \cdot pn}{pn - x_n} \quad \cdots \quad \frac{x_{n-1} \cdot pn}{pn - x_n} \quad 0 \quad 1}_{n+1} \right]
$$

All the elements in the projection matrix's main diagonal will have *pn* as its value, except the element in the position (n,n) that will be zero. The element in the row n and column n+1 will be equal to –1. Resuming, the perspective projection takes place when multiplying the n-1 coordinates by $\frac{pn}{pn - x_n}$, leaving to the n-th coordinate as zero.

Now, consider the k-th coordinate such that $k < n$ (because the previous generalizations are based in the fact that the projection hyperplane is $X_n=0$, the popularly used for this end, where the X_n-axis corresponds to the last coordinate) and let x_k be its respective axis. In this way, the Parallel Projection nD - (n-1)D substitutes the k-th coordinate by zero and its matrix representation is:

$$\underbrace{[x_1 \quad \cdots \quad x_{k-1} \quad x_k \quad x_{k+1} \quad \cdots \quad x_n \quad 1]}_{n+1} \cdot \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(n+1)\times(n+1)}$$

$$= \underbrace{[x_1 \quad \cdots \quad x_{k-1} \quad 0 \quad x_{k+1} \quad \cdots \quad x_n \quad 1]}_{n+1}$$

All the elements in the matrix's main diagonal are 1's except the position (k,k) which is zero (in this way, the k-th coordinate is replaced by zero). The matrix's remaining elements will be all zero.

And the Perspective Projection nD – (n-1)D, when the projection center is on the X_k axis at a distance pk from the origin, is defined by the matrix representation:

$$
\underbrace{[x_1 \quad \cdots \quad x_{k-1} \quad x_k \quad x_{k+1} \quad \cdots \quad x_n \quad 1]}_{n+1} \cdot \begin{bmatrix} pk & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & pk & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & pk & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & pk & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pk \end{bmatrix}_{(n+1)\times(n+1)}
$$

$$
= \underbrace{[x_1 \cdot pk \quad \cdots \quad x_{k-1} \cdot pk \quad 0 \quad x_{k+1} \cdot pk \quad \cdots \quad x_n \cdot pk \quad pk - x_k]}_{n+1}
$$

$$
= \left[\underbrace{\frac{x_1 \cdot pk}{pk - x_k} \quad \cdots \quad \frac{x_{k-1} \cdot pk}{pk - x_k} \quad 0 \quad \frac{x_{k+1} \cdot pk}{pk - x_k} \quad \cdots \quad \frac{x_n \cdot pk}{pk - x_k} \quad 1}_{n+1} \right]
$$

In this case, all the elements in the projection matrix's main diagonal will have *pk* as their value, except the element in the position (*k*,*k*) which will be zero. The element in the row *k* and column n+1 will be equal to –1. The matrix's remaining elements will be all zero. Resuming, the perspective projection takes place when multiplying all the coordinates by $\frac{pk}{pk - x_k}$ except the *k*-th which is replaced by zero.

Through the generalization of the parallel and perspective projections, it is possible to obtain, in a simple way, the required matrices for visualizing polytopes beyond the 4D space. For example, in **Figure 3.3** is presented a 5D Hypercube's projection, which has its center at the origin. Also, all the projections, applied to it, (5D-4D, 4D-3D and 3D-2D) were perspective projections. As can be seen, that projection results to be the 5D hypercube's central projection, which can be considered as a 4D hypercube inside another 4D hypercube (the interior 4D hypercube was remarked to facilitate the visualization).

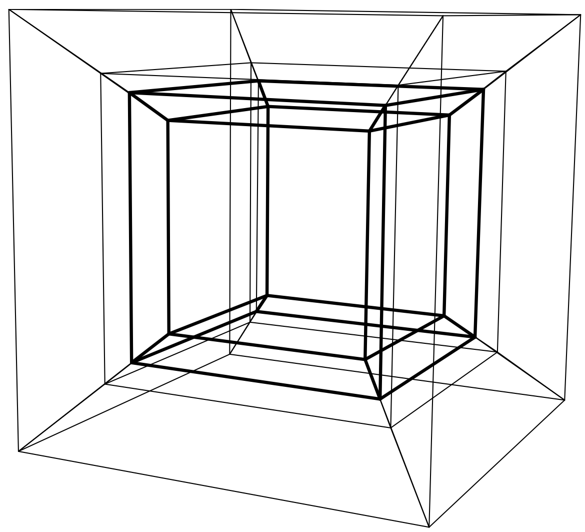


FIGURE 3.3
The 5D Hypercube's central projection (own elaboration).

3.2 The 4D Polytopes' Unravelings and Unfoldings

3.2.1 Unraveling the 4D Hypercube

3.2.1.1 The Hypercube's Unravelings

In section 1.3 were discussed the methods for visualizing 4D polytopes. One of them is the visualization through the unravelings. We remember that the six faces on the boundary of a cube can be unraveled as a 2D cross (**Figure 3.4**). The set of unraveled faces is called the unravelings of the cube.

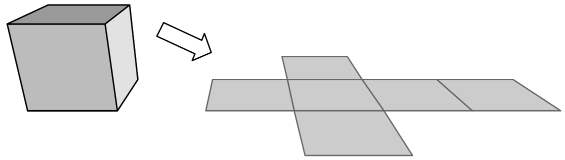


FIGURE 3.4
Unraveling the cube (taken from [Aguilera & Pérez, 02]).

In analogous way, the eight cubes on the boundary of a hypercube can be unraveled as a 3D cross (the hypercube's properties were discussed in chapter 2). This 3D cross was named ***tesseract*** by C. H. Hinton in the XIX century (**Figure 3.5**).

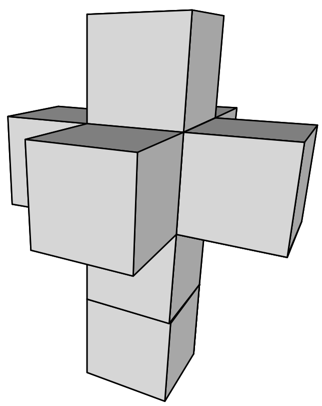


FIGURE 3.5
The unraveled hypercube: the tesseract
(taken from [Aguilera & Pérez, 02]).

We also discussed that a flatlander will visualize the 2D cross, but he will not be able to assembly it back as a cube (even if the specific instructions are provided). This fact is true because of the needed face-rotations in the third dimension around an axis which are physically impossible in the 2D space. However, it is possible for the flatlander to visualize the raveling process through the projection of the faces and their movements onto the 2D space where he lives.

Analogously, we can visualize the tesseract but we won't be able to assembly it back as a hypercube. We know this because of the needed volume-rotations in the fourth dimension around a plane which are physically impossible in our 3D space.

[Kaku, 94] and [Banchoff, 96] describe with detail the representation model for the hypercube through their unravelings. They also mention the physical incapacity of a 3D being to ravel the hypercube back, because the required transformations are not possible in our 3D space (**Figure 3.6**).

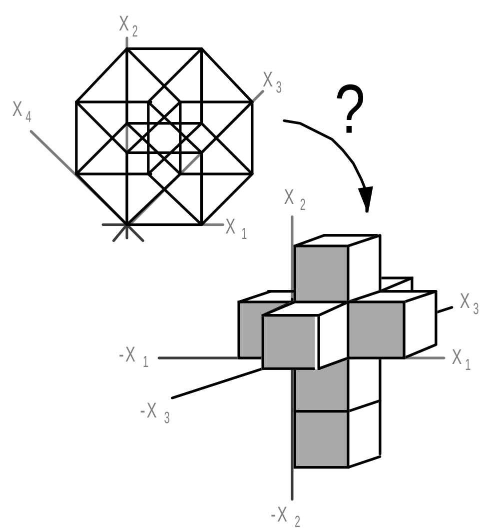


FIGURE 3.6
The hypercube's unraveling result (taken from [Aguilera & Pérez, 02c]).

[Kaku, 94] and [Banchoff, 96] also describe that if we witness the raveling process, seven of eight cubes that compose the tesseract will suddenly disappear, because they have moved in the direction of the fourth dimension. However, they don't provide a methodology that indicates the transformations and their parameters to execute the raveling process. In spite of our physical incapacity, we can visualize a projection onto our 3D space of the cubes on the hypercube's boundary through the unraveling and raveling processes.

3.2.1.2 The Cube's Unraveling Methodology

Although this process is absolutely trivial, it is included here to underline some key points that will be very useful when extending it to the 4D case in section 3.2.1.3.

The unraveling process for a cube can be resumed in the following steps:

1. Identify a face that is "naturally embedded" into the plane where all the cube's faces will be positioned. This face will be called "central face". Because the central face is "naturally embedded" in the selected final plane (for example, the X_1X_2 plane), it will not require any transformation.

2. Identify those faces that share an edge with the central face. There are four of such faces and they will be called "adjacent faces".

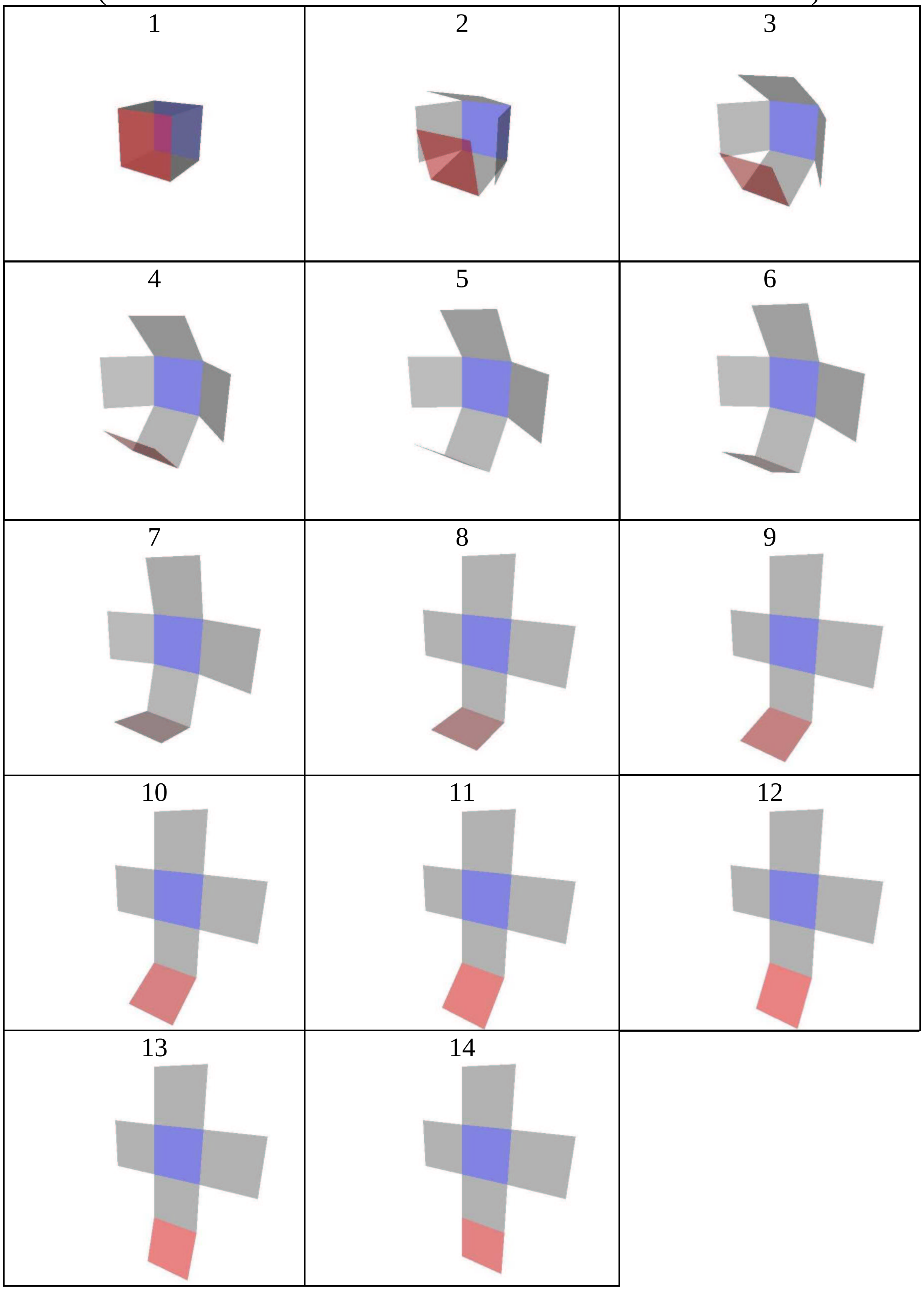
3. After the identification of the central and adjacent faces there will be a last face whose supporting plane is parallel to central face's supporting plane. This face will be called "satellite face" because its movements will be around an edge that is shared with any arbitrary selected adjacent face (and the selected adjacent face will rotate around an edge that is shared with the central face).

4. The adjacent faces will rotate around those edges that share with the central face.

5. When the central, adjacent and satellite faces are identified, it must be determined the rotating angles and their directions. All four adjacent faces will rotate right angles, however two opposite adjacent faces will have opposite rotating directions; otherwise, one of them will end in the same position as the central face.

Table 3.1 presents some snapshots from the cube's unraveling sequence. In snapshots 1 to 7 (except 5), the applied rotations are 0°, ±15°, ±30°, ±45°, ±60° and ±75° (the rotation's sign depends of the adjacent face). In snapshot 5, the applied rotation is ±53°; the satellite face looks like a straight line -an effect due to the selected 3D-2D projection. In snapshot 8, the applied rotation is ±90°; the adjacent faces have finished their movements. In snapshots 9 to 14, the satellite face moves independently and the applied rotations are +15°, +30°, +45°, +60°, +75° and +90°.

TABLE 3.1
Unraveling the cube (taken from [Aguilera & Pérez, 02c])
(the red face is the satellite face and the blue one is the central face).



3.2.1.3 Hypercube's Unraveling Methodology

The process will be easier if we take the following considerations:

- Select the hypercube's position in the 4D space.
- Select the hyperplane (a 3D subspace embedded in the hyperspace) where the volumes will be directed to.
- Establish the angles which guarantee that all volumes will be totally embedded in the selected hyperplane.
- All the volumes through their movement into the selected hyperplane must maintain a face adjacent to another volume.

The hypercube's position in the 4D space is essential, because it will define the rotating planes used by the volumes to be positioned onto a hyperplane. For simplicity, one vertex of the hypercube will coincide with the origin, six of its faces will coincide each one with some of the X_1X_2, X_1X_4, X_2X_3, X_2X_4, X_3X_1 and X_3X_4 planes and all the coordinates will be positive. The coordinates to use are presented in **Table 3.2**.

TABLE 3.2

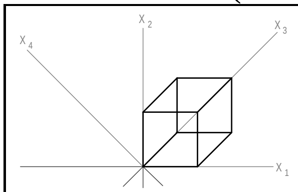
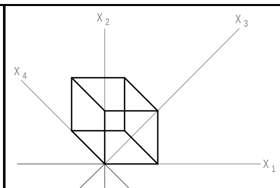
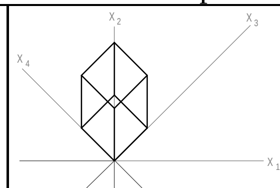
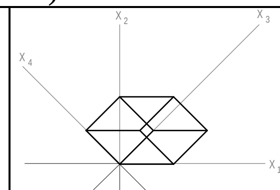
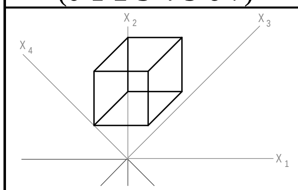
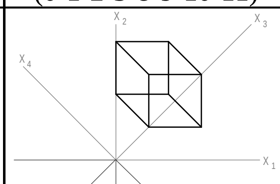
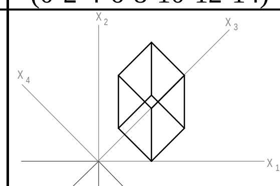
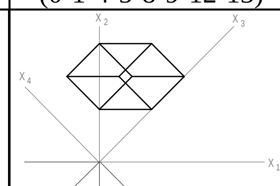
The hypercube's coordinates (Reproduction of Table 2.3).

X_1	X_2	X_3	X_4	***Binary Representation***	***Vertices' decimal representation***
0	0	0	0	0000	0
1	0	0	0	0001	1
0	1	0	0	0010	2
1	1	0	0	0011	3
0	0	1	0	0100	4
1	0	1	0	0101	5
0	1	1	0	0110	6
1	1	1	0	0111	7
0	0	0	1	1000	8
1	0	0	1	1001	9
0	1	0	1	1010	10
1	1	0	1	1011	11
0	0	1	1	1100	12
1	0	1	1	1101	13
0	1	1	1	1110	14
1	1	1	1	1111	15

We know now why the hypercube's position in the 4D space is important, since it will define the rotating planes to use. The situation is the same for the selected hyperplane, because it is where all the volumes will be finally positioned. Observing the hypercube's coordinates we can see that eight of them present their fourth coordinate value (X_4) equal to zero. This fact represents that one of the hypercube's volumes (formed by vertexes 0-1-2-3-4-5-6-7) has X_4=0 as its supporting hyperplane. Selecting the hyperplane X_4=0 is useful because one of the volumes is "naturally embedded" in the 3D space and it won't require any transformations.

Now, it is also useful to identify the hypercube's volumes through their vertices and to label them for future references. Until now we have one identified volume, it is formed by vertexes 0-1-2-3-4-5-6-7, and it will be called volume A. See **Table 3.3**.

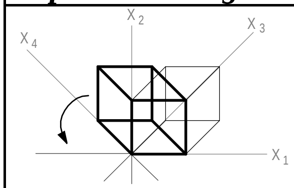
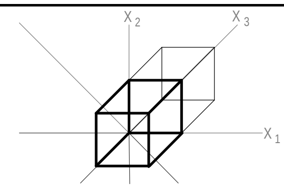
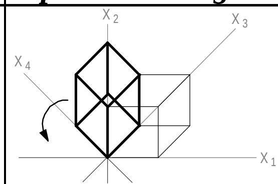
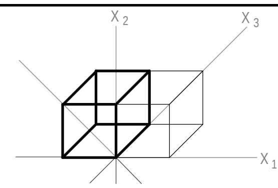
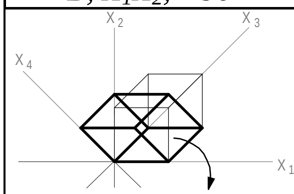
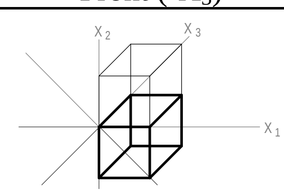
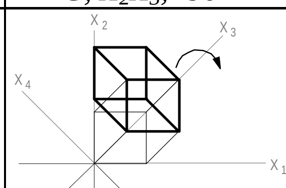
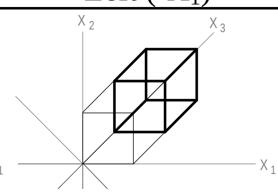
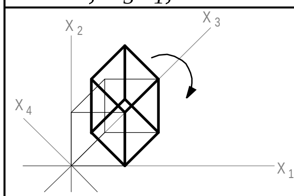
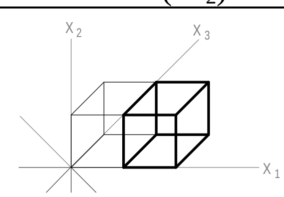
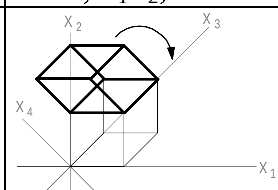
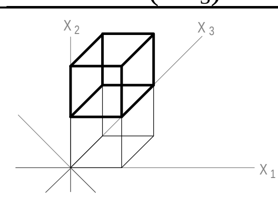
TABLE 3.3
The hypercube's volumes (taken from [Aguilera & Pérez, 01])
(the numbers indicate the vertices that compose them).

			
Volume A (0-1-2-3-4-5-6-7)	Volume B (0-1-2-3-8-9-10-11)	Volume C (0-2-4-6-8-10-12-14)	Volume D (0-1-4-5-8-9-12-13)
			
Volume E (8-9-10-11-12-13-14-15)	Volume F (4-5-6-7-12-13-14-15)	Volume G (1-3-5-7-9-11-13-15)	Volume H (2-3-6-7-10-11-14-15)

We have already described volume A as "naturally embedded" in the 3D space, because it won't require any transformations. Volume A will occupy the central position in the 3D cross and it will be called the "central volume".

From the remaining volumes, six of them will have face adjacency with the central volume. Due to this characteristic they can easily be rotated toward our space because their rotating plane is clearly identified. Each of these volumes will rotate around the supporting plane of its shared face with central volume. They will be called "adjacent volumes". Adjacent volumes are B, C, D, F, G and H. The remaining volume E will be called "satellite volume" and it will be discussed later on.

TABLE 3.4
Applied transformations to the adjacent volumes
(taken from [Aguilera & Pérez, 01]).

Adjacent volume (previous to rotation), rotation plane and angle	***Position in the 3D space and in the tesseract after rotation***	***Adjacent volume (previous to rotation), rotation plane and angle***	***Position in the 3D space and in the tesseract after rotation***
 *B, X_1X_2, +90°*	 Front ($-X_3$)	 *C, X_2X_3, -90°*	 Left ($-X_1$)
 *D, X_3X_1, +90°*	 Down ($-X_2$)	 *F, X_1X_2, -90°*	 Back ($+X_3$)
 *G, X_2X_3, -90°*	 Right ($+X_1$)	 *H, X_3X_1, -90°*	 Up ($+X_2$)

All of the adjacent volumes will rotate right angles. In this way we guarantee that their X_4 coordinate will be equal to zero. As in the 3D case, it is also important to consider their rotating directions, because the volumes, after the rotations, could otherwise coincide with the central volume. The direction and rotating planes for each adjacent volume are presented in **Table 3.4** (the central volume is also included in each image as a reference for the initial and final position of the volume being analyzed).

At this point, we have seven of the eight hypercube's volumes placed in their final positions (volumes A, B, C, D, F, G and H). Volume E will perform a rather more complex set of transformations. There are two reasons that justify this conclusion:

- The supporting hyperplane for volume E is parallel to the supporting hyperplane for the central volume. Consequently, there are no adjacencies between volume E and central volume (this is the reason for not calling "adjacent volume" to volume E).
- In the tesseract, we still have an empty position. This position corresponds to the most distant volume from the central volume (the inferior position, **Figure 3.5**). This position will be occupied by volume E. This is the reason for calling E the "satellite volume".

At the beginning of this section its is mentioned the need for maintaining a face adjacency between all the volumes while they rotate towards the selected hyperplane. Volumes B, C, D, F, G and H share a face with central volume (remember that central volume is static during the whole unraveling process). In order to determine the needed transformations for the satellite volume, we must first select the volume which will share a face with it. Any volume, except the central one, can be selected for this. In this work, volume D will be selected to share a face with satellite volume through the hyper-flattening process.

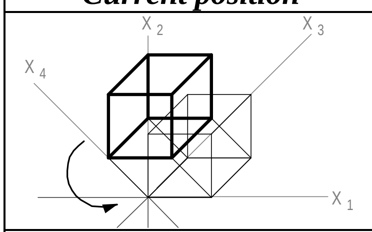
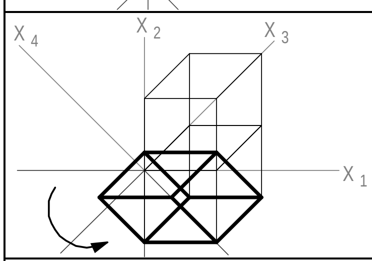
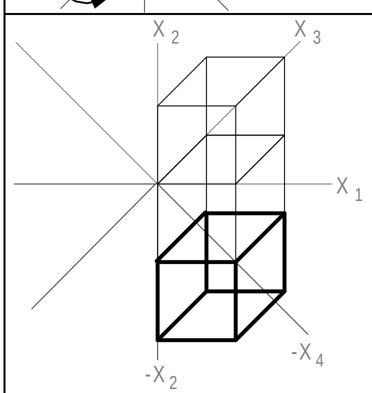
The direction and the rotation plane for volume D was determined before (X_3X_1 plane +90°). These transformations will take it to its final position. During the beginning of the unraveling process, the same transformations will be applied to satellite volume. In this way, we ensure that volumes E and satellite will share a face at all times.

When volume D has finished its movement, it will be placed in its final position in the tesseract. At this moment, the satellite volume's supporting hyperplane will be perpendicular to the selected hyperplane and the shared face will be parallel to X_3X_1 plane. The last movement to apply to the satellite volume will be a +90° rotation around the supporting plane of the shared face with volume D.

The set of movements to be executed for the satellite volume are resumed in the **Table 3.5** (Central volume and volume D are shown too).

Now, all the transformations to unravel the hypercube have been determined. To ravel it back, the same process must be applied in an inverse way (the angles' signs must be changed). This methodology was originally presented in [Aguilera & Pérez, 01].

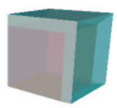
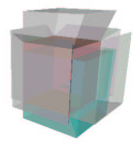
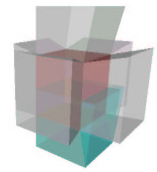
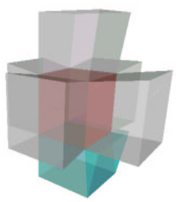
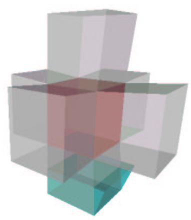
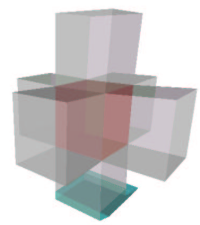
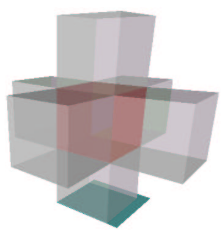
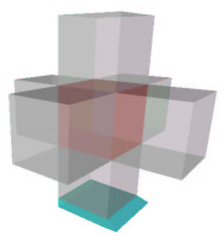
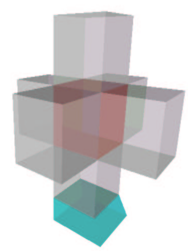
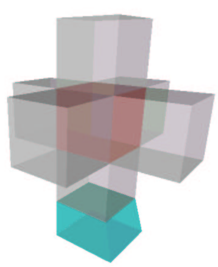
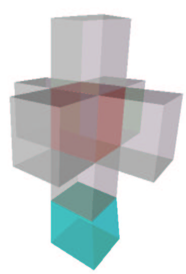
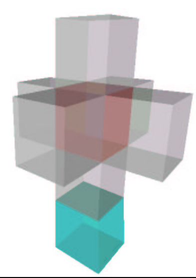
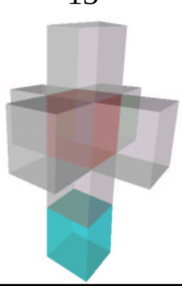
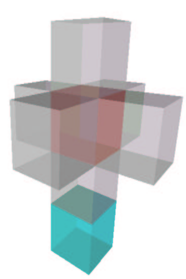
TABLE 3.5
Associated transformations to satellite volume
(taken from [Aguilera & Pérez, 01]).

Current position	***Transformations***
	Rotation of volumes D and satellite around the plane X_3X_1 (+90°).
	Volume D is in its final position. Rotation of sate-llite volume of +90° around the shared face with volume D (parallel plane to X_3X_1).
	Satellite volume in its final position (inferior position in the 3D cross on $–X_2$ axis).

3.2.1.4 Visualizing the Hypercube's Unraveling Process

Table 3.6 presents some snapshots from the hypercube's unraveling sequence. In snapshots 1 to 6, the applied rotations are ±0°, ±15°, ±30°, ±45°, ±60° and ±75° (the rotation’s sign depends of the adjacent volume). In snapshot 7, the applied rotation is ±82°; the satellite volume looks like a plane -an effect due to the selected 4D-3D projection. In snapshot 8, the applied rotation is ±90°; the adjacent volumes finish their movements. In snapshots 9 to 14, the satellite volume moves independently and the applied rotations are +15°, +30°, +45°, +60°, +75° and +90°.

TABLE 3.6
Unraveling the hypercube (taken from [Aguilera & Pérez, 02])
(satellite volume is shown in blue and central volume in red).

1 	2 	3 
4 	5 	6 
7 	8 	9 
10 	11 	12 
13 	14 	

3.2.1.5 The n-Dimensional Hyper-Tesseract

The properties of the unravelings for the parallelotopes in 2D, 3D and 4D space can be resumed in the following way (see **Table 3.7**):

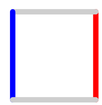
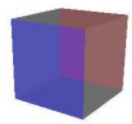
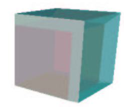
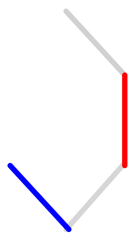
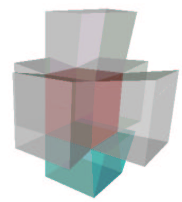
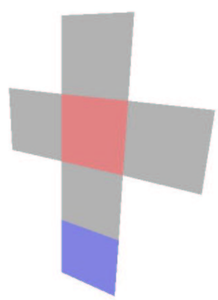
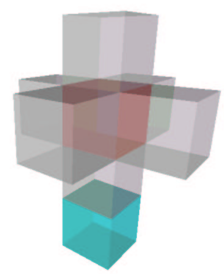
- Square: A central segment surrounded by other two through a vertex adjacency with each one; a satellite segment adjacent to any other of the segments except the central. Completely immersed in a 1D space (a straight line).

- Cube: A central face surrounded by other four through an edge adjacency with each one; a satellite face adjacent to any other of the faces except the central. Completely immersed in a 2D space (a plane).

- Hypercube: A central volume surrounded by other six through a face adjacency with each one; a satellite volume adjacent to any other of the volumes except the central. Completely immersed in a 3D space (a hyperplane).

TABLE 3.7

Analogies between the unravelings of the square, the cube and the hypercube
(the central cell is shown in red and the satellite cell is shown in blue; own elaboration).

	2D Space Square	**3D Space Cube**	**4D Space Hypercube**
Parallelotope			
Unraveling process			
Unravellings			

Observing the unravelings for the square (C_2), the cube (C_3) and the 4D hypercube (C_4) and the fact a nD parallelotopes-family share analogous properties, [Aguilera & Pérez,02] generalize the ***n-dimensional hyper-tesseract*** ($n \geq 1$) as the result of the (n+1)-dimensional parallelotope's unraveling with the following properties:

- The (n+1)-dimensional hypercube will have 2(n+1) n-dimensional cells on its boundary.
- A central cell will be static during the unraveling/raveling process.

- 2(n+1)-2 cells are adjacent to central cell. All of them will share a (n-1)-dimensional cell with central cell.
- A satellite cell won't be adjacent to central cell because their supporting hyperplanes are parallel. It will be adjacent to any of the adjacent cells (it will share a (n-1)-dimensional cell with the selected adjacent cell).
- All the adjacent cells and satellite cell during the unraveling/raveling process will rotate ±90° around the supporting hyperplane of the (n-1)-dimensional shared cells.

Then, for n = 4 we have the 4D hyper-tesseract as the result of the 5D hypercube's unraveling. The 4D hyper-tesseract will be composed by 10 hypervolumes, where one of them will be the central hypervolume (static), eight of them are adjacent to central hypervolume (they share a volume) and the last one will be the satellite hypervolume (it shares a volume with any of the adjacent hypervolumes). See **Figure 3.7**. The adjacent hypervolumes and the satellite hypervolume will rotate around a volume or a hyperplane during the unraveling/raveling process.

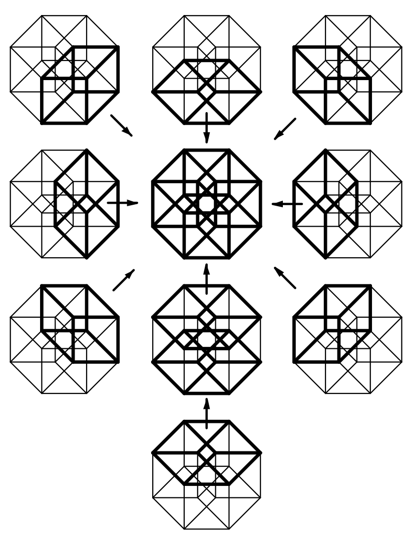


FIGURE 3.7
The adjacency relations between the 4D *hyper-tesseract*'s hypervolume's
(taken from [Aguilera & Pérez, 02]).

3.2.2 Unraveling the 4D Simplex

3.2.2.1 Introduction

In the previous sections was presented the methodology used by [Aguilera & Pérez,01] for unraveling the 4D hypercube. Such method is based in the cube's unraveling process for obtaining the analogous one. The idea is reconsidered again for determining the unraveling process for other polytopes, as the 4D Simplex (which was analyzed in chapter 2), **Figure 3.8**. That means that the tetrahedron's unraveling process will be first analyzed, and through it, a process for unraveling the 4D simplex will be proposed. As the hypercube's unraveling process, we will visualize a projection onto our 3D space of the volumes (tetrahedrons) on the 4D simplex's boundary through its unraveling and raveling processes.

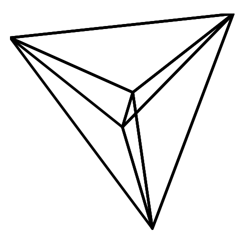


FIGURE 3.8
The 4D simplex (taken from [Aguilera & Pérez, 02c]).

3.2.2.2 The 3D Simplex (Tetrahedron) Unraveling Methodology

Although the tetrahedron's unraveling process is trivial, we will consider here some key points that will be extended later in the 4D simplex unraveling:

1 Identify a face that is "naturally embedded" into the plane where all the tetrahedron's faces will be positioned. This face will be called "central face". Because the central face is "naturally embedded" in the selected plane, it will not require any transformation.

2 Each of the remaining faces shares an edge with the central face. These faces will be called "adjacent faces".

3 The adjacent faces will rotate around those edges that share with the central face.

4 When the central and adjacent faces are identified, it must be determined the rotating angles and their directions. The rotating angle is the supplement of the tetrahedron's dihedral angle. Finally, the tetrahedron’s unravelings will compose a ***stellated triangle***.

TABLE 3.8
Unraveling the 3D simplex (taken from [Aguilera & Pérez, 02c]).

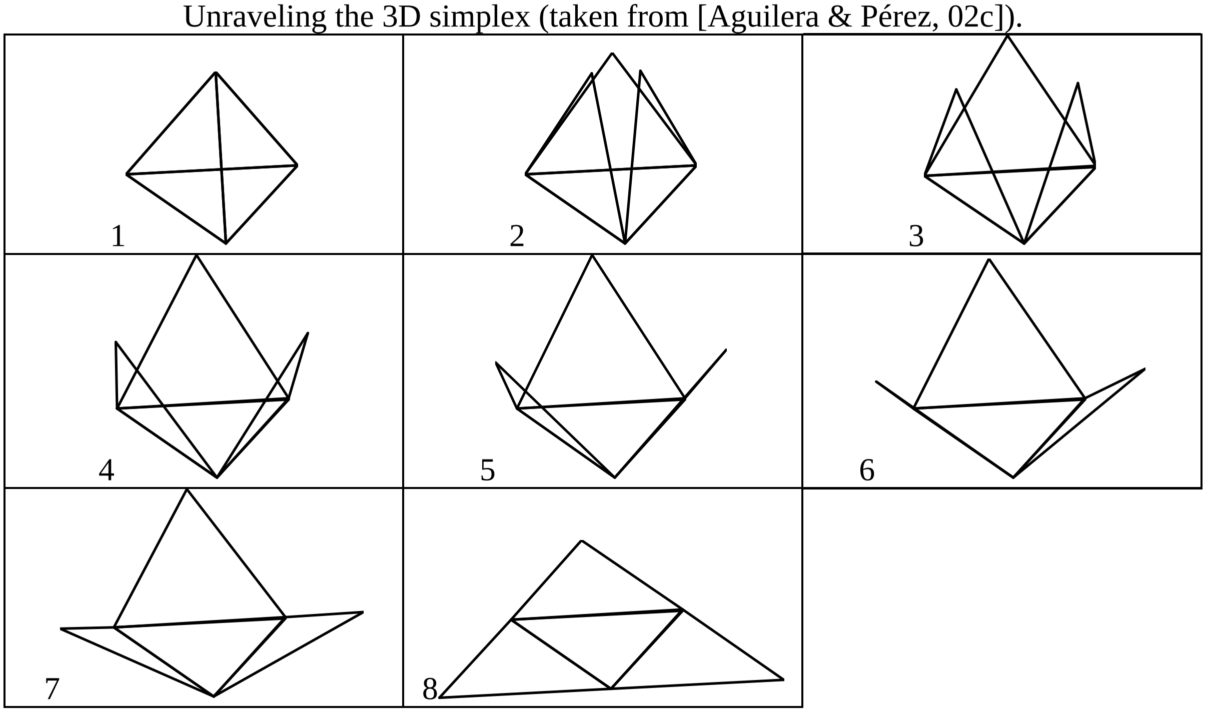


Table 3.8 presents some snapshots from the 3D simplex's unraveling sequence. In snapshots 1 to 4, the applied rotations are ±0, ±10.94°, ±27.35° and ±43.76° (the rotation's

sign depends of the adjacent face). In snapshots 5 and 6, the applied rotations are ±54.7° and ±65.64°; in each snapshot one adjacent face looks like a straight line -an effect due to the selected 3D-2D projection. In snapshots 7 to 8, the applied rotations are ±76.58 and ±109.4.

3.2.2.3 The 4D Simplex's Unraveling Methodology

Because the 4D simplex boundary is composed by five tetrahedrons [Coxeter, 63], we can expect, by analogy, that the unravelings of the 4D simplex will be a tetrahedron surrounded by four other tetrahedrons and sharing a face with each one (the unravelings of the tetrahedron are a triangle surrounded by other three triangles and sharing an edge with each one). Aguilera and Pérez refer to the unravelings of the 4D simplex as a ***stellated tetrahedron*** (as the unravelings of the hypercube are referred as the tesseract) [Aguilera & Pérez, 02c].

We will consider and adapt the same recommendations proposed by [Aguilera & Pérez, 01] to unraveling the simplex:

- Select the simplex's position in the 4D space.
- Select the hyperplane (a 3D subspace embedded in the hyperspace) where the volumes will be directed to.
- Establish the angles which guarantee that all volumes will be totally embedded in the selected hyperplane.
- All the volumes through their movement into the selected hyperplane must maintain a face adjacent to another volume.

We consider that the simplex will have a position with the following characteristics:

- One vertex of the simplex will be the origin.
- An edge will coincide with X_1 axis.
- A face will coincide with X_1X_2 plane.
- All the coordinates will be positive.

The coordinates to use are presented in **Table 3.9**.

TABLE 3.9

The 4D simplex coordinates (taken from [Aguilera & Pérez, 02c]).

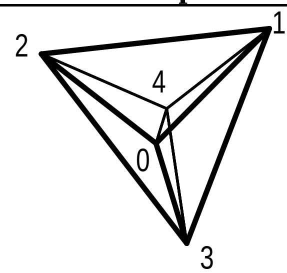
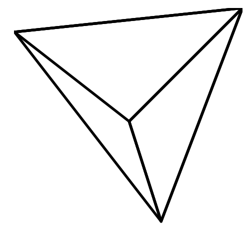
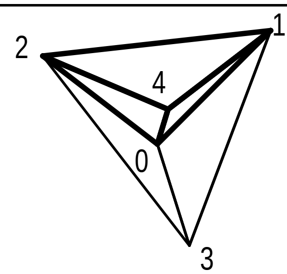
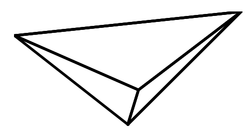
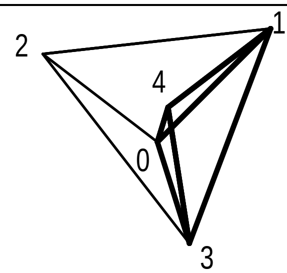
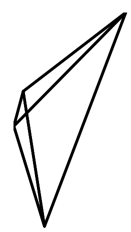
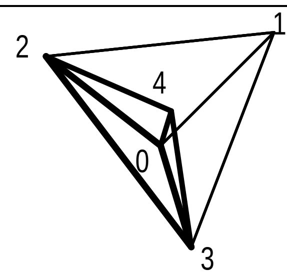
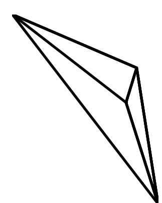
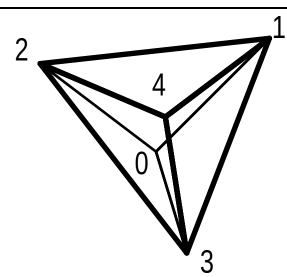
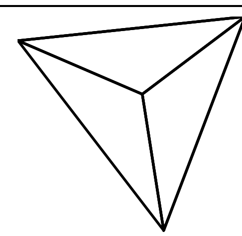
Vertex	X_1	X_2	X_3	X_4
0	0	0	0	0
1	1	0	0	0
2	$1/2$	$\sqrt{3}/2$	0	0
3	$1/2$	$\sqrt{3}/6$	$\sqrt{2/3}$	0
4	$1/2$	$\sqrt{3}/6$	$\sqrt{2}/4\sqrt{3}$	$\sqrt{5/8}$

Observing the 4D simplex's coordinates we can see that four of them present their fourth coordinate value (X_4) equal to zero. This fact represents that one of the simplex's volumes (formed by vertexes 0-1-2-3) has $X_4=0$ as its supporting hyperplane. Selecting the hyperplane $X_4=0$ is useful because one of the volumes is "naturally embedded" in the 3D space and it won't require any transformations.

Now, it is also useful to identify the simplex's volumes through their vertices and to label them for future references. Until now we have one identified volume, it is formed by vertexes 0-1-2-3, and it will be called volume A. See **Table 3.10**.

TABLE 3.10

The 4D simplex's boundary volumes (taken from [Aguilera & Pérez, 02c]).

Volume's label and vertices	Volume's position in the 4D simplex	Volume
Volume A 0-1-2-3		
Volume B 0-1-2-4		
Volume C 0-1-3-4		
Volume D 0-2-3-4		
Volume E 1-2-3-4		

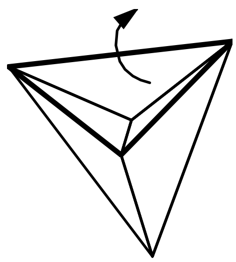
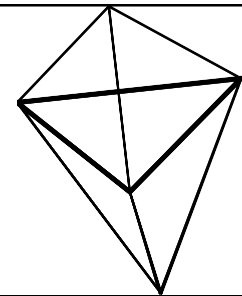
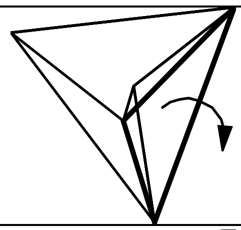
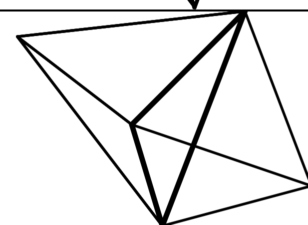
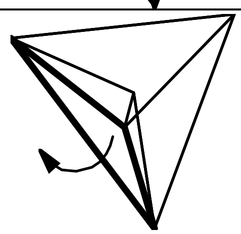
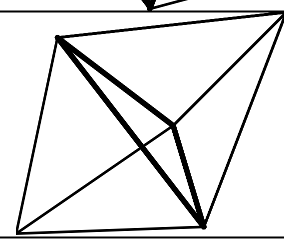
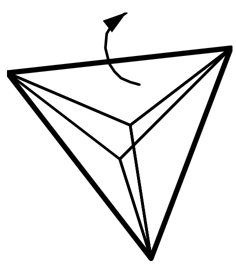
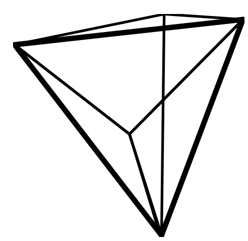
We have already described volume A as "naturally embedded" in the 3D space, because it won't require any transformations. Volume A will occupy the central position in the stellated tetrahedron and it will be called the "central volume".

All of the remaining volumes will have face adjacency with the central volume. Due to this characteristic they can "easily" be rotated toward our space because their rotating plane is clearly identified. Each of these volumes will rotate around the supporting plane of its shared face with central volume. They will be called "adjacent volumes".

Although the rotating planes are clearly identified, the main difference between the hypercube and simplex's unraveling is that the rotating planes don't correspond to 4D space main planes (X_1X_2, X_2X_3, X_3X_1, X_1X_4, X_2X_4 and X_3X_4) in the simplex's unraveling. Due to this situation, the volume's rotations will be a composition of rotations around the 4D space main planes. The objective taken for us was to position a volume's face in the X_1X_2 plane, and then rotate it 104° 29'. This angle corresponds to the supplement of the simplex's dihedral angle that is 75° 31' [Coxeter, 63]. In this way we guarantee that their X_4 coordinate will be equal to zero. The direction and rotating planes for each adjacent volume are presented in **Table 3.11** (the central volume is also included in each image as a reference for the initial and final position of the volume being analyzed).

TABLE 3.11

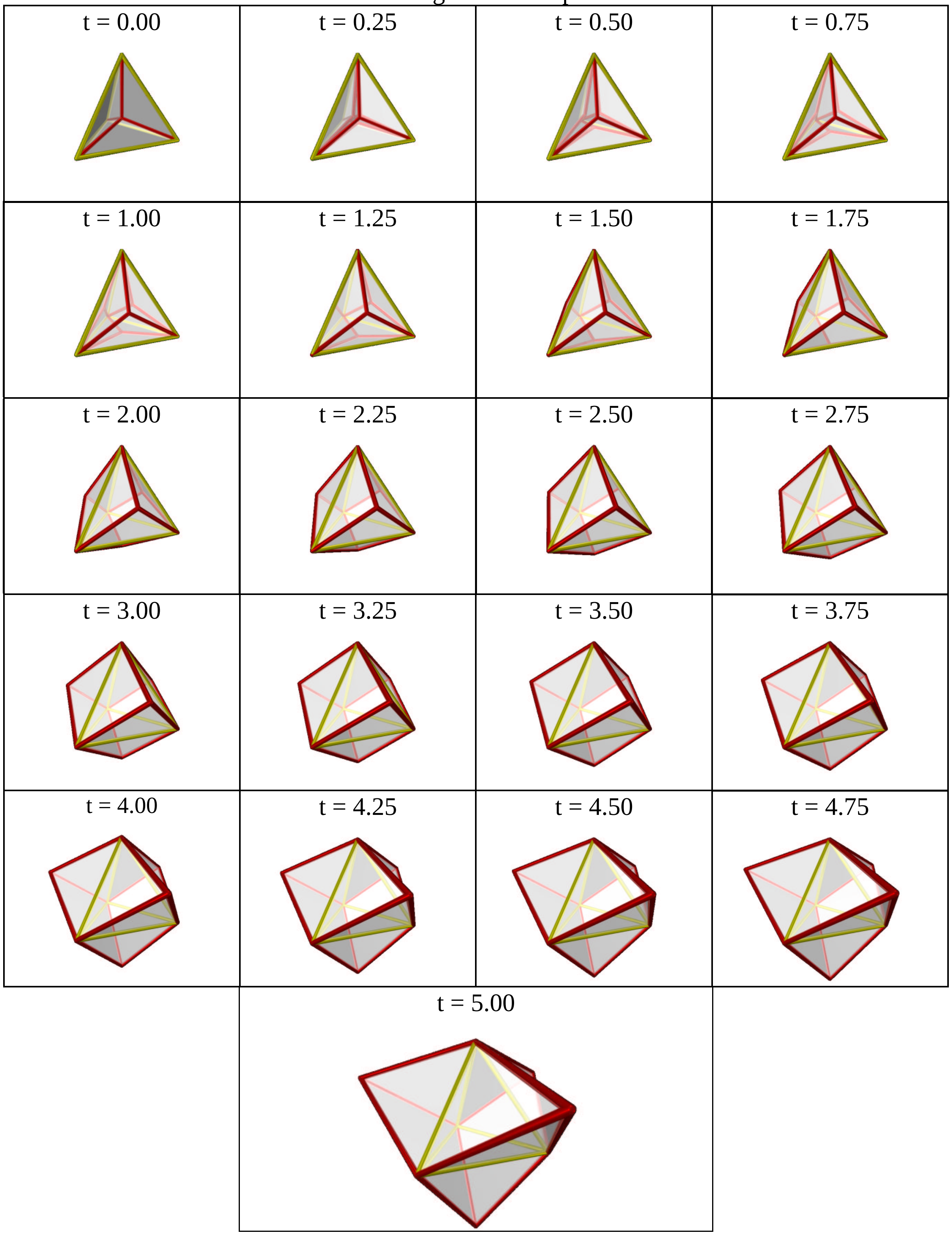
Applied transformations to the adjacent volumes. Rotation around X_1X_2 plane is the same for all volumes (taken from [Aguilera & Pérez, 02c]).

Adjacent volume previous to rotation	**Transformations**	**Position in the stellated tetrahedron after the transformations**
	Volume B **X_1X_2 104° 29'**	
	Volume C X_1X_4 109° 30' **X_1X_2 104° 29'** X_1X_4 -109° 30'	
	Volume D X_3X_4 -60° X_1X_4 70° 30' **X_1X_2 -104° 29'** X_1X_4 -70° 30' X_3X_4 60°	
	Volume E T(-1,0,0,0) X_3X_4 60° X_1X_4 70° 30' **X_1X_2 -104° 29'** X_1X_4 -70° 30' X_3X_4 -60° T(1,0,0,0)	

Now, all the transformations to unravel the simplex have been determined. To ravel it back, the same process must be applied in an inverse way but only the angles' signs for rotations around X_1X_2 plane must be changed, because the remaining rotations only position the volumes with a face on X_1X_2 plane.

TABLE 3.12

Unraveling the 4D simplex[1].



[1] This sequence of images was originally rendered and kindly provided by this thesis' advisor. The original wireframe model based sequence can be consulted in [Aguilera & Pérez, 02c] (see the appendix E).

3.2.2.4 Visualizing The 4D Simplex's Unraveling Process

Table 3.12 presents some snapshots from the 4D simplex's unraveling sequence. From t = 0.00 until t = 0.75, the adjacent volumes (in red) are projected inside the central tetrahedron (in yellow). When t = 1.00, adjacent volumes are projected on the central tetrahedron’s faces (they look like planes) -an effect due to the selected 4D-3D projection. From t = 1.25 until t = 5.00, the adjacent volumes are projected outside the central tetrahedron. When t = 3.00 an interesting phenomenon arises, the projected volumes form an hexadron (a cube) –again, an effect due to the selected projection. When t = 5.00 the stellated tetrahedron is finally composed.

3.2.2.5 The Stellated n-Dimensional Simplex

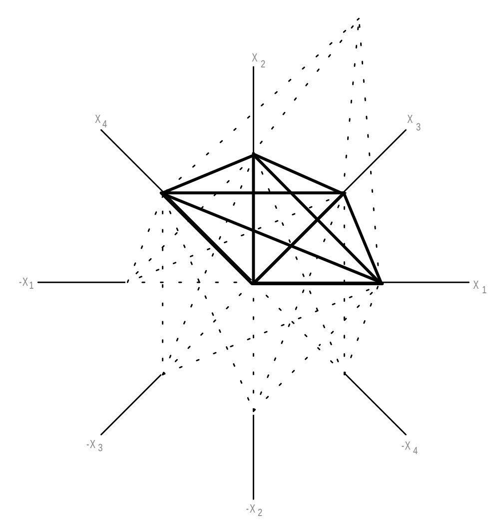
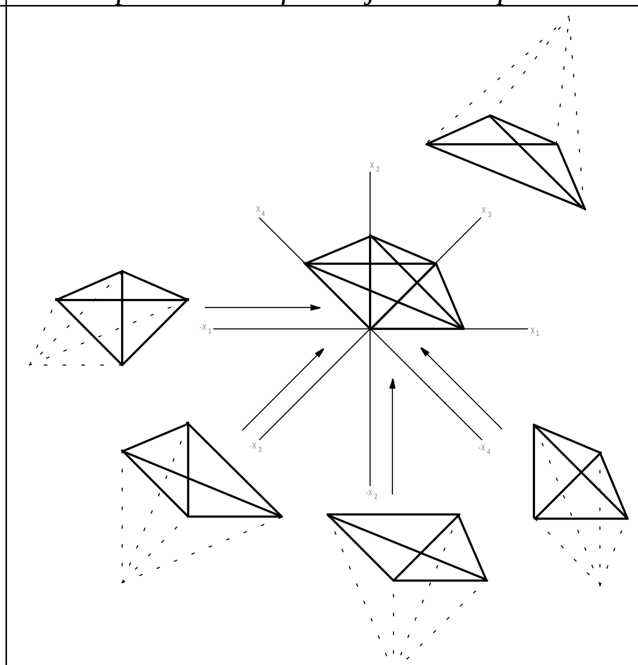
Analyzing the unravelings for the triangle (a 2D simplex), the tetrahedron (a 3D simplex) and the 4D simplex and the fact a nD simplexes-family share analogous properties [Coxeter, 63], [Aguilera & Pérez, 02c] generalize the ***stellated n-dimensional simplex*** (n≥1) as the result of the (n+1)-dimensional simplex's unraveling with the following properties:

- The (n+1)-dimensional simplex will have (n+2) n-dimensional cells on its boundary.
- A central cell will be static during the unraveling/raveling process.
- (n+1) cells are adjacent to central cell. All of them will share a (n-1)-dimensional cell with central cell.

- All the adjacent cells during the unraveling/raveling process will rotate the supplement of the simplex's dihedral angle around the supporting hyperplane of the (n-1)-dimensional shared cells.

For example, the unraveling process for the 5D simplex will generate a *stellated 4D simplex* (**Table 3.13.a**) which will be composed by six 4D simplexes, one of them will be the central 4D simplex (with continuous lines in the figure) and the remaining 5, the adjacent cells, will share a tetrahedron with it (**Table 3.13.b**. Due to the selected projection, is that the 4D simplexes look as distorted).

TABLE 3.13
The unravelings for the 5D simplex (see text for details).

a) The stellated 4D simplex.	*b) The adjacencies between the central 4D simplex and the five adjacent simplexes.*
	

3.3 Polytope’s Intersection with Three-Dimensional Space

As introduced in chapter 1, the intersections were the method used by Abbott in *Flatland* to describe the interactions between two and three-dimensional spaces

[Banchoff,96]. Furthermore, the conic sections were presented as a daily example about the use of the intersections between an object, in that case a cone, with a plane. Section 3.3.1 will treat the aspects related to the 4D hypercube's intersections with our 3D space. In section 3.3.2, the [Rucker, 77]'s method for visualizing the 4D hypersphere will be discussed.

3.3.1 The Intersections Between a 4D Hypercube and the 3D Space

[Banchoff, 96] identifies three of the most important ways to intersect a cube with a 2D space: (1) when one if its faces is parallel to *Flatland*, (2) when one of its edges is parallel to *Flatland*, and (3) when the cube's main diagonal coincides with the *Flatland*'s normal vector.

In the first case (**Figure 3.9.a**), and while the intersection takes place, a *Flatlander* will only appreciate a square whose size or shape don't change (**Figure 3.9.b**).

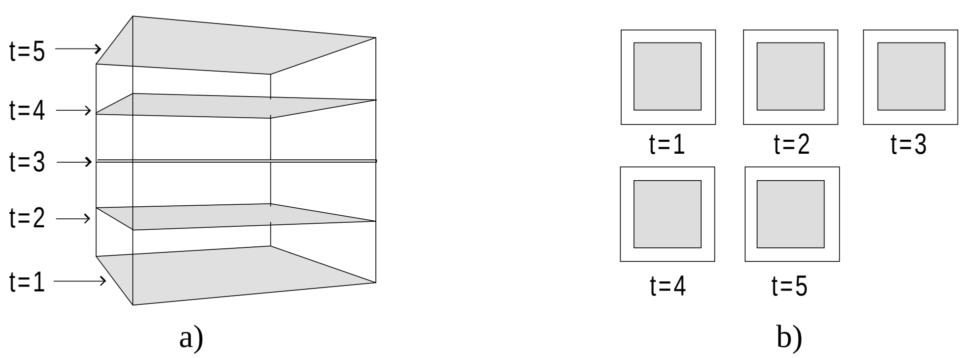
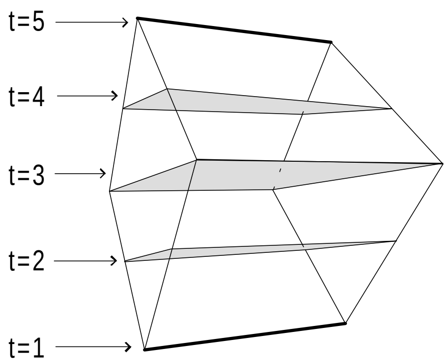
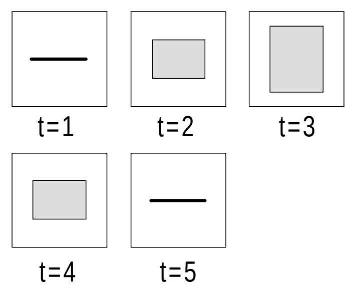


FIGURE 3.9
Intersections between a cube and Flatland when one of its faces is parallel to the 2D space. a) Three-Dimensional Space's View. b) *Flatland*'s View (own elaboration).

The second case, when an edge is parallel to *Flatland* (**Figure 3.10.a**), provides a more interesting visualization. Because at the beginning, the *Flatlander* will visualize in first place a segment (the edge that is parallel to his 2D space, **Figure 3.10.b**, t=1) which will become a rectangle whose two of its parallel edges, through all the process, will have the first visualized segment's length, while the remaining two will be minor and perpendicular to it (**Figure 3.10.b**, t=2). However, in the middle of the process, these two edges will be greater than the two edges whose length is constant (**Figure 3.10.b**, t=3). Starting from this moment, the process is inverted taking place the reduction of the visualized rectangle (**Figure 3.10.b**, t=4), until finally, again is visualized an edge (**Figure 3.10.b**, t=5).



a)



b)

FIGURE 3.10

Intersections between a cube and Flatland when one of its edges is parallel to the 2D space. a) Three-Dimensional Space's View. b) Flatland's View (own elaboration).

The third case, when the cube's main diagonal coincides with the 2D space's normal vector (**Figure 3.11**), results to be one of the most interesting. In first place a point is visualized: one of the cube's vertices that compose its main diagonal's boundary (**Figure 3.12**, t=0.00). In the following instants, for t=0.01 until t=1.00, a triangle will be visualized, whose size will increase, and its vertices belong to the three edges that are incident to the vertex visualized when t=0.00. When t=1.10 and until t=1.40 (**Figure 3.12**), the intersection between the cube and *Flatland* will generate an irregular hexagon whose vertices belong exclusively to those six edges that are not incident to the main diagonal's vertices. When t=1.50, a regular hexagon will be visualized, whose six vertices are the middle points of those six edges not incident to the main diagonal's vertices. From t=1.60 until t=1.90 an irregular hexagon will be visualized again. Finally, in t=2.00 until t=2.80 (**Figure 3.12**) a triangle will be visualized and its vertices belong to the three edges that are incident to the second vertex that defines the cube's main diagonal, which will be visualized when t=2.90 (**Figure 3.12**).

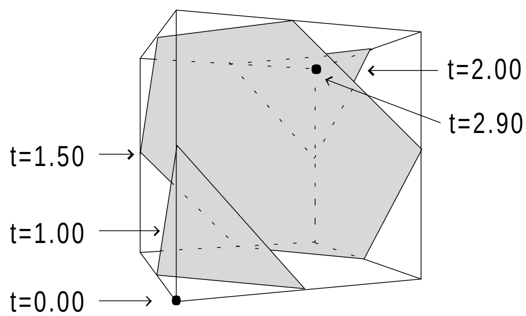


FIGURE 3.11
Intersections between a cube and *Flatland* when its main diagonal coincides with the 2D space's normal vector (3D Space's View. Own elaboration).

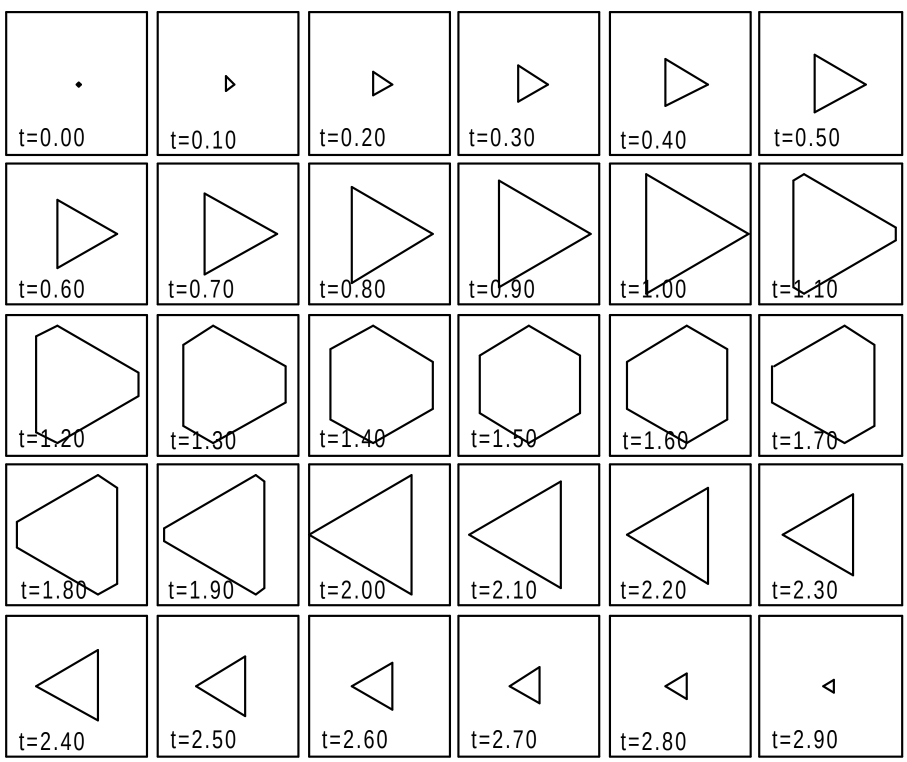


FIGURE 3.12
Visualizing in Flatland its intersections with a cube whose main diagonal coincides with 2D space's normal vector (taken from [Aichholzer, 97]).

[Banchoff, 96] points out the existence of four important ways to intersect a 4D hypercube through the first cell that makes first contact with our space: (1) when a volume (cube), a face (2), an edge (3) or a vertex (4) starts the contact with three-dimensional space.

In the first case, when a volume starts the contact with 3D space, it is only visualized, in all the instants where the intersection takes place, a cube (**Figure 3.13**) [Banchoff, 96]. This situation is analogous to the intersections between a cube and *Flatland* when one of its faces is parallel to 2D space (**Figure 3.9.b**).

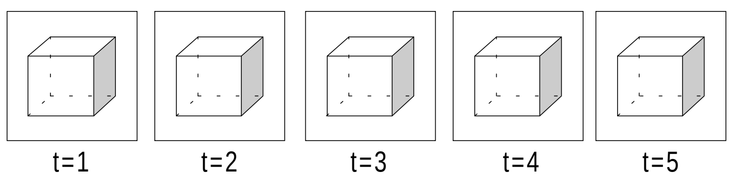


FIGURE 3.13

Visualizing the intersections between a 4D hypercube with 3D space: the first element that makes contact with 3D space is a volume (own elaboration based in an illustration presented in [Banchoff, 96]).

In the second case, in first place a face will be visualized which starts the contact with the 3D space (**Figure 3.14**, t=1). The face expands in a series of rectangular prisms whose bases are all equal to the first visualized face (t=2). The height of those prisms increases until it has the same length of the main diagonal of a hypercube's face [Banchoff,96] (t=3). Later on, the prisms' height starts to decrease (t=4) until it is zero (t=5).

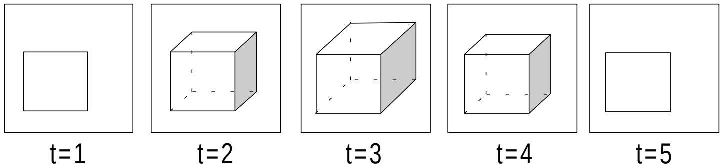


FIGURE 3.14
Visualizing the intersections between a 4D hypercube with 3D space: the first element that makes contact with 3D space is a face (own elaboration based in an illustration presented in [Banchoff, 96]).

When an edge starts the contact with 3D space (**Figure 3.15**) it will be visualized an edge (t=1) that expands until compose a triangular prism (t=2,3) whose height is equal to the main edge's length [Banchoff, 96]. The triangular prism then becomes a prism with hexagonal base (t=4). The process is then inverted, the prism becomes a triangular one (t=5,6) and finally to be a segment (t=7).

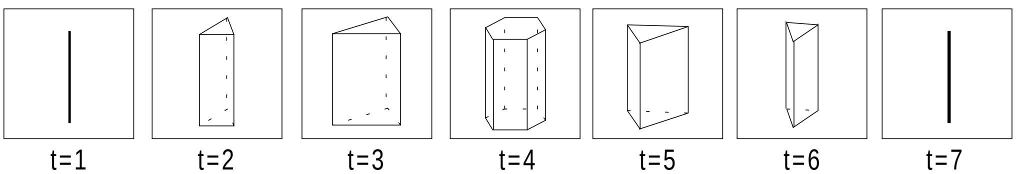


FIGURE 3.15
Visualizing the intersections between a 4D hypercube with 3D space: the first element that makes contact with 3D space is an edge (own elaboration based in an illustration presented in [Banchoff, 96]).

The fourth sequence of intersections is obtained by moving the 4D hypercube along the normal vector of the 3D space (a hyperplane) [Aichholzer, 97], in such way that the 4D hypercube's main diagonal coincides with the normal vector. In this way, the first element which makes contact with our space will be a vertex (**Figure 3.16**, t=0.00). This vertex will expand to compose a tetrahedron (t=0.20 until t=1.00). Later on, the tetrahedron will start to experiment a truncation's process in their corners, which induces the visualization of a polyhedron with eight faces, where four of them are triangular and the remaining are hexagonal [Banchoff, 96] (t=1.20 until t=1.80). Finally, the four hexagonal faces become triangular (t=2.00) taking place a regular octahedron's visualization ([Aichholzer, 97] & [Banchoff, 96]). The process' second half is the reversal respect the first half. The four triangular faces become again hexagonal (t=2.20 until t=2.80), it is visualized a tetrahedrons' sequence whose size decreases (t=3.00 until t=3.80); and finally, the second vertex that compose the hypercube's main diagonal is the last one to be visualized.

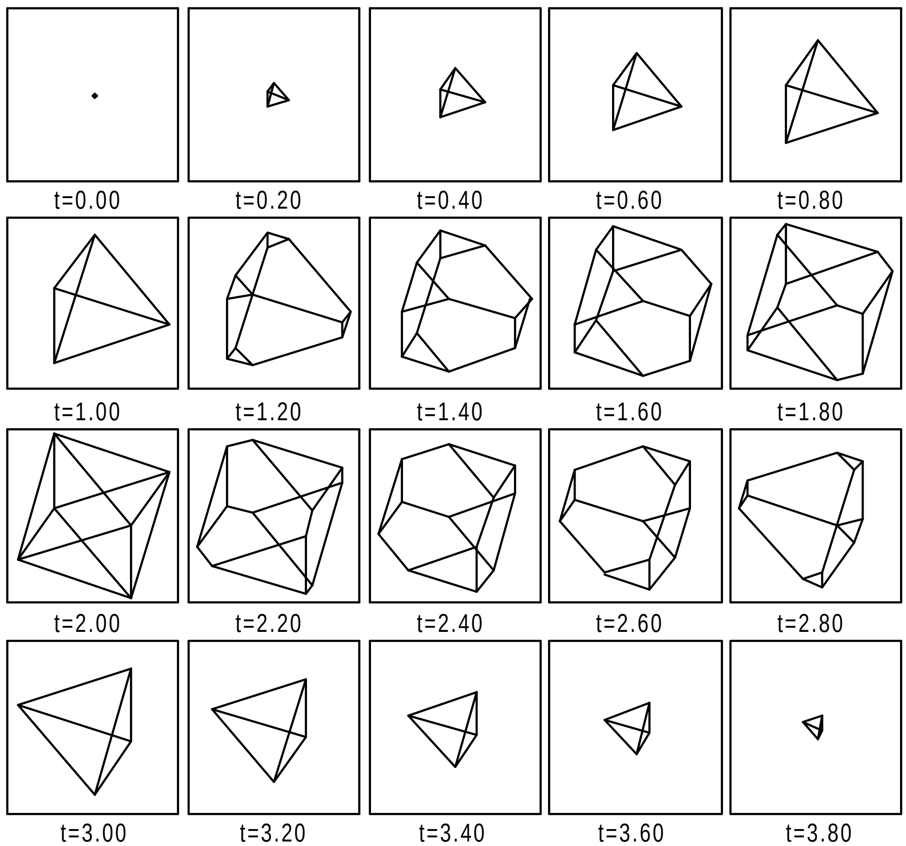


FIGURE 3.16
Visualizing the intersections between a 4D hypercube with 3D space: the first element that makes contact with 3D space is a vertex (taken from [Aichholzer, 97]).

3.3.2 Visualizing the 4D Hypersphere

In section 1.3 it was described an example of the intersection between a sphere and a plane (the *A.Sphere & Flatland's* relation). Basically, the sphere is visualized by *A.Square* as a point which, through the time, becomes a circumference whose diameter increases. Later on, the circumference starts to decrease its size to become, again, a point (**Figure 3.17**).

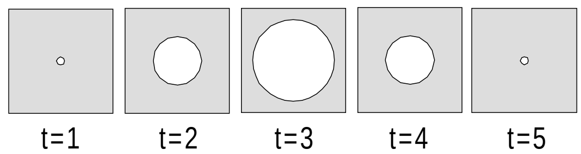


FIGURE 3.17
Visualizing the sphere's intersections with *Flatland*
(own elaboration).

As illustrated in section 1.3, the situation is analogous to the intersection between a 4D hypersphere and our three-dimensional space. In the first instant, a point would appear, which through the time, will be visualized as a sphere that increases its size. Later on, the sphere starts to decrease to finally become, again, a point (**Figure 3.18**).

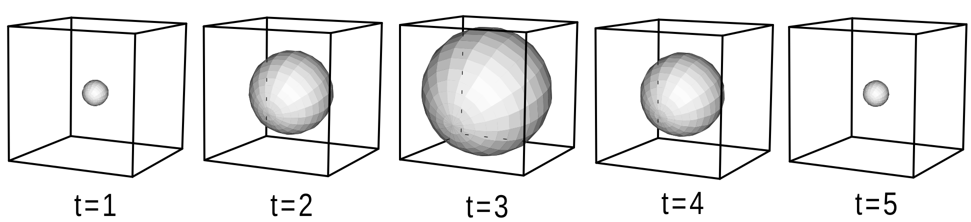
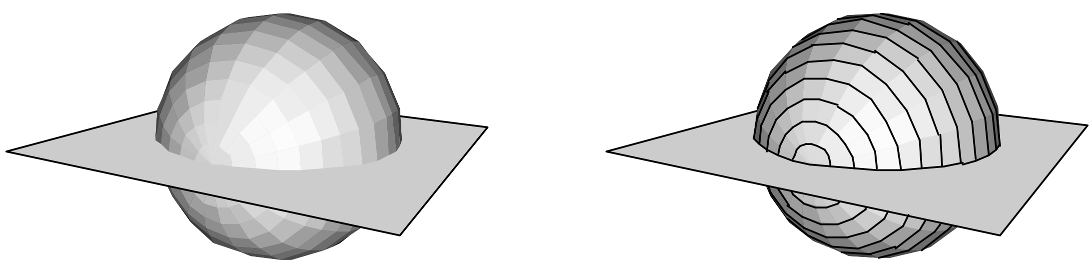


FIGURE 3.18
Visualizing the intersections between a 4D hypersphere and the 3D space
(own elaboration).

[Rucker, 77] points out that a sphere's surface can be considered as a set with an infinite number of circumferences. The method presented in **Figure 3.17** only shows one of these circumferences in turn, in fact, when t = 3, the circumference with the greatest diameter is shown. In analogous way, due to the 4D hypersphere's hypersuface will be composed by an infinite number of spheres, its intersection with our 3D space will show only one sphere at the time. However, [Banchoff, 96] points out that visualizing the sphere and 4D hypersphere by these ways don't provide more information about their boundary. It does not matter if they are rotated, because the visualized sequence is the same: just one circle/sphere increasing and decreasing its size.

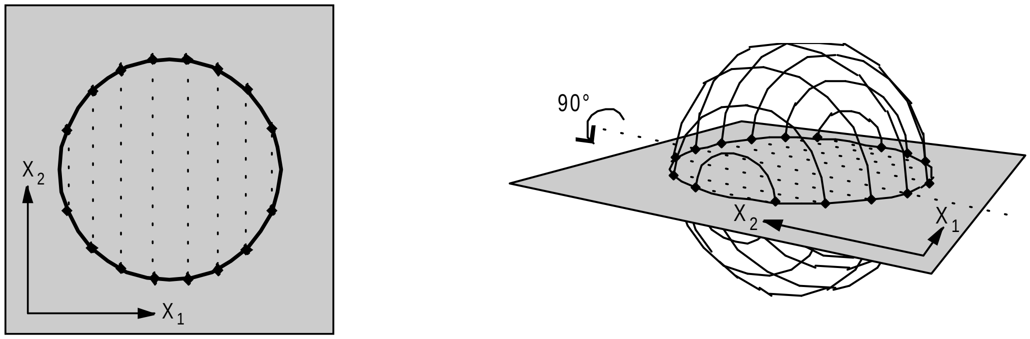
[Rucker, 77] presents a method by means of which it is possible to visualize a greater number of spheres that compose the 4D hypersphere's hypersurface. In first place, we will describe such method for visualizing a sphere in *Flatland*. It was before mentioned that the sphere's surface can be considered as composed by an infinite number of circumferences, because of that, we will consider the sphere's intersection with *Flatland* so that the circumference with the greatest diameter must be embedded in the 2D plane (**Figure 3.19.a**). From the set of circumferences only will be considered those that are perpendicular to Flatland (**Figure 3.19.b**).



a) b)

FIGURE 3.19
A sphere's intersection with Flatland and considering some circumferences on its surface (own elaboration).

In *Flatland*, only the circumference embedded in the plane and two points for each selected circumference will be visualized (**Figure 3.20.a**). Those two points are the result of the intersection between a perpendicular circumference and *Flatland* (**Figure 3.20.b**).



a) b)

FIGURE 3.20
Visualizing some surface's circumferences on a sphere from *Flatland* and from the 3D space (own elaboration).

The pair of points for each perpendicular circumference to *Flatland* describe a straight line in the 2D space, or a rotation's axis in the 3D space (**Figures 3.20.a and b**). Each circumference can be rotated $\pm 90°$ around the axis described by its two intersection's points. In this way, now, those circumferences will coincide with the plane, and all of them will be observable, together with the originally embedded circumference, by a two-dimensional being (**Figure 3.21**).

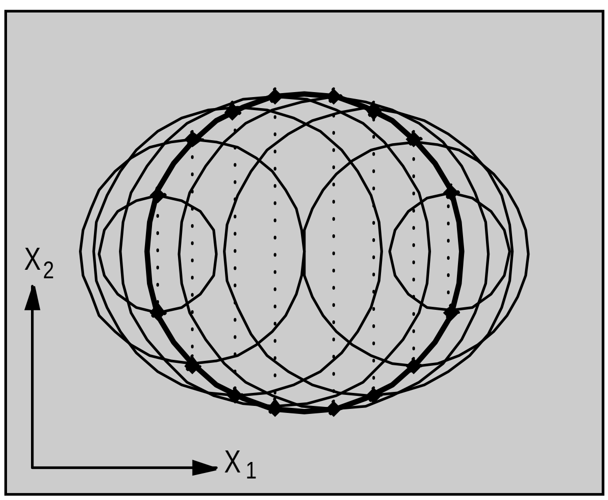


FIGURE 3.21
Visualizing the circumferences, now embedded in *Flatland*, that compose the 3D sphere (own elaboration).

Now, we will show [Rucker, 77]'s method for visualizing the 4D hypersphere. It will be considered the hypersphere's intersection with our 3D space in such way that the sphere with the greatest volume will be the first embedded (for example, in **Figure 3.18**, t = 3). It is known that the hypersphere's hypersurface is composed by an infinite number of spheres (as the sphere's surface is composed by an infinite number of circumferences). Only those spheres (initially embedded in the 4D space) that are perpendicular to our 3D space will be considered.

Each one of the selected spheres can be considered as a 3D subspace embedded in the 4D space (in analogous way, the initially perpendicular circumferences to Flatland could be considered as 2D spaces embedded in our 3D space, **Figure 3.20.b**). It is known by [Sommerville, 58] that the intersection between two perpendicular (n-1)-dimensional subspaces describe a (n-2)-dimensional subspace. By instantiation, in 4D space the intersection between two 3D subspaces will define a 2D subspace. In our current context, the intersection between each one of the spheres with our space will describe a plane (in analogous way, the intersection of each sphere's circumferences with Flatland, both describing 2D subspaces, describe a straight line). Such intersections will be visualizable in our 3D space as parallel circles embedded in the main sphere. See **Figure 3.22** (analogously, the intersections between the circumferences and *Flatland* could be visualizable as straight lines inside the main circumference, see **Figure 3.20.a**).

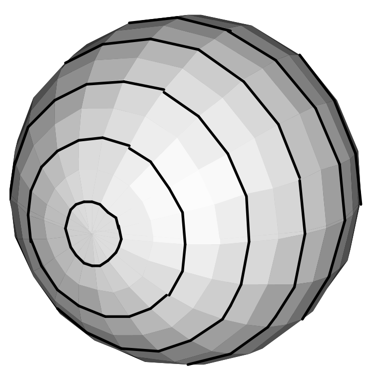
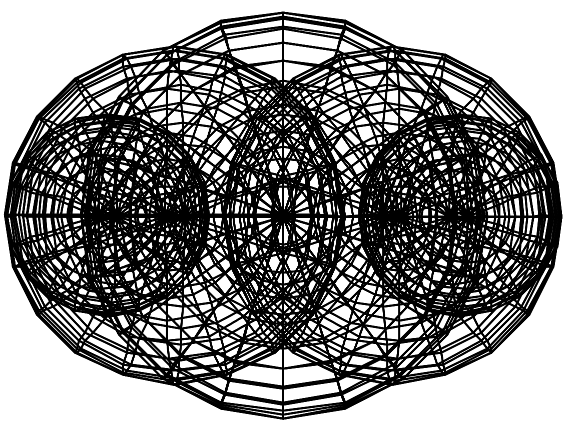
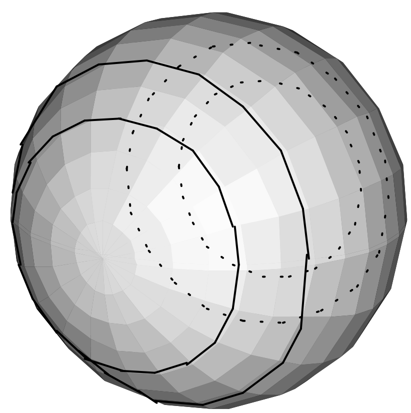


FIGURE 3.22
The intersections (the parallel circles embedded in the sphere) between the 3D space and some spheres on the 4D hypersphere's boundary (own elaboration).

The intersections between our space and the selected spheres describe a 2D subspace, which can be considered a rotation's plane in the 4D space (in the same way the straight lines were considered rotation's axis in the 3D space). Therefore, Since the selected spheres are perpendicular to our 3D space, it is sufficient to apply to each sphere a rotation of $\pm 90°$ around the intersection's plane, so that, they become embedded, and consequently, visualizable in our space. **Figure 3.23.a** shows five selected 4D hypersphere's spheres: the central sphere is the one embedded in the 3D space that didn't require any transformation (also shown in **Figure 3.23.b**), while the remaining four were rotated around the planes defined by the circumferences (shown in **Figure 3.23.b**) which are the product of the intersection between those four spheres with the 3D space.



a)



b)

FIGURE 3.23
Visualizing in the 3D space 4D Hypersphere's five selected spheres
(see text for details. Own elaboration).