# Chapter 1 <br> Introduction 

### 1.1 Historical Overview

In the ancient Greece, Euclid said that "a point has no dimension at all. A line has only one dimension: length. A plane has two dimensions: length and breadth. A solid has three dimensions: length, breadth, and height. And there it stops. Nothing has four dimensions".

Plato's allegory of the cave is presented In "The Republic" (370 b.C.). [Gutiérrez, 95] resumes it as follows: in a dark cave there are some prisoners chained since they were children. They can't see the daylight, the objects nor the people from the exterior. They just see the shadows that are projected onto the bottom's cave. Outside the cave there are a road and a torch that originates these shadows. The prisoners consider the shadows as their only reality. One of the prisoners escapes and discovers the real world. He returns to the cave and tries to convince the others. They don't believe him. An important aspect of Plato's allegory, is that it introduces the notion of a two-dimensional world and the experience of a being that discovers the existence of a three-dimensional world which includes him and his partners [Rucker, 84].

It is since the $18^{\text {th }}$ century when the scientific and philosophic communities began to consider the idea of a geometric fourth dimension. In 1747, Immanuel Kant in his first published paper "Thoughts on the True Estimation of Living Forces", questioned why our
space was three-dimensional [Pickover,99]. After more than one hundred years, in 1895 H. G. Wells recaptured this question in his classic novel "The Time Machine" [Wells, 55].

The first important approach to the fourth dimension (4D) was made by August Möbius in 1827. He speculated that rotations could work as reflections if any body (or figure) is passed through a higher dimension (one higher than the body or figure). For example, a right hand silhouette (a 2D figure) can be turned into a left hand silhouette passing it through the 3D space [Robbin, 92]. Möbius proposed that a 4D space is needed to turn right-handed three-dimensional crystals into left-handed crystals. A pair of objects which are congruent but not superimposable, unless we rotate one of them 180 degrees, is called an enantiomorphic pair [Pickover, 99].

In England, Arthur Cayley and John J. Sylvester described an Euclidean geometry of four dimensions where hyperplanes are determined by noncoplanar quadruples of points. They were able to move into a higher dimension because they added a new axiom: "outside any given three-dimensional hyperplane, there are other points" [Banchoff, 96].

In 1854, George Bernhard Riemann broke the cult position that the Euclidean geometry had for two thousand years with the introduction of the theory of higher dimensions. In "On the hypotheses which lie at the foundation of geometry", Riemann exposed the novel properties of higher dimensional space and demonstrated that Euclid's geometry is based only in the perception [Kaku, 94].

In 1855, Ludwig Schläffi established that regular polytopes' boundary is composed by a finite number of solid cells (like polyhedra's boundary is composed by a finite number of polygons) in different hyperplanes and placed in the form that every cell's face is shared with another cell [Coxeter, 84]. Schläffi determined all six regular 4D polytopes and their numerical and metrical properties [Robbin, 92]. Unfortunately, Schläffi's work does not have any illustration. Chapter 2 presents the properties of the 4D hypercube and simplex (some of the 4D regular polytopes).

The first steps for the visualization of 4D polytopes were made in the 1880's. In 1880, William Stringham presented for the first time illustrations of many 4D polytopes in the American Journal of Mathematics [Robbin, 92]. In the same years, Charles Howard Hinton, in the Oxford University, published "What is the fourth dimension?" where he presented three methods to visualize 4D polytopes: examining their shadows, their cross sections and their unravelings [Kaku, 94].

Finally in 1884, Edwin A. Abbot proposed a method to conceptualize the fourth dimensional space in his book "Flatland", where a 2D being (A. Square) tries to understand 3D objects that appear to him by means of analogy (like we would try to understand 4D polytopes) [Rucker, 77]. The notion of a bidimensional universe was introduced popularly by Abbot. However, it was earlier presented by the psychologist Gustave Fechner in his "Space has four dimensions" where he studied the interactions between human shadows generated by projection [Pickover, 99]. Anyway, the work that considered for the first time a 2D universe was Plato's allegory of the cave.

The term "hyperspace" was created in 1934 by John W. Campbell in his tale "The Mightiest Machine". Since that time the term is popularly used to reference spaces with more than three dimensions. Also the term "superspace" has been used as synonym of "hyperspace". It was introduced by the physicist John A. Wheeler. The prefix "hyper" is accepted by the scientific community to describe those entities with more than three dimensions [Pickover, 99].

### 1.2 Theoretical Physics and Hyperdimensional Geometry

In the last two decades of the $20^{\text {th }}$ century, Physics is one of the sciences that has used substantially the advances produced after the rupture of the dogma imposed by three dimensional Euclidean Geometry. During the last years of his life, Albert Einstein centered his efforts in building a theory that was called by him a "Theory of Everything" (today that effort still continues). This theory should explain all the forces in the nature, including light and gravity [Kaku, 94].

The first theory that related our universe with hyperspace was developed in 1919 by Theodr Kaluza, a mathematician from the University of Königsberg, Germany. Kaluza unified the Einstein's results about gravity with Maxwell's light theory by introducing a fifth dimension [Kaku, 94]. In this way, the universe was described with four geometric dimensions and one temporal dimension.

The main objection imposed by physicists, including Einstein, was the consideration of a fifth geometric dimension from which there was no experimental evidence [Kaku, 94]. In 1926, the mathematician Oskar Klein improved Kaluza's theory by calculating the length of the fifth dimension in $10^{-33}$ centimeters (this length is today known as Plank's length [Hawking, 01]). This very small length disabled any detection by the experimental way. The Kaluza-Klein Theory (as known today) was relegated by the hidden fifth dimension and the emergence of Quantum Mechanics [Kaku, 94].

In the 1920's, Erwin Schrödinger and Paul Dirac defined Quantum Mechanics which presents a different vision of the universe. While Einstein's General Relativity is a theory in which the universe is composed by stars and galaxies whose interactions are defined by a Space-Time geometry; Quantum theory defines a microcosm whose laws govern the interactions between atoms and subatomic particles (protons, electrons, etc) [Hawking, 01].

The history of physics in the second half of the $20^{\text {th }}$ century can be resumed in the study of the four forces that govern the universe [Kaku, 94]: gravity, electromagnetism, the strong force (the force that keeps the subatomic particles joined to form atoms [Hawking, 96]) and the weak force (responsible for the radioactive decay of elements as the uranium [Greene, 99]). Relativistic physics or Quantum physics considers each of these forces, which are incompatible. Therefore, the efforts for unifying them, at that time, gave unsatisfactory results [Greene, 99].

During that time, the universe was not reconsidered as a hyperspace again, because the use of the higher dimensions could not be justified. In fact, in the $19^{\text {th }}$ century, Riemann (who broke with three dimensional Euclidean geometry) manifested that higher dimensions had not physical application [Kaku, 94]. However, as seen before, Kaluza provided an application: the unification of the laws of physics [Kaku, 94]. Kaluza unified two forces (gravity and electromagnetism) by adding a higher dimension to Einstein's Space-Time geometry.

Since the 1980 's, the Kaluza-Klein theory was reconsidered only by what [Kaku, 94] has called the physics' new premise: "The laws of nature become simpler and more elegant when expressed in higher dimensions".

In 1984, due to a paper published by physicists John Schwarz and Mike Green (Queen Mary College, London), it was considered that a theory denominated as String Theory was the only way to combine the gravitational force and Quantum Theory. The fundamental objects in this theory are one-dimensional strings, they have only length [Hawking, 96].

The strings, in this theory, are moving inside a Space-Time geometry. Their movements correspond to vibrations (analogously to musical instruments with chords) which code different particles. It is known that matter is composed by atoms which, in turn, are composed by protons, neutrons and electrons which are composed by fundamental particles as the quarks. According to this theory, these particles are very small vibrating strings [Greene, 99] (Figure 1.1).


FIGURE 1.1
Strings as fundamental objects (taken from [Greene, 99]).

The strings can be considered as segments (open strings) or closed loops (closed strings). The intensity of the vibrations produce different wavelengths. According to the wavelength the string will have specific mass and forces. If the wavelength is short, then the mass of the particle will be greater. Since the strings occupy a line in space at every moment in time; inside a Space-Time geometry the strings compose a two-dimensional surface. Two strings can interact so that they can join and separate [Hawking,96].

String Theory is valid only if the Space-Time geometry where they move has 10 or 26 dimensions. One of the functions that define the vibrations of the strings is called the Ramanujan Function, in honor to the mathematician Srinivasa Ramanujan, who determined it since the $19^{\text {th }}$ century [Kaku, 94]. The function has 24 modes, one of each possible string's vibration, which can be reduced to 8 . The two additional dimensions were incorporated by physicists by considering relativist factors.

Due to the obtained results, it was concluded by the first time in physics' history, that String Theory was the "Theory of Everything" or "Ultimate Theory" or "Final Theory", just as Einstein was looking for [Greene, 99]. However, later on, it was found that there exists at least five valid String Theories (one with open strings and four with closed strings) [Hawking, 01]:

- Type I (Open) String Theory.
- Type IIA (Closed) String Theory.
- Type IIB (Closed) String Theory.
- Heterotic-O (Closed) String Theory.
- Heterotic-E (Closed) String Theory.

Since 1994 physicists started to consider strings as a class of objects that can be extended to more than one dimension. Paul Townsend (Cambridge) called these objects "p-branes". A p-brane extends in p directions (dimensions). In this way, a particle can be considered a 0-brane while strings are a special case, therefore they are denominated as 1-branes. It has been found that p-branes provide unifications for the fundamental forces (gravity, electromagnetism, strong and weak interactions) in spaces with 10 or 11 dimensions [Hawking, 01].

In spite of the diverse string and p-branes theories, it has been found that, between them, there exists a set of similarities that carry to the same physical results. These similarities have been called dualities between these theories. Furthermore, each theory offers advantages over the others in the calculus of different situations. Due to those
theories have some relations and specializations, physicists have considered them as pieces of a fundamental theory called the M-Theory. Currently, there is no approximation to this theory [Hawking, 01].

The String and p-brane Theories have been useful in unifying the fundamental forces, however, there still is the opposition referent to experimental evidence about higher dimensions. The answer that has been provided is the same given for Kaluza-Klein Theory: the length of higher dimensions is extremely small to be detected. For this answer it has been added a new factor: the anthropic principle [Hawking, 96].

The anthropic principle, in a resumed way, proposes that the universe's physical constants are determined so that it is possible the development of life [Kaku, 94]. Its relation with the number of dimensions in the universe is given by the following examples:

- One-dimensional beings only would have contact with their lateral neighborhoods, this would disable their transit [Coxeter, 84].
- A two-dimensional being could not have a digestive tract, because it would divide him in two separate parts [Kaku, 94] (see [Rucker, 84] for a solution to this problem). Furthermore, the existence of systems like the circulatory would be impossible [Hawking, 96].
- In a four-dimensional space, the gravitational force between two bodies would decrease faster with distance than in the three-dimensional space. This has as a consequence that a planet rotating around a star could affect its movement with a small interaction with
another massive body. A planet could collapse with its star or move away from it ([Hawking, 96] \& [Pickover, 99]).

Due to the anthropic principle, it has been established that during the beginning of the universe, only three geometric and one temporal dimensions were preserved, the necessary for the existence of life forms as ours. Higher dimensions were minimized during the same process to the least possible length, in other words, to the Plank length ( $10^{-33}$ centimeters) [Hawking, 96]. During the 1990's the building of particle accelerators started with the objective of confirming the existence of higher dimensions. One of them, the SSC (Superconducting Super Collider), in the United States, was cancelled in 1994, while others like the LHC (Large Hadron Collider) is currently in construction at Geneva, Switzerland [Hawking, 01].

Since the emergence of p-branes, a new approach was proposed for modeling the universe. It is established that our space with three geometric dimensions and one temporal dimension is the boundary (a p-brane) of a five dimensional region. The events in this 5D region are codified in the p-brane's region (the space we perceive) composing its actual state. This would be possible thanks to the Holographic Principle. This principle establishes that all the information associated with all phenomena in a region can be stored in its boundary from which it is possible to recover the original information [Hawking, 01].

The procedure for creating holograms can be resumed as follows: the light produced by a laser is divided into two beams. The first beam is directed toward an object (3D) whose reflexes are directed toward an holographic plate (2D). The second beam is directed
toward the holographic plate so that it collides with the first beam. The collision creates an interference that is stored by the plate (Figure 1.2). In the beginning, the image in the plate doesn't have relation with the original object. However, when a new laser or light source insides on the plate, a three-dimensional image of the original object appears, which can be observed from any angle [Talbot, 92].


FIGURE 1.2
Procedure for creating an hologram for storing a 3D object's information into a 2D plate (taken from [Talbot, 92]).

In this way, our universe can be seen as a great bubble where the events are taking place inside it. The reality we perceive, including us, is a set of patterns stored in the bubble's surface (a great holographic "plate") [Hawking, 01].

### 1.3 Methods for Visualizing 4D Polytopes

It is known that one of the most important contribution of Charles Howard Hinton was the three methods to visualize 4D polytopes in our 3D space: examining their shadows, their unravellings and their cross sections. The method of the shadows consists in that if it is possible to make drawings of 3D solids when they are projected onto a plane, then it is possible to make drawings or 3D models of 4D polytopes when they are projected onto a hyperplane [Coxeter, 84]. Let us follow the analogy presented in "Flatland" [Abbot, 84]. If a 3D being wants to show a cube to a 2D being (a flatlander) then the first one must project the cube's shadow onto the plane where the flatlander lives. For this case, the projected shape could be, for example, a square inside another square (Figure 1.3).


FIGURE 1.3
Projecting a cube on a plane (taken from [Kaku, 94]).

If a 4D being wants to show us a hypercube, he must project the shadow onto the 3D space where we live. The projected body could be a cube inside another cube [Kaku,94]
called central projection [Banchoff, 96] (Figure 1.4). We know that a projected cube onto a plane is just an approximation of the real one. Analogously, the hypercube projected onto our 3D space is also a mimic of the real one.


FIGURE 1.4
Hypercube's central projection onto the 3D space (taken from [Aguilera \& Pérez, 02]).

The six faces on the boundary of a cube can be unraveled as a 2D cross (Figure 1.5). The set of unraveled faces is called the unravelings of the cube. This is Hinton's second method for visualizing 4D polytopes.


FIGURE 1.5
Unraveling the cube (own elaboration).

In analogous way, the eight cubes on the boundary of a hypercube can be unraveled as a 3D cross [Kaku, 94]. This 3D cross was named tesseract by C. H. Hinton (Figure 1.6).


## FIGURE 1.6

The tesseract (taken from [Aguilera \& Pérez, 02]).

A flatlander will visualize the 2D cross, but he will not be able to assembly it back as a cube (even if the specific instructions are provided). This fact is true because of the needed face-rotations in the third dimension around an axis, which are physically impossible in the 2 D space. However, it is possible for the flatlander to visualize the raveling process through the projection of the faces and their movements onto the 2 D space where he lives.

Analogously, we can visualize the tesseract but we won't be able to assembly it back as a hypercube. We know this because of the needed cube-rotations in the fourth dimension around a plane which are physically impossible in our 3D space. However, it is possible for us to visualize the raveling process through the projection of the volumes and their movements onto our 3D space. Chapter 3 will present the Aguilera \& Pérez's methods for unraveling the hypercube and the 4 D simplex (analogous to the tetrahedron) and to visualize the process.

The slicing was the method used for Abbott in "Flatland" to describe the communication between 2D and 3D spaces [Banchoff, 96]. In that case, the 3D visitor, A. Sphere was perceived by A. Square (the flatlander) as a circle changing in size through the time (Table 1.1) because the first one was moving throught Flatland (a plane).

TABLE 1.1
Sphere's plane intersections with Flatland (own elaboration).

| 3D view | Flatland view | 3D view |
| ---: | ---: | ---: |
|  |  |  |

In analogous way, if a 4D hypersphere visits our 3D space, we would see a point that increases its size in all directions to take the shape of a sphere (Figure 1.7). During these movements, the 4D hypersphere was moving though our 3D space and we visualize its 3D-slicings.

$$
t=1
$$

$t=2$


## FIGURE 1.7

A 4D hypersphere seen in our 3D space (own elaboration).

The most common application of the slicings of a body are the conic sections.
Slicing a cone we produce the hyperbola, the parabola, the ellipse and the circle (Table 1.2). A 2 D slicing can be geometrically defined as the plane intersection with a body's surface. For analogy, a 3D slicing is the hyperplane intersection with a polytope's boundary.

TABLE 1.2
Slicing the cone and generating the conic sections (own elaboration).


### 1.3.1 Polytope's Visualization Related Works

At the Bell Labs, in 1966, Michael Noll created the first computer images of 4D hypercubes. Important features of Noll's programs were the use of stereo vision and 4D perspective projection ([Robbin, 92] and [Hollasch, 91]). The Noll's method was the generation of the pictures via plotter and the transference onto film [Noll, 67].

Thomas Banchoff (Mathematics Department at Brown University) has written computer programs that allow interactive manipulation (for example, with a joystick) of higher dimensional polytopes. Banchoff's technique of visualization is the projection of the polytopes' shadows onto 2D computer screens [Robbin, 92]. Banchoff and Charles Strauss are authors of the film "The Hypercube: Projections and Slicing", which was presented at the International Congress of Mathematicians in Helsinki in 1978.

In [Hollasch, 91] it is mentioned the work of Scott Carey and Victor Steiner. They have rendered 4D polytopes to produce 3D "images" (like the rendering of a 3D object produces a 2D image). Finally, the results of 4D-3D rendering are 3D voxel fields.
[Hollasch, 91] proposes a 4D ray-tracer that supports four-dimensional lighting, reflections, refractions, and also solves the hidden surfaces and shadowing problems in 4D space. The proposed ray-tracing method employs true four-space viewing parameters and geometry. Finally, the produced 3D field of RGB values is rendered with some of the existing methods.

In [Zhou, 91] is described a method for visualizing curves, surfaces and hypersurfaces embedded in the 4D space. Such method is mainly based in the 4D-3D-2D projection technique. It presents methodologies for applying quaternions in the definition of rotations, and describes algorithms for polygonization of surfaces in detail. Finally, it demonstrates some geometric properties and phenomena characteristic of 4 D space.
[Banks, 92] presents techniques for interaction with 4D-surfaces projected in the computer screen. Banks describes the ways to recover lost information that the 4D-3D-2D projection causes by means of visualization cues like depth. Also, ten degrees of freedom in 4D space are identified (6 rotations and 4 translations) and the use of devices to control the interaction.
[Gunn, 93] describes a software implementation for visualizing 4D hyper-surfaces. Its application doesn't consider only the 4D Euclidean Space, because it is possible the visualization in 4D hyperbolic and spherical hyperspaces. Furthermore, it is possible to visualize the hyper-surfaces from an intrinsic point of view (with the observer embedded in the hyper-surface) or from an extrinsic point of view (with the observer outside the hypersurface).

In [Hanson, 94] is presented a summary of the works achieved during the decade 1984-1994, without ignoring classical contributions (some of them are mentioned in section 1.1). Furthermore, [Hanson, 94] presents the concept of Visualizable Geometry, which can
be understood as the set of systems, concepts and methodologies related to the visualization of hyperspaces under distinct geometries and its applications.

In [D'Zmura, 00] and [D'Zmura, 01] is described the Hyper system, which was developed by researchers of the Department of Cognitive Sciences at the University of California, Irvine. The system's main objective is the creation of 4D virtual worlds and the users' interaction through Virtual Reality devices. In [Seyranian, 01] is described an experiment where the users are randomly positioned in a 4D virtual world (which was generated by Hyper system). The users must find a target in the minor possible time. Each user repeated the experiment several times. By the obtained results, [Seyranian, 01] concludes that an individual is able to navigate efficiently in environments with four dimensions.
[Aguilera \& Pérez, 01], [Aguilera \& Pérez, 02] and [Aguilera \& Pérez, 02c] discuss the method for visualizing 4D polytopes through their unravelings and present methodologies (which are described in chapter 3) for unraveling the hypercube and 4D simplex.

### 1.4 Dimensional Analogies

In the previous sections it has been mentioned with regularity the term "analogy". This has its foundation in the Method of the "Dimensional Analogies" ([Sagan, 80] called
them "Interdimensional Contemplations"). We know that it was popularly presented by Abbott in Flatland but it was previously considered since Plato's time.

When we are trying to visualize and understand the 4D space, the situation is similar for Flatland's inhabitants (flatlanders) trying to visualize 3D space. Due to this, it results very useful to consider the analogous situations with a reduced number of dimensions [Zhou, 91]. For example, try to answer the following question: What is a 4D being able to see in the 3D beings? In order to get the answer, first it must be referenced the interaction between a 3D being with a 2D being. A.Sphere is the 3D being that makes contact with A.Square in Flatland. From his 3D space, A.sphere can visualize the Flatland polygons' boundary, but additionally, he is able to see their interior (and therefore, their internal organs, if they have them, Figure 1.8). But in Flatland it is also referred Lineland, a onedimensional universe. Lineland's inhabitants were segments whose interior was visualized by A.Square. By analogy, we can expect that a 4D being, interacting with our 3D universe, could visualize our "boundary" (the skin), but furthermore, he could visualize our internal organs (in other words, the 4D being's vision could work as the systems of X rays, tomography or magnetic resonance [Pickover, 99], Figure 1.9).


FIGURE 1.8
A.Square seen by a three-dimensional being (its boundary and one of its internal organs: its "heart", are visualized. Taken from [Rucker, 77]).


FIGURE 1.9
Possible visualization by a 4D being of a human cranium (internal organs as the Central Nervous System could be visualized. Magnetic Resonance taken from [Olivera, 02]).

Fundamentally, the method of the analogies considers the contemplation of an analogy between 1D and 2D spaces, as well as between 2D and 3D spaces, then (through some extrapolation) between 3D and 4D spaces; and so forth. In this way the expected results can be suggested (a hypothesis is established) [Coxeter, 63]. Once the hypothesis is demonstrated, it is possible to suggest a generalization of the characteristic that has been demonstrated in n -dimensional space.

At this point, the relation between the method of the analogies and the scientific method results obvious:

1. Analysis: Observation of the analogies between 1 D and 2 D spaces; and between 2 D and 3D spaces.
2. Hypothesis: Proposal of an analogy between 3D and 4D spaces.
3. Synthesis: Selection of a mechanism to demonstrate the analogy.
4. Validation: The process of demonstration.
5. Argumentation: The proposal of an n-dimensional generalization based in the analogies previously observed and the demonstration already achieved.

Along this document the continuous application of the Method of the Analogies could be contemplated (except when the opposite is indicated).

### 1.5 The Concepts of Dimension

[Banchoff, 96] points out that the term "dimension" is commonly used for specifying characteristics which are feasible to be measured. For example, an object's list of dimensions would include width, height, weight, color, brightness, temperature, etc. The elements periodic table' list of dimensions includes atomic number, atomic mass, oxidation degree, etc.

The list of dimensions for a determined phenomenon can compose a space in which each point corresponds to a possible combination of the considered dimensions' values. The following are some examples:

- In [Feiner, 90] is presented the $n$-Vision system for the visualization of n-dimensional spaces. Its applications are related to the visualization and control of multidimensional financial data.
- [Wegenkittl, 97] presents a visualization interactive tool for exploring and analyzing multidimensional dynamical systems. Such systems include chemical reactions and statistical models.
- [Lees, 99] describes Geotouch, a Geographical Information System (GIS) which includes the time as a fourth dimension with the objective of visualizing earthquake hypocenters, volcanic eruptions or other time sequences of events.
- In [Weeks, 02] a set of educative tools for visualizing and understanding 2 and 3-manifolds are referred, whose main objective is to analyze the possible topologies of our universe.

Another perspective is offered by the Albert Einstein's Relativity Theory and the Space-Time Geometry as one of its main contributions. For the relativists, time is considered as the fourth dimension [Russell, 84] and it is fully linked with space. Einstein proposed that time and space are not independent because an event must be described in terms of the place and the time at which it occurs [Kaku, 94] (in other words, time and space compose the event's list of dimensions). For example, for a meeting it is necessary to specify a place in 3D space (a restaurant, a park, etc.) and the time (12:30 p.m., tomorrow, next Sunday, etc.). Consequently, space is an arbitrary 3D cross section of the 4D ST where 3D objects are moving forward in the direction of the remaining dimension, the time [Rucker, 77].

In strict terms, the fourth dimension is spatial, represented by a line perpendicular to each of three other perpendicular lines and it leads out of the space defined by the other three and never intersects them [Robbin, 92]. [Coxeter,84] considers Euclidean 4D space as the space with four coordinates $(x, y, z, w)$ instead of habitual two ( $x, y$ ) or three ( $x, y, z$ ). And it is established by him that two distinct points determine a straight line, three vertices of a triangle determine a plane and four vertices of a tetrahedron determine a hyperplane which has only a lineal equation that relates to the four coordinates.

### 1.6 Objectives

### 1.6.1 Problem's Definition

When working with multidimensional data, it is necessary to lay the foundation of the theoretical basis referent to the spaces where these data are embedded. In this way, it can be guaranteed the validity of the visualization and analysis to perform [Herman, 98]. Starting from this premise, fundamental in the methodology we have followed, is that in [Pérez, 01] we have presented some main properties related to Orthogonal Polytopes in 4D Spaces. This research was directly focused to a space with four geometric dimensions.

Currently, the main results obtained from our research in its first phase (which are resumed in chapters 2,3 and 4 ) and those presented in this work will allow the extension of the Extreme Vertices Model (EVM), presented by Aguilera \& Ayala in [Aguilera, 97], to the 4D space (EVM-4D). After this step, we will count with a representation model for 4D Orthogonal Polytopes, which will supply us a tool to perform queries and operations on these polytopes. Although we will define a Polytopes' representation model in a purely geometric and four-dimensional fashion, it won't limit our research's coverage because it could be applied over geometries like the space-time.

### 1.6.2 General Objectives

The general objective of our research is to propose and demonstrate how the numerical and geometrical properties for polygons (in 2D space) and for polyhedra (in 3D space) can be extended to define, in analogous way, the properties of 4D polytopes. Using these extensions, we will propose and demonstrate the generalizations that define the geometric and numerical properties of nD polytopes [Pérez, 01]. Moreover, it will be considered the use of the 4D polytope's geometrical and topological properties for representing multidimensional data and events under geometries like the space-time.

### 1.6.3 Specific Objectives

The study topics to be considered in our research are included, but not restricted to:

- 4D geometric transformations.
- Analysis and study of 4D-3D-2D projections.
- Numerical and geometrical properties of 4D Orthogonal Polytopes: A 4D Orthogonal Polytope is a polytope whose edges, faces and volumes (its boundary) are oriented in four orthogonal directions to $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ and W axis of the 4D space [Pérez, 01]
- Boundary analysis for 4D Orthogonal Polytopes.
- Modeling of 4D Orthogonal Polytopes: Boundary Representations and Hyperspatial Partitioning Representations.
- Boolean operations for 4D Orthogonal Polytopes.

Without leaving the application of 4D Polytopes' modeling and their Boolean operations in the analysis of multidimensional data and events not only embedded in a purely 4D Euclidean Geometry but by considering Space-Time Geometry (defined by [Hawking, 02] as that one where exists the equivalence, respect to 4D Euclidean Space, $w=t$ where $t$ is the time and the events, the points in 4 D space, are defined by three geometric coordinates and one temporal coordinate).

### 1.6.4 Organization

Besides this chapter, the structure for this document is the following:

- Chapter 2: Geometry of Four-Dimensional Space. This chapter presents the concepts of polytope and pseudo-polytope with their geometrical and topological properties. Moreover, it presents the numerical and geometrical properties of some of the probably most known 4D polytopes: the hypercube, the simplex, the cross polytope and the 0/1-polytopes. Finally, some 4D geometric transformations are mentioned.
- Chapter 3: Techniques for Visualizing the Four-Dimensional Space. The 4D-3D projections are presented as an extension of those applied in the visualization of 3D objects. Methods for hyper-flattening the 4D hypercube and simplex's boundaries in order to obtain their unravelings are also presented. Furthermore, four ways of intersection of a hypercube with 3D space are presented and it is mentioned a method for visualizing a 4D hypersphere.
- Chapter 4: Four-Dimensional Orthogonal Polytopes. This chapter presents an experimental and exhaustive analysis about the 4D Orthogonal Polytopes' boundary and their properties. Furthermore, the generalizations of these properties to be applied in the nD Orthogonal Polytopes are proposed.
- Chapter 5: Determining the Configurations for the nD-OPP's ( $n \geq 4$ ). Where it is described the "Test-Box" heuristic that gives a solution to the problem of determining the configurations that can represent the nD Orthogonal Pseudo-Polytopes. Moreover, there are presented some formulations that describe properties of the heuristic and these configurations. Finally, there are presented the main differences between the procedures for obtaining the 402 Hill's configurations and the 253 Aguilera \& Pérez's configurations for the 4D-OPP's.
- Chapter 6: Some Schemes for the Modeling of n-Dimensional Polytopes. This chapter analyses two categories for the representation of nD Polytopes: n-dimensional Boundary Representations and the Hyperspatial Partitioning Representations.
- Chapter 7: Future Work. This chapter describes the steps to follow in order to propose the Extreme Vertices Model in the 4D and 5D spaces (EVM-4D \& EVM-5D). There are proposed two applications, for the first experimental results related to the EVM-4D and the EVM-5D, under the contexts of 2D and 3D animations' managing and GIS.
- Conclusions. Where the main contributions of this work are summarized.

