TABLE I. Comparison of the measured optical constants with values taken from the literature.

λ (Å)	Direction and mode	n_1 Present	n_1 Previous	n_2 Present	n_2 Previous
		Silicon			
6328	[100], L	3.89	3.92 [a]	⋯	⋯
	[100], T	3.92		⋯	⋯
4880	[100], L	4.38	4.36 [a]	0.06	0.051 [b]
	[111], L	4.35		0.07	
		Germanium			
6328	[100], L	5.56		0.67	
	[111], L	5.61	5.53,[c] 5.43[d]	0.70	0.69,[c] 0.82,[d] 0.85[e]
	[111], T	5.55		0.65	
4880	[100], L	5.76	4.55,[c] 4.31[d]	2.38	2.56,[c] 2.30,[d] 1.55[e]

[a]Refs. 5, 6. [b]Ref. 7. [c]Ref. 8. [d]Ref. 9 [e]Ref. 10.

the peaks are absorption broadened, as distinct from Raman scattering where the optical phonon energy is virtually independent of wave vector. Analysis of the Brillouin spectra allows the measurement of both the optical constants (assuming that the hypersound velocity is known). These Brillouin results have been shown to agree closely with values reported in the literature, and are probably at least as reliable in this high absorption region. The multipassed interferometer is seen to be a powerful tool for the investigation of Brillouin scattering in opaque materials.

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Critical Exponents in 3.99 Dimensions*

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Critical exponents are calculated for dimension $d=4-\epsilon$ with ϵ small, using renormalization-group techniques. To order ϵ the exponent γ is $1+\frac{1}{6}\epsilon$ for an Ising-like model and $1+\frac{1}{5}\epsilon$ for an XY model.

A generalized Ising model is solved here for dimension $d=4-\epsilon$ with ϵ small. Critical exponents[1] are obtained to order ϵ or ϵ^2. For $d>4$ the exponents are mean-field exponents[1] independent of ϵ; below $d=4$ the exponents vary continuously with ϵ. For example, the susceptibility exponent γ is $1+\frac{1}{6}\epsilon$ to order ϵ for $\epsilon>0$, and 1 exactly for $\epsilon<0$. The definitions for nonintegral d are trivial for the calculations reported here but may be more difficult for exact calculations to higher orders in ϵ. The exponents will be calculated using a recursion formula derived elsewhere[2] which represents critical behavior approximately in three dimensions but turns out to be exact to order ϵ (see the end of this paper). Exponents will also be obtained for the classical planar Heisenberg model (XY model) and a modified form of Baxter's eight-vertex model.[3]

The background for the recursion formula is as follows.[4] Let $s_{\vec{r}}$ be the spin at site $\vec{r}$, and let its range be $-\infty < s_{\vec{r}} < \infty$. Let $\sigma_{\vec{k}}$ be the Fourier-transform variable $\sum_{\vec{r}} \exp(i\vec{k}\cdot\vec{r})s_{\vec{r}}$. Define a "block-spin" variable $s_l(\vec{x})$ to be

$$s_l(\vec{x}) = 2^{l(d/2-1)}\int_l \sigma_{\vec{k}} \exp[-i\vec{k}\cdot(2^l\vec{x})]\,d^dk, \quad (1)$$

where $\int_l$ means the integration is restricted to the range $|\vec{k}| > 2^{-l}$. The variable $s_l(\vec{x})$ is, very roughly, the sum over all spins $s_{\vec{r}}$ in a block of length 2^l surrounding the point $2^l\vec{x}$.[5] There is an effective interaction (in Kadanoff's sense[6]) of Landau-Ginsberg form for the block-spin variable[7]:

$$\mathcal{H}_l = -\int[\tfrac{1}{2}\nabla s_l(\vec{x})\cdot\nabla s_l(\vec{x}) + Q_l(s_l(\vec{x}))]\,d^3x. \quad (2)$$

For $l=0$ this specifies the interaction of interest. The function Q_l is obtained from the recursion formulas

$$Q_{l+1}(y) = -2^d \ln[I_l(2^{1-d/2}y)/I_l(0)], \quad (3)$$

$$I_l(z) = \int_{-\infty}^{\infty} dy \exp[-y^2 - \tfrac{1}{2}Q_l(y+z) - \tfrac{1}{2}Q_l(-y+z)]. \quad (4)$$

The initial function $Q_0(y)$ may be chosen as

$$Q_0(y) = r_0 y^2 + u_0 y^4, \quad (5)$$

where the constant r_0 is varied to locate the critical point of the model, and the y^4 term is present so that the model is not the Gaussian model.[8] One must choose $u_0 \geq 0$ to avoid a divergent integral in (4). The effective interaction $\mathcal{H}_l$ determines the spin-spin correlation function through

$$\hat{G}(\vec{k}) \sim (k^2)^{-1} I_{l(k)}^{-1}(0) \int_{-\infty}^{\infty} dy\, y^2 \exp[-y^2 - Q_{l(k)}(y)], \quad (6)$$

where $l(k) \sim -\log_2(ka)$ and a is the lattice spacing. (This formula is only an order-of-magnitude estimate.[2]) To derive these results, the partition function Z was first defined as a functional integral over all $\sigma_{\vec{k}}$ ($|\vec{k}| \lesssim 1$) of the initial Boltzmann factor $\exp(\mathcal{H}_0)$.[9] The recursion formula was obtained by performing the functional integral over $\sigma_{\vec{k}}$ for a factor-of-2 range of $|\vec{k}|$; qualitative approximations were made to ensure the simple form (2) for the effective interaction $\mathcal{H}_l$.[2,10]

General considerations (confirmed by a numerical study[2]) show the following. At the critical point $r_0 = r_c$ (with u_0 fixed), the function $Q_l(y)$ normally approaches a limit $Q_c(y)$ for $l\to\infty$.[11] The function $Q_c(y)$ is a "fixed point" of the recursion formula, namely an l-independent solution. The fixed point is unstable to changes in r_0; for $r_0 \simeq r_c$ and reasonably large l, $Q_l(y)$ has the form

$$Q_l(y) \simeq Q_c(y) + (r_0 - r_c)\lambda^l R_c(y), \quad (7)$$

where λ is a constant. The critical exponents[1] ν and γ are given by $2\nu = \gamma = 2(\ln 2)/\ln\lambda$; the approximations made in deriving the recursion formula enforce $\eta = 0$. For $d=3$ the numerical work gave $2\nu = 1.217$.

If $u_0 = 0$, the solution of the recursion formula approaches a different fixed point. For this "Gaussian" fixed point,[2] $Q_l(y)$ has the form (7) for any r_0, but now $Q_c(y) \equiv 0$, $R_c(y) = y^2$, $r_c = 0$, $\lambda = 4$, and $2\nu = \gamma = 1$.

There is a common expectation that critical exponents become the mean-field exponents at $d = 4$.[12] This suggests that the nontrivial fixed point coincides with the Gaussian fixed point for $d = 4$ and can be calculated analytically for $d \simeq 4$. This is correct; the calculation is remarkably easy to perform and will be summarized here.

Let ϵ be small; let the initial constants r_0 and u_0 be of order ϵ. Then by induction in l one finds that $Q_l(y)$ has the form

$$Q_l(y) = r_l y^2 + u_l y^4 + O(\epsilon^3), \quad (8)$$

where r_l and u_l are of order ϵ. The recursion formulas for r_l and u_l are[13]

$$r_{l+1} = 4[r_l + 3u_l(1+r_l)^{-1} - 9u_l^2] + O(\epsilon^3), \quad (9)$$

$$u_{l+1} = (1 + \epsilon\ln 2)u_l - 9u_l^2 + O(\epsilon^3). \quad (10)$$

A fixed point is a solution $r_l = r$, $u_l = u$ of the recursion formulas independent of l. It is evident from Eq. (10) that there are two fixed points: the Gaussian fixed point $r = u = 0$, and a nontrivial fixed point

$$u = u^* = \tfrac{1}{9}\epsilon\ln 2 + O(\epsilon^2), \quad (11)$$

$$r = r^* = -\tfrac{4}{9}\epsilon\ln 2 + O(\epsilon^2). \quad (12)$$

Clearly the two fixed points coincide for $d = 4$ and u^* is small for ϵ small.

Arbitrary initial values r_0 and u_0 will not correspond to either fixed point. Therefore the two fixed points compete to determine the asymptotic behavior for $l\to\infty$ of $Q_l(y)$ for a given initial value of u_0. In consequence there are *domains* of initial values associated with each fixed point. In the present case it is easy to determine the domains of u_0 from Eq. (10). The value of r_0 must be fixed at its critical value $[r_c = -4u_0 + O(\epsilon^2)]$ once u_0 is given. One sees from Table I that the nontrivial fixed point wins the competition for $\epsilon > 0$ (unless $u_0 = 0$), and the Gaussian fixed point wins for $\epsilon < 0$. To obtain γ and ν for $d < 4$ one

TABLE I. Domains of initial values for the two fixed points of the generalized Ising model. Only $u_0 \lesssim \epsilon$ is considered; "unphysical" means negative values of u_0.

Dimension	Fixed point Gaussian	Nontrivial
$d<4$	$u_0=0$	$0<u_0$
$d>4$	$0\leq u_0$	unphysical

must determine the constant λ for the nontrivial fixed point. This one does by linearizing the recursion formulas for small departures from the fixed point and looking for solutions of the form $\Delta r_l = A\lambda^l$, $\Delta u_l = B\lambda^l$, where A and B are independent of l and proportional to $(r_0 - r_c)$. One gets an eigenvalue equation for λ; to order ϵ this equation is

$$\lambda\begin{bmatrix}A\\B\end{bmatrix}=\begin{bmatrix}4-12u & 12\\ 0 & 1+\epsilon\ln 2-18u\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix}. \tag{13}$$

The largest eigenvalue λ determines the unstable solution of Eq. (7) and is used in the calculation of ν and γ. For $\epsilon<0$ one gets the Gaussian exponents. For $\epsilon>0$ one gets $\lambda = 4-12u^*$ giving γ $1+\frac{1}{6}\epsilon$. A more accurate calculation[14] gives

$$2\nu=\gamma=1+\tfrac{1}{6}\epsilon+\epsilon^2(\tfrac{1}{36}+\tfrac{1}{54}\ln 2),\quad \epsilon>0. \tag{14}$$

For $\epsilon=1$ this differs from the numerical result 1.217 by only 0.010. These results are plotted in Fig. 1.

The above analysis is easily extended to (classical) models where the spin $s_{\vec{r}}$ has n components $s_{\vec{r}j}$, as in the Heisenberg model. The recursion formulas are still Eqs. (3) and (4) except that y and z are vectors $\vec{y}$ and $\vec{z}$, $\int dy$ is replaced by $\int d^n y$, and $y^2=\vec{y}\cdot\vec{y}$. Consider the following initial form for $Q_0(y)$, for $n=2$:

$$Q_0(y)=r_0(y_1^2+y_2^2)+u_0(y_1^4+y_2^4)+g_0y_1^2y_2^2. \tag{15}$$

For $g_0=0$ one has two independent Ising-like models. For $g_0=2u_0$ the model has the rotational symmetry of the XY model. For $g_0=6u_0$ the model turns out again to be two independent Ising-like models if one uses the variables $x_1=(y_1+y_2)/\sqrt{2}$ and $x_2=(y_1-y_2)/\sqrt{2}$. For other values of g_0 the model involves two Ising-like models with $g_0y_1^2 \times y_2^2$, providing, in the language of Kadanoff and Wegner,[15] an energy-energy–type coupling of the two: The model resembles the reformulation[15] of Baxter's eight-vertex model.[3]

The critical behavior has been computed to order ϵ for these models. The essential recursion formulas are

$$u_{l+1}=(1+\epsilon\ln 2)u_l-9u_l^2-\tfrac{1}{4}g_l^2, \tag{16}$$

$$g_{l+1}=(1+\epsilon\ln 2)g_l-6u_lg_l-2g_l^2. \tag{17}$$

There are four fixed points:

$u=g=0$ (Gaussian),

$u=\frac{1}{9}\epsilon\ln 2,\quad g=0$ (Ising-like),

$u=\frac{1}{18}\epsilon\ln 2,\quad g=6\lambda$ (Ising-like),

$u=\frac{1}{10}\epsilon\ln 2,\quad g=2\lambda$ (XY-like).

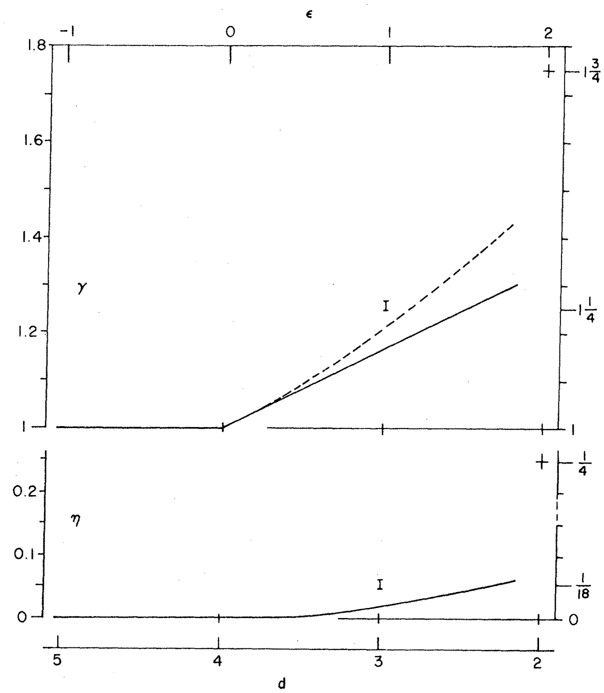

FIG. 1. Plot of the susceptibility exponent γ and the critical-point correlation exponent η versus dimension d to leading order in $\epsilon=4-d$. The dashed curve represents the truncated expansion (14). The special values indicated by "+" and "I" are for the standard spin-$\frac{1}{2}$ Ising models in two and three dimensions.

The XY-like fixed point is the most stable one for $\epsilon>0$ and gives the critical behavior for any initial condition with g_0 in the range $0<g_0<6u_0$. The critical exponents for this regime are $2\nu=\gamma=1+\frac{1}{5}\epsilon$. The Ising-like roots are less stable and give the critical behavior only for $g_0=0$ or $g_0=6u_0$. The Gaussian root is the least stable root giving the critical behavior only for $u_0=g_0=0$. The ranges $g_0<0$ and $g_0>6u_0$ are anomalous; with this type of initial condition, iteration of the recursion formula gave values of g_l which increase without limit in magnitude. This takes one outside the range of validity of the recursion formulas (16) and (17); we do not know the critical behavior for this range of g_0.

To check these calculations an *exact* calculation was performed to determine γ to order ϵ, and η to order ϵ^2. The effective interaction $\mathcal{H}_l$ was allowed to be the general form

$$\mathcal{H}_l = -\iint d^3x\, d^3y\, u_{2l}(\vec{x}-\vec{y}) s_l(\vec{x}) s_l(\vec{y}) - \int d^3x_1 \cdots \int d^3x_4\, u_{4l}(\vec{x}_1, \cdots, \vec{x}_4) s_l(\vec{x}_1) \cdots s_l(\vec{x}_4) - \cdots. \tag{18}$$

Exact recursion formulas for u_{2l}, u_{4l}, etc. were obtained as power series in the non-Gaussian terms, by integrating exactly the functional integral over $\sigma_{\vec{k}}$ with $|\vec{k}|$ restricted to the range $b^{-l} < |\vec{k}| < b \cdot b^{-l}$, where b was left arbitrary. To obtain a fixed point it was necessary to use a more general scale factor $b^{l(d-2+\eta)/2}$ in Eq. (1). By induction it was shown that u_{4l} is of order ϵ, u_{6l} of order ϵ^2, u_{8l} of order ϵ^3, etc. A nontrivial fixed point was found for $\epsilon > 0$ with exponents $2\nu = \gamma = 1 + \frac{1}{6}\epsilon$ to order ϵ and $\eta = \frac{1}{54}\epsilon^2$ to order ϵ^2 (as plotted in Fig. 1).[16] Thus the result for 2ν and γ from the approximate recursion formulas is exact to order ϵ, but the result $\eta = 0$ is incorrect in order ϵ^2. The calculations for the Heisenberg and modified Baxter models are exact to order ϵ.

The exact results obtained here for d near 4 complement the exact solutions of two-dimensional models. Qualitative results concerning the competition between different fixed points and corresponding sets of exponents are probably true more generally. The analysis described here is a simple and powerful method for obtaining such results.

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[5]The definition of $S_l(\vec{x})$ involves several scale factors: $2^{l(d/2-1)}$, 2^l, and a factor (not shown) depending on kT. These factors are introduced so that $s_l(\vec{x})$ is of order 1 (at the critical point) and so that $\vec{x}$ is measured in units of the block spacing.

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[11]The function $Q_c(y)$ is plotted in Fig. 4 of Ref. 2.

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[13]These equations are Eqs. (4.29) and (4.30) of Ref. 2.

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[16]This is a very small value for η; see, in this connection, A. A. Migdal, Zh. Eksp. Teor. Fiz. 59, 1015 (1970) [Sov. Phys. JETP 32, 552 (1971)].