BLACK HOLES AS WINDOWS TO EXTRA DIMENSIONS

Aharon DAVIDSON and David A. OWEN
Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

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We derive the static three-dimensional spherically symmetric solution of the Einstein equations in an empty five-dimensional universe, and analyse the effective four-dimensional perspective. The radius of the compact fifth dimension is regarded as a local quantity. Our main conclusion is that the extra dimension becomes visible, and the associated Kaluza–Klein gauge interaction singular, when approaching the modified Schwarzschild horizon. The possibility of "massless" black holes is noted.

The Schwarzschild solution [1] can be regarded as one of the main achievements of general relativity in the field of celestial mechanics. Not only is it the key for understanding gravitational collapse and black holes, but it is first of all the geometrical generalization of Newton's $1/r^2$ force law. At present, when extra dimensions are commonly invoked to generate local gauge invariance, and the idea of electro/nuclear grand unification [2] requires a mass scale in an intriguing vicinity to the Planck mass [3], we find it quite tempting to re-examine the geometry surrounding a massive point particle, this time in the presence of compact extra dimensions. For our discussion to be non-trivial, it is essential that the length scale associated with the extra dimensions be regarded as a local quantity. This way, following the generalized Kaluza–Klein trail of unification, we would like to extract the effects of interference between the gravitational field and the emerging gauge interactions. One of the striking conclusions we are led to is that the extra-dimensional space becomes fully visible, and the underlying gauge interaction singular, when approaching the Schwarzschild horizon. Black holes may thus play the unexpected role of windows to the extra dimensions.

We start our discussion by considering a five-dimensional metric tensor which is static, three-dimensional spherically symmetric, and above all independent of the extra spatial dimension. Using isotropic coordinates, the corresponding line element can be written as

$$ds^2 = -A^2(r)\,dt^2 + B^2(r)\,dx^i\,dx^i + C^2(r)\,dy^2\,, \quad (1)$$

with $r^2 \equiv \Sigma_{i=1}^3 (x^i)^2$. It is assumed, following the general Kaluza–Klein idea, that the y coordinate compactify. But in the absence of any first symmetry principle, the corresponding radius of compactification $\sim C(r)$ is a priori treated as a local quantity, meaning that the associated gauge coupling constant may not be a constant any more. It should be emphasized that, for the sake of simplicity, we have chosen to sacrify some generality and to make our point without introducing an off-diagonal $\sim dy\,dt$ potential type term in eq. (1).

The various scale functions $A(r)$, $B(r)$, $C(r)$ are determined from the following Einstein equations

$$R_{tt} = -\frac{AA'}{B^2}\left(\frac{2}{r}+\frac{A''}{A'}+\frac{B'}{B}+\frac{C'}{C}\right) = 0\,, \quad (2a)$$

$$R_{yy} = \frac{CC'}{B^2}\left(\frac{2}{r}+\frac{A'}{A}+\frac{B'}{B}+\frac{C''}{C'}\right) = 0\,, \quad (2b)$$

$$R_{ij} = \left[\frac{1}{r}\left(\frac{A'}{A}+3\frac{B'}{B}+\frac{C'}{C}\right)+\frac{B''}{B} + \frac{B'}{B}\left(\frac{A'}{A}+\frac{C'}{C}\right)\right]\delta_{ij} + \left[\left(\frac{A''}{A}+\frac{B''}{B}+\frac{C''}{C}\right) - \left(\frac{1}{r}+2\frac{B'}{B}\right)\left(\frac{A'}{A}+\frac{B'}{B}+\frac{C'}{C}\right)\right]\frac{x^i x^j}{r^2} = 0\,, \quad (2c)$$

with the prime denoting differentiation with respect to r. The RHS of eq. (2) vanishes as a matter of prin-

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ciple. No five-dimensional energy–momentum tensor is introduced following the observation that the locality of the compactification radius of the fifth dimension is sufficient for effectively generating proper four-dimensional pressure and radiation density. Also the cosmological term Λ is taken to be zero not only for supporting the elegance of the theory. In a Robertson–Walker-type local Kaluza–Klein theory [4], which is the cosmological analog of the present scheme, the gauge coupling constancy can be asymptotically ($t \to \infty$) respected only provided $\Lambda = 0$.

The first two equations (2a,b) are easily integrated, leading to the relation

$$A'/A = -kC'/C\,, \tag{3}$$

with $-k$ being an integration constant. Consequently, $A \sim C^{-k}$ and $B \sim C^k/r^2C'$. The substitution of eq. (3) into the δ_{ij} part of eq. (2c) gives rise to

$$\frac{C^{k-1}}{r^3}\left(\frac{r^3B'}{C^{k-1}}\right)' = (k-1)\frac{BC'}{rC} \sim (k-1)\frac{C^{k-1}}{r^3}\,. \tag{4}$$

This in turn implies $(B/C^{k-1})' \equiv -(k-1)BC'/C^k + B'/C^{k-1} \sim 1/r^3$, and after some algebra one arrives at the simple differential equation $C'/C = 1/(\alpha + \beta r^2)$ for some α, β. Using a more convenient set of integration constants ϵ, k, a we can finally present our analytic solution, namely

$$A(r) = \left(\frac{ar-1}{ar+1}\right)^{\epsilon k}\,, \tag{5a}$$

$$B(r) = \frac{1}{a^2r^2}\frac{(ar+1)^{\epsilon(k-1)+1}}{(ar-1)^{\epsilon(k-1)-1}}\,, \tag{5b}$$

$$C(r) = C_0\left(\frac{ar+1}{ar-1}\right)^{\epsilon}\,. \tag{5c}$$

This solution is subject to a crucial consistency condition

$$\epsilon^2(k^2 - k + 1) = 1\,, \tag{6}$$

imposed by the off-diagonal $\sim x^i x^j/r^2$ part of eq. (2c). The scales of $A(r)$ and $B(r)$ have been absorbed by redefining the variables t and x, respectively, yet the overall scale of $C(r)$ is physically meaningful once the period of y is fixed. Notice that the solution (5) is invariant under

$$\epsilon \to -\epsilon, \quad k \to k, \quad a \to -a, \tag{7}$$

so that the convention $a > 0$ can be adopted without any lose of generality.

At spatial infinity we recover the ground state of the traditional Kaluza–Klein theory. That is a direct product of a flat Lorentz geometry and a circle. It is encouraging to verify that it is indeed ordinary electromagnetism which is selected out by special relativity. The radius of compactification turns out to be a constant at the $\epsilon \to 0$ limit. It is important to notice that, by virtue of eq. (6), $\epsilon \to 0$ indicates $k \to \infty$ in such a way that $\epsilon k \to 1$. Consequently,

$$A(r) \to (ar-1)/(ar+1)\,, \tag{8a}$$

$$B(r) \to (1/a^2r^2)(ar+1)^2\,, \tag{8b}$$

and of course

$$C(r) \to C_0\,. \tag{8c}$$

This is how the usual Schwarzschild geometry, in isotropic disguise, makes its appearance. At any rate, it is the new physics associated with $\epsilon \neq 0$ which we are after.

The time is ripe now to establish the effective four-dimensional picture. A four-dimensional observer who is not directly aware of the existence of an extra dimension, perhaps because $C_0 \ll 1$, naturally interprets the scale functions $A(r)$ and $B(r)$ as if they are governed by an effective energy–momentum tensor $T^{\mathrm{eff}}_{\mu\nu}$. More precisely, for the effective four-dimensional action to contain the exact Einstein lagrangian, i.e. to deal with a constant effective gravitational constant, the dictionary must read

$$g^{\mathrm{eff}}_{\mu\nu} = [C(r)/C_0]g_{\mu\nu}\,. \tag{9}$$

Such a factorization is a standard Kaluza–Klein technique [5].

At this stage, all the physics is concentrated in the effective metric tensor

$$-g^{\mathrm{eff}}_{00} = \left(\frac{ar-1}{ar+1}\right)^p\,,$$

$$g^{\mathrm{eff}}_{ij} = \frac{1}{a^4r^4}\frac{(ar+1)^{2(p+1)}}{(ar-1)^{2(p-1)}}\delta_{ij}\,, \tag{10}$$

where p is defined by

$$p \equiv \epsilon(k - \tfrac{1}{2})\,. \tag{11}$$

The newtonian $1/r$ potential can be identified now by a direct comparison with the "standard form" of the metric far away from a stationary source. Indeed, since $-g_{00}^{\text{eff}} \sim 1 - 4p/ar$ and $g_{ij}^{\text{eff}} \sim (1 + 4p/ar)\delta_{ij}$, the mass parameter is nothing but

$$M = 2p/a \,. \tag{12}$$

The novel feature has to do with the modified location of the Schwarzschild horizon. $r_s \equiv 1/a = M/2p$ differs from the conventional value by a factor of $1/p$. It is then interesting to notice that the absolute value of p is severely restricted by the consistency relation (6), i.e. $|p| \leqslant 1$, so that r_s cannot be smaller than $M/2$. On the other hand, $p = 0$ is a perfectly allowed value, telling us that a proper horizon can be established even at the $M \to 0$ limit. It has not escaped our attention that this property may suggest the existence of "massless" black holes.

To complete the four-dimensional interpretation, one must calculate the effective energy–momentum tensor $T_{\mu\nu}^{\text{eff}} = R_{\mu\nu}^{\text{eff}} - \frac{1}{2} g_{\mu\nu}^{\text{eff}} R^{\text{eff}}$. The explicit expressions for the density and pressure are somewhat lengthy, and here we give just their asymptotic behavior at large distances, namely

$$\rho = m^2/r^4 + \dots \,,$$

$$P^i{}_j = (m^2/r^4)(-\delta_{ij} + 2x^i x^j/r^2) + \dots \,, \tag{13}$$

with the "equation of state"

$$P + \tfrac{1}{3}\rho = 0 \,. \tag{14}$$

The necessarily positive quantity m^2 obeys the relation

$$m^2 + M^2 = 4/a^2 \,. \tag{15}$$

We see that two independent (mass)^2 contributions are involved in our solution. It is the combination $m^2 + M^2$ which fixes the area of the Schwarzschild surface, whereas the long-range newtonian curvature is dominated exclusively by M. Being aware of the conventional four-dimensional theory of charged black holes [6], and recalling the Kaluza–Klein electromagnetic interpretation of the compact fifth dimension, it is quite tempting to speculate that m is somehow related to the electric charge, that is $m \sim Q$, and that the above is a piece of a more general electromagnetic effect. Obviously, apart from the y-periodicity, the present version is electromagnetically impotent since the potential term $\sim \mathrm{d}y\, \mathrm{d}t$ has never been introduced. We hope to gain a better understanding once the general solution is fully analysed, and the Reissner–Nordstrøm limit [7] is met.

Having in mind the Kaluza–Klein ground state at large distances, our attention is naturally focused on the quantity $C(r)$ which signifies spatial variations of the "fine structure constant". Such an effect may show up at relatively large distances since $C(r) = C_0(1 + 2\epsilon/ar + \dots)$ and $m^2 \sim \epsilon^2$, provided ϵ is big enough. The identification of the mass parameter M dictates the positivity of $p = \epsilon(k - \frac{1}{2})$, and the consistency relation (6) tells us that $|\epsilon| \leqslant 2/\sqrt{3}$, yet the overall sign of ϵ is by far more provocative. A positive ϵ would imply $C(r) \to \infty$ for $r \to r_s$, whereas a negative ϵ means $C(r) \to 0$ at the same limit. Each of the two alternatives is dramatic from the Kaluza–Klein point of view, and is unavoidable as long as ϵ is not strictly zero. However, the clue cannot be extracted from the effective four-dimensional action. In fact, it comes from cosmology. The Robertson–Walker-type solution in an empty five-dimensional universe, the cosmological analog [4] of the theory in discussion, tells us, without any sign ambiguity, that when the ordinary three-space undergoes the Big Bang singularity, the extra dimension explodes. Consequently, the physical choice is

$$\epsilon > 0 \,, \tag{16}$$

and the overall conclusion is the following. If extra dimensions do exist, they were visible at the very early universe, and are still visible at the neighborhood of black holes.

Altogether, starting from an empty five-dimensional universe, we have derived the generalized Schwarzschild geometry. The solution is parametrized by two independent (mass)^2 contributions. The location of the horizon is sensitive to the $m^2 + M^2$ combination, whereas the Newton mass M dominates the curvature at large distances. As a consequence, the theory seems to allow for $M = 0$ black holes. The conventional Schwarzschild solution is obtained for $m = 0$. As far as the Kaluza–Klein idea is concerned, the length scale of the otherwise compact extra dimension explodes when nearing the Schwarzschild radius. The extra space becomes then visible, while dictating the singularity of the associated local gauge interaction. The most general case, where an off-diagonal potential term is included, is currently under intensive investigation.

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